Theories of Analytic Monads

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(joint work with Stanisław Szawiel)

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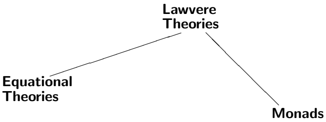

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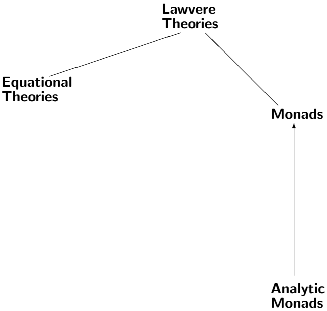
Morphism of monads = nat. transf. commuting with units and multiplications



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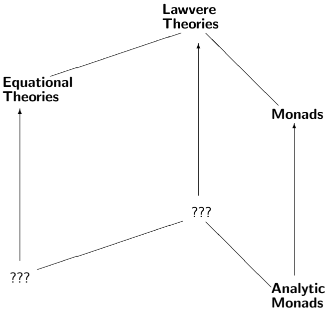
Analytic Monad = functor part preserves weak wide pullbacks; unit and multiplication are weakly cartesian nat. transf.

Morphism of analytic monads = weakly cartesian nat. transf. commuting with units and multiplications

Categories of Equational Theories



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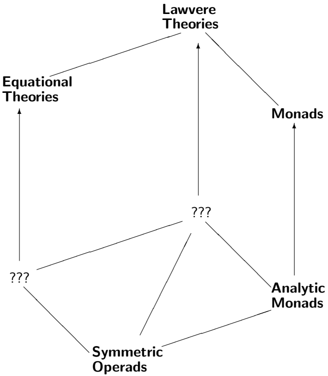

Symmetric Operad = usual symmetric operad in *Set*

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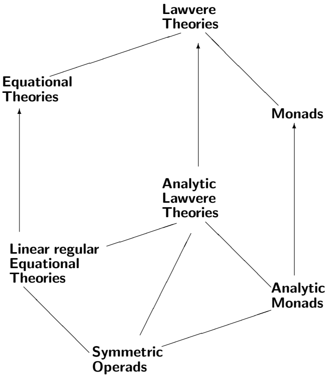

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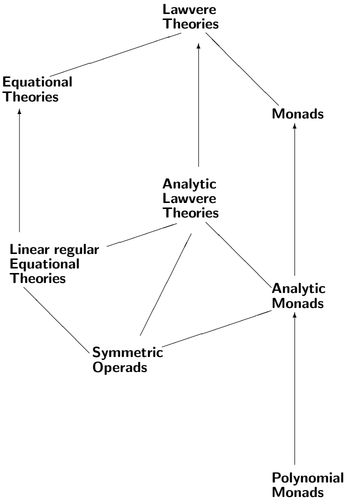

Polynomial Monad = functor part preserves wide pullbacks; unit and multiplication are (weakly) cartesian nat. transf.

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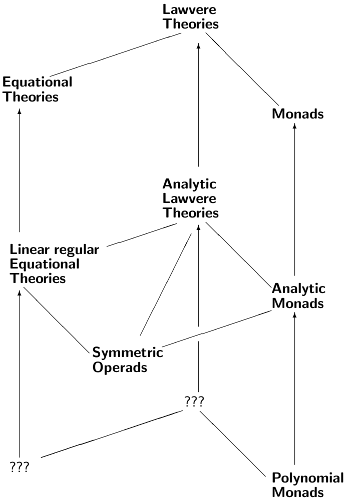

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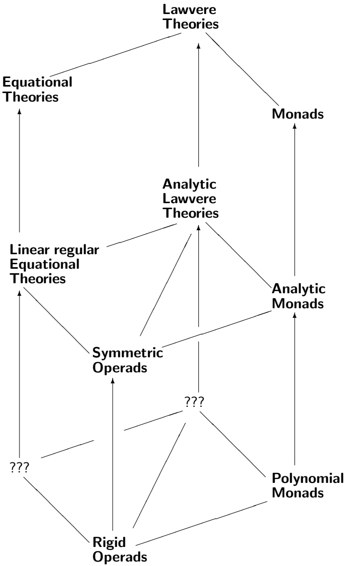

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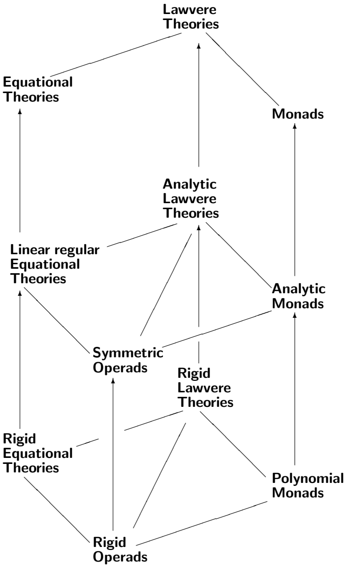

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- $Aut(n)$ is the set of automorphisms of n in T
- We have functions
$$\rho_n : S_n \times Aut(1)^n \longrightarrow Aut(n)$$
such that
$$(\sigma, a_1, \ldots, a_n) \mapsto a_1 \times \ldots \times a_n \circ \pi_\sigma$$

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Analytic morphisms

A morphism in T is *analytic* iff it is right orthogonal to all structural morphisms.

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Rigid Lawvere theory

Lawvere theory T is *rigid* iff

- T is analytic;
- the actions of symmetric groups

$$S_n \times T(n,1) \to T(n,1)$$

$$\langle \sigma, f \rangle \mapsto f \circ \pi_\sigma$$

are free on analytic morphisms.

Interpretations of Analytic Lawvere theories

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The embedding of the category of analytic Lawvere theories into all Lawvere theories has a right adjoint which is monadic.

- $\vec{x}^n = x_1, \ldots, x_n$

Equational theories
linear-regular theories

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- A term in context

$$t : \vec{x}^n$$

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Linear-regular theory

A an equational theory T is *linear-regular* iff it has a set of linear-regular axioms.

- A linear-regular term in context

$$t(x_1, \dots, x_n) : \vec{x}^n$$

is *flabby* in T iff

$$T \vdash t(x_1, \dots, x_n) = t(x_{\sigma(1)}, \dots, x_{\sigma(n)}) : \vec{x}^n$$

for some $\sigma \in S_n$, $\sigma \neq id_n$.

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An example of a flabby term

In the theory T_{cm} of commutative monoids the term $x_1 \cdot x_2$ is flabby as

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Rigid theory

A an equational theory T is *rigid* iff it is linear-regular and has no flabby terms.

Linear-regular interpretation

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- The category of rigid theories and linear-regular interpretations is equivalent the category of polynomial monads.

Theorem[M.Bojanczyk, S.Szawiel, M.Z.]

The problem whether a finite set of linear-regular axioms defines a rigid equational theory is undecidable.

Monoids

The theory of monoids has two operations $\cdot$ and e, of arity 2 and 0, respectively, and equations

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$$n \longrightarrow 1$$

is analytic iff it is of form

$$\langle x_1, \ldots x_n \rangle \mapsto x_{\sigma(1)} \cdot \ldots \cdot x_{\sigma(n)}$$

for some $\sigma \in S_n$.

Monoids with anti-involution

The theory of monoids with anti-involution in a theory of monoids that has an additional unary operation s and additional two axiom

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$$\langle x_1, \ldots x_n \rangle \mapsto s^{\varepsilon_n}(x_{\sigma(1)}) \cdot \ldots \cdot s^{\varepsilon_n}(x_{\sigma(n)})$$

for some $\sigma \in S_n$ and $\varepsilon_i \in \{0, 1\}$.

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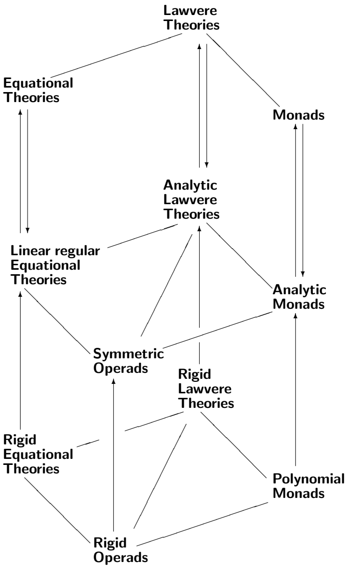
T_{cm} is the terminal analytic Lawvere theory.

Categories of Equational Theories (again)



Thank you!