Category Theory
List of problems

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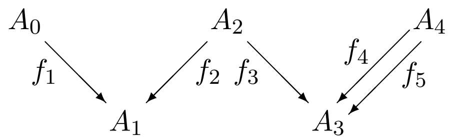
Pick 9 problems (out of 13) and provide written solutions till January 31st, 2020.

1. Express the limit of the following diagram



   using products and equalizers.

2. Let $\mathbf{C}$ be a small category, $F : \mathbf{C}^{op} \to Set$ a functor. Show that F is representable if and only if the category $\int_{\mathbf{C}} F$ of elements of F has the terminal object.

3. Let $G : \mathcal{A} \to Set$ be a functor from a locally small category $\mathcal{A}$.

   (a) Show, that if a functor G has a left adjoint, it is representable.

   (b) Show that if $\mathcal{A}$ has limits, then G is representable if and only if G preserves limits and satisfies the following solution set condition: there is a set I and an I-indexed family $\{A_i\}_{i \in I}$ objects of $\mathcal{A}$ such that for any object $A \in \mathcal{A}$ and element $x \in G(A)$ there is $i \in I$, morphism $h : A_i \to A$ and $y \in G(A_i)$ 2 such that $G(h)(y) = x$.

4. Let $\mathbf{C}$ be a small category, $\mathcal{E}$ a (locally small) cocomplete category, $K : \mathbf{C} \to \mathcal{E}$ a functor. The functor K induces a functor

$$R : \mathcal{E} \to Set^{\mathbf{C}^{op}}$$

   defined on objects E in $\mathcal{E}$ as

$$R(E) : \mathbf{C}^{op} \to Set$$

   so that for $f : C \to C'$ is $\mathbf{C}$ we have

$$R(E)(C) = \mathcal{E}(K(C), E)$$

   and for $h : K(C') \to E$ in $R(E)(C')$

$$R(E)(f)(h) = h \circ K(f).$$

   (a) Show that the functor R has a left adjoint $L \dashv R$. Hint: use the fact that presheaf in $Set^{\mathbf{C}^{op}}$ is a colimit of the representable functors.

(b) Let $K : \mathbf{C} \to (Set^{\mathbf{C}})^{op}$ be the dual of the Yoneda embedding $Y : \mathbf{C}^{op} \to Set^{\mathbf{C}}$. Show that in this case both functors in the above adjunction $L \dashv R$ send representable functors to representable functors.

5. Let $\{A_i\}_{i \in I}$ be a family of sets. We define a functor

$$A : Set \to Set$$

by

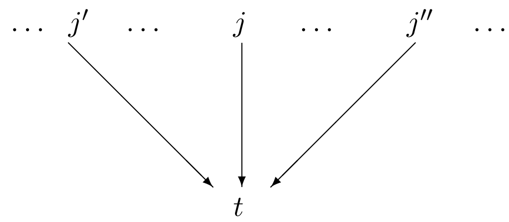
$$A(X) = \coprod_{i \in I} X^{A_i}, \qquad A(f) = f \circ h \in Y^{A_i}$$

for set X, functions $f : X \to Y$ and $h \in X^{A_i}$.

(a) Show that the functor A preserves wide pullbacks.

(b) For what kind of families of sets $\{A_i\}_{i \in I}$ the functor A preserves all limits.

A wide pullback is a limit over small category J whose non-identity arrows are as displayed below



i.e., J has a terminal object t, and the full subcategory spanned by the other objects is discrete.

6. Describe (binary) products and coproducts in category

(a) abelian groups Ab;

(b) commutative rings $CRng$. Hint: Check that the coproducts are tensor products over the ring of integers Z.

7. Show that the forgetful functor

(a) $U_1 : Mod_k \to Ab$, forgetting the action of the ring k, from the category of modules over a commutative ring k to abelian groups has a left adjoint;

(b) $U_2 : Rng \to Mon$, forgetting the addition from the category of rings to the category of monoids has a left adjoint.

8. Let $\mathbf{C}$ be a small category, $f : A \to B$ be a morphism in presheaves on $\mathbf{C}$, i.e. in $Set^{\mathbf{C}^{op}}$, X a subobject of A and Y a subobject of B. Check that we have the following equality of subobjects of B

$$\exists_f(X) \cap Y = \exists_f(X \cap f^{-1}(Y))$$

Show that, for some category $\mathbf{C}$ and objects A, B, X, Y, as above, the equality of subobjects of B

$$\forall_f(X) \cup Y = \forall_f(X \cup f^{-1}(Y))$$

do NOT hold (even if it always hold in Set. Hint: it is enough to consider $\mathbf{C}$ be $\mathbf{2} = \{0 \to 1\}$.