Representing multicategories as monads

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Fibrations,
fibered functors, morphisms of fibrations,
fibered natural transformations

$$
\begin{tikzcd}
\mathcal{E} \arrow[rr, shift left=2, "F"] \arrow[rr, shift right=2, "G"'] \arrow[rr, phantom, "\Downarrow\tau"] \arrow[dr, bend right, "p"'] & & \mathcal{F} \arrow[dl, bend left, "q"] \\
& \mathbb{B} &
\end{tikzcd}
$$

Lax monoidal fibrations

$$
\begin{tikzcd}
\mathcal{E} \times_{\mathbb{B}} \mathcal{E} \arrow[r, "\otimes"] \arrow[dr, bend right] & \mathcal{E} \arrow[d] & \mathbb{B} \arrow[l, "I"'] \arrow[dl, bend left] \\
& \mathbb{B} &
\end{tikzcd}
$$

$\otimes, I$ - fibered functors

α, λ, ρ - fibered nat. isomorphisms

The fibration of monoids

$$
\begin{tikzcd}
\mathcal{E} \arrow[dr, bend right] & & Mon(\mathcal{E}) \arrow[ll, "U"'] \arrow[dl, bend left] \\
& \mathbb{B} &
\end{tikzcd}
$$

Example 1 **Exponential fibrations**

$\begin{array}{c} \mathcal{C}^{\to} \\ \downarrow \\ \mathcal{C} \end{array}$ - bifibration, $\mathcal{C}$ - with p.b's

$\begin{array}{c} \mathrm{End}(\mathcal{C}) \\ \downarrow \\ \mathcal{C} \end{array} = \begin{array}{c} \mathcal{C}^{\to} \\ \downarrow \\ \mathcal{C} \end{array} \begin{array}{c} \mathcal{C}^{\to} \\ \downarrow \\ \mathcal{C} \end{array}$ in $Cat/\mathcal{C}$

Object over $C \in \mathcal{C}$: endofunctor on $\mathcal{C}_{/C}$

Morphism $\tau : F \to G$ over $u : C \to D$

natural transformation: $\tau : Fu^* \longrightarrow u^*G$

$F \circlearrowleft \mathcal{C}_{/C} \xleftarrow{u^*} \mathcal{C}_{/D} \circlearrowleft G$

$I = id$, $\otimes = \circ$

Prone morphism over $u : C \to D$ with the codomain G is

$$u^*G(\varepsilon^u) : u^*Gu_!u^* \longrightarrow u^*G$$

NB. For supine morphism use η^u.

Example 2 **Burroni fibrations**

M monad (possibly cartesian) on $\mathcal{E}$ with pb's

$$\begin{tikzcd} \mathrm{Gph}(M) \arrow[d] \\ \mathcal{E} \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dl, "\gamma"'] \arrow[dr, "\partial"] & \\ O & & M(O) \end{tikzcd} \quad \longmapsto \quad O$$

unit tensor

$$\begin{tikzcd} & O \arrow[dl, "1_O"'] \arrow[dr, "\eta_O"] & \\ O & & M(O) \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & & A \otimes B \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner"] & & & \\ & A \arrow[dl, "\gamma"'] \arrow[dr, "\partial"] & & M(B) \arrow[dl, "M(\gamma)"'] \arrow[dr, "M(\partial)"] & & \\ O & & M(O) & & M^2(O) \arrow[dr, "\mu_O"] & \\ & & & & & M(O) \end{tikzcd}$$

Actions (of Burroni fibrations)

$$\begin{tikzcd} \mathrm{Gph}(M) \times_{\mathcal{E}} \mathcal{E}^{\to} \arrow[rr, "*"] \arrow[dr] & & \mathcal{E}^{\to} \arrow[dl] \\ & \mathcal{E} & \end{tikzcd}$$

$$\left(\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ O & & M(O) \end{tikzcd}, \begin{tikzcd} X \arrow[d, "d"] \\ O \end{tikzcd} \right) \longmapsto \begin{tikzcd}[column sep=small] & & A * X \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner"] & \\ & A \arrow[dl] \arrow[dr] & & M(X) \arrow[dl, "M(d)"] \\ O & & M(O) & \end{tikzcd}$$

Representations (of Burroni fibrations)

(by adjunction)

$$\begin{tikzcd} \mathrm{Gph}(M) \arrow[rr, "\mathrm{rep}"] \arrow[dr] & & \mathrm{End}(\mathcal{E}) \arrow[dl] \\ & \mathcal{E} & \end{tikzcd}$$

Example 3 **The free monoid monad M on** *Set*

operations

$$\begin{tikzcd} & A \arrow[dl, "\gamma"'] \arrow[dr, "\partial"] & \\ O & & M(O) = \coprod_{n \in \omega} O^{[n]} \end{tikzcd}$$

$$\gamma_a - a \begin{cases} \partial_a(1) \\ \vdots \\ \partial_a(n) \end{cases}$$

signatures — Lambek's multicats

$$\begin{tikzcd} Gph(M) \arrow[dr, bend right] & & Mon(M) \arrow[ll] \arrow[dl, bend left] \\ & Set & \end{tikzcd}$$

γ, ∂ – typing

$[n] = \{1, \ldots, n\}$

unit — I_O $\circ - \circ - \circ$

tensor — $A \otimes B$

$$-a \begin{cases} b_1 \\ \vdots \\ b_n \end{cases}$$

Action

$$Gph(M) \times_{Set} Set^{\to} \longrightarrow Set^{\to}$$

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & & X \arrow[d] & \mapsto & A * X \arrow[d] \\ O & & M(O) & O & & O \end{tikzcd}$$

$$-a \begin{cases} x_1 \\ \vdots \\ x_n \end{cases}$$

Representation

$$Gph(M) \xrightarrow{Rep} End(Set)$$

not full on isos!

The symmetrization monad $\mathcal{S}$ **on** $\mathrm{Gph}(M) \to \mathrm{Set}$

(or any polynomial functor on any lccc)

$$\begin{array}{c} A \\ {\scriptstyle \gamma}\swarrow \quad \searrow{\scriptstyle \partial} \\ O \qquad M(O) \end{array} \longmapsto \begin{array}{c} \mathcal{S}(A) \\ \swarrow \quad \searrow \\ O \qquad M(O) \end{array}$$

all 'versions' of the operations from A

$\mathcal{S}(A) = \{ \langle a, \sigma \rangle : a \in A,\ \sigma \in S_{|a|} \}$

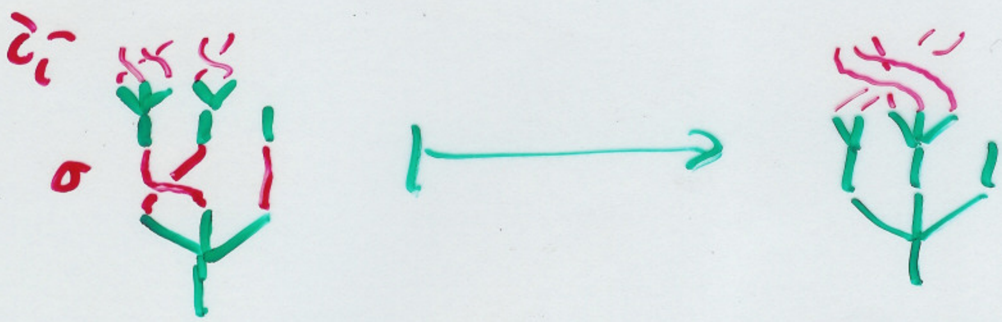
permutation group

$\langle a, \sigma \rangle \mapsto \gamma_a$, $\langle a, \sigma \rangle \mapsto \partial_a \circ \sigma$

$\mathcal{S}$ is a monoidal monad and more... η, μ

$$I \xrightarrow[\cong]{\dot{\varphi}} \mathcal{S}(I)$$

$$\mathcal{S}(A) \otimes \mathcal{S}(B) \xrightarrow{\varphi_{A,B}} \mathcal{S}(A \otimes B)$$



Little combing!

Monoidal functors acts on actions

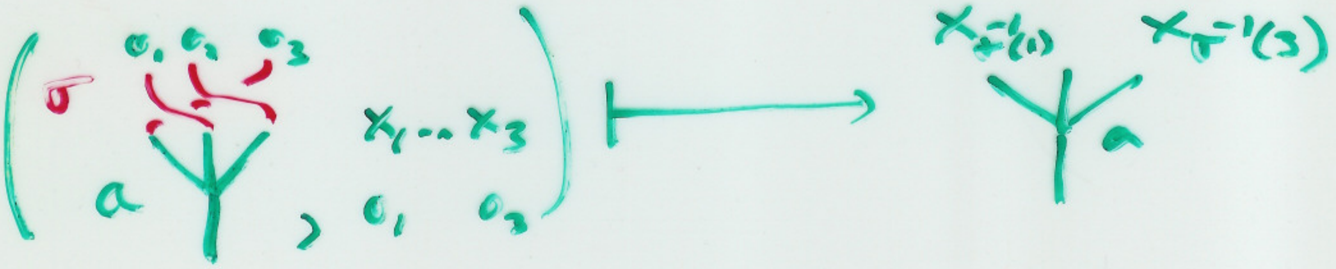
so

$$\begin{tikzcd} \mathrm{Gph}(M) \arrow[rr, "S"] \arrow[dr] & & \mathrm{Gph}(M) \arrow[dl] \\ & \mathrm{Set} & \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Gph}(M)\times_{\mathrm{Set}}\mathrm{Set}^{\to} \arrow[rr, "*"] \arrow[dr] & & \mathrm{Set}^{\to} \arrow[dl] \\ & \mathrm{Set} & \end{tikzcd}$$

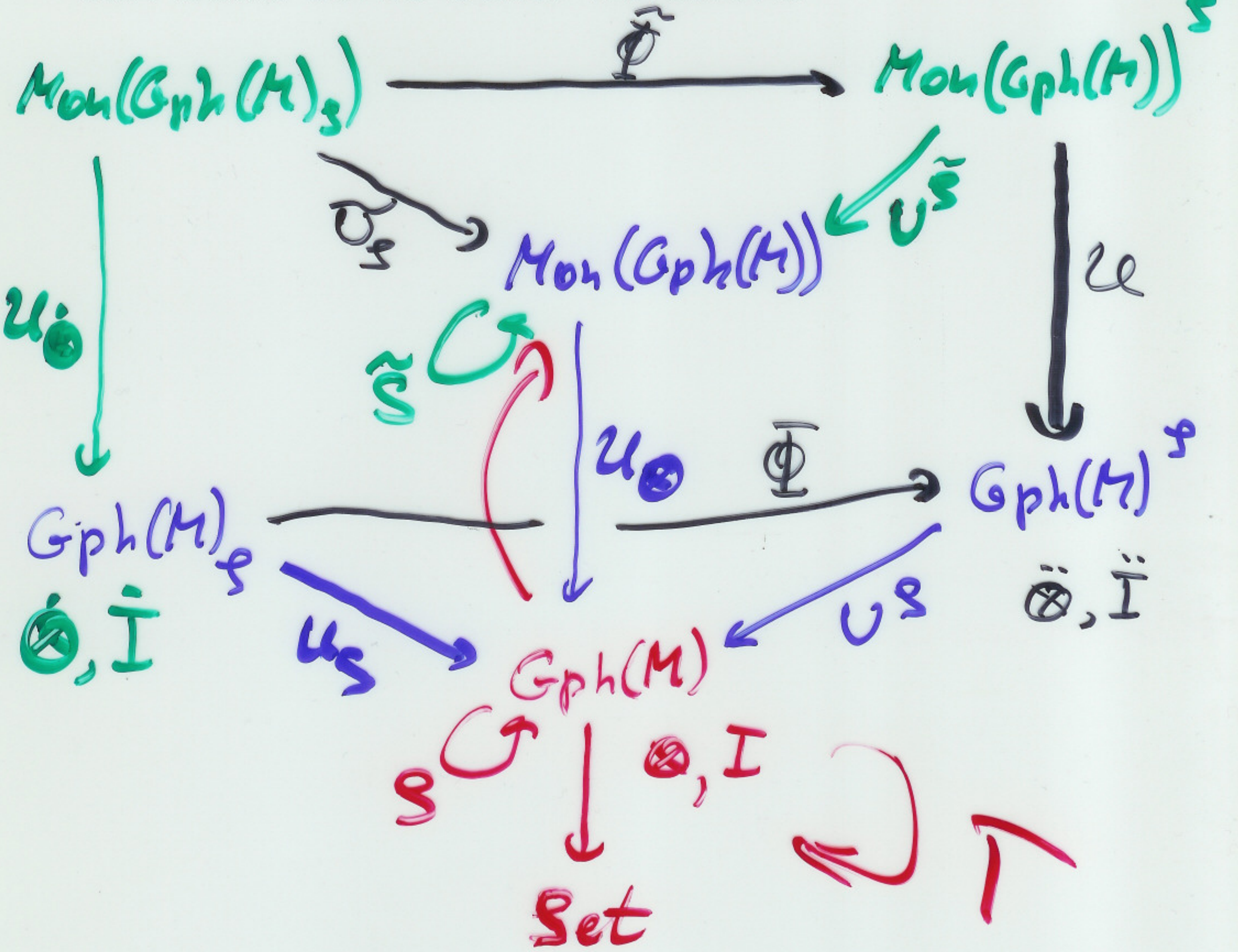
$$\begin{tikzcd} \mathrm{Gph}(M)\times_{\mathrm{Set}}\mathrm{Set}^{\to} \arrow[rr, "S(-)*(=)"] \arrow[dr] & & \mathrm{Set}^{\to} \arrow[dl] \\ & \mathrm{Set} & \end{tikzcd}$$

and there is an S -algebra ξ on $*$ along $⁂$

$$\underline{S(A)}*\underline{X} \xrightarrow{\ \xi_{A,X}\ } A*X$$



Having a fibered monoidal monad (like $\mathcal{S}$) we can build some other fibrations



$$\widetilde{\mathcal{S}}(M, m, e) = (\mathcal{S}(M) \otimes \mathcal{S}(M) \xrightarrow{\varphi} \mathcal{S}(M \otimes M) \xrightarrow{\mathcal{S}(m)} \mathcal{S}(M),$$

$$I \xrightarrow{\dot{\varphi}} \mathcal{S}(I) \xrightarrow{e} \mathcal{S}(M))$$

NB. We have a distributive law ...

$TS \longrightarrow ST$

We can product the previous diagram by any fibrations, e.d $Set^{\rightarrow} \longrightarrow Set$ and we get

$$\begin{tikzcd}
Gph(M)_S \times_{Set} Set^{\rightarrow} \arrow[rr, "\phi \times 1"] \arrow[dd, dashed, "*"'] & & Gph(M)^S \times_{Set} Set^{\rightarrow} \arrow[ddll, dashed, "\ddot{*}"] \\
& Gph(M) \times_{Set} Set^{\rightarrow} \arrow[ul, "F_S \times 1"'] \arrow[ur, "F^S \times 1"] \arrow[dl, "\wr\ *"'] \arrow[dd] & \\
Set^{\rightarrow} \arrow[dr] & & \\
& Set &
\end{tikzcd}$$

ξ + coeq

$\otimes$, S

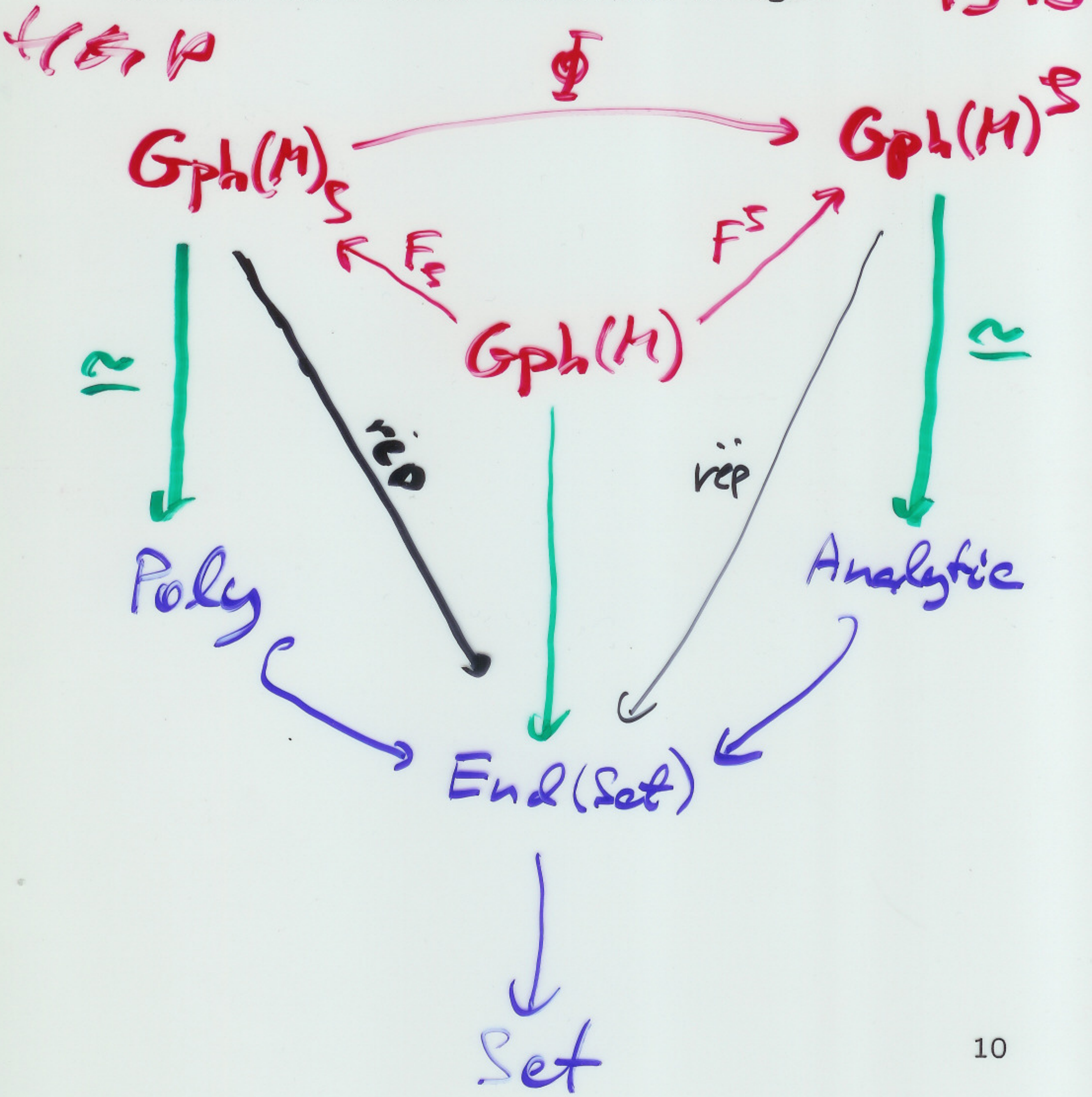
since

$\xi : S \circledast * \longrightarrow *$

is an S-algebra

By exponential adjunction, we get representations of these lax monoidal fibrations... and we can take their essential images

BD



The full picture