Categorically-algebraic topology: theory and applications☆

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Motivated by the recent developments in the theory of lattice-valued topology, the talk will present a new framework for studying topological structures. The approach is based on category theory and universal algebra, and incorporates the most important settings of both many-valued and classical topology, ultimately erasing the border between them. The main underlying idea comes from the *point-set lattice-theoretic* (*poslat*) topological theories of S. E. Rodabaugh [20], which rely on the concepts of powerset operator and semi-quantale. We replace semi-quantales with algebras from an arbitrary variety and show that the arising setting is more than promising. The new framework is called *categorically-algebraic* (*catalg*) to underline its motivating theories, on one hand, and to distinguish it from the poslat approach, on the other. Currently, the catalg theory is being developed in several directions, which influence each other dramatically:

(1) The huge amount of various topological structures (topological space, closure space, *etc.*) motivated the concept of *catalg space*, which provided a common setting for many approaches to topology [21].

(2) The study of functorial relationships between topological spaces and topological systems of S. Vickers [28] (introduced as a unifying framework for both topological spaces and their underlying algebras – locales) done by J. T. Denniston *et al.* [6, 7] and C. Guido [11, 12], together with state property systems of D. Aerts [2] (serving as the basic mathematical structure in the Geneva-Brussels approach to foundations of physics) induced the notion of *catalg system* [27], which extended the concept of catalg space.

(3) The theory of natural dualities developed by D. Clark, B. Davey, M. Haviar, H. Priestley, *etc.* [4], which gives a machinery for obtaining topological representations of algebraic structures (like the well-known representations of Boolean algebras and distributive lattices of M. Stone and H. Priestley), gave rise to the notion of *catalg duality*, which streamlined the classical theory, showing its dependence on powerset operators and dropping several essential restrictions on the structures involved [22, 23].

(4) Categorical frameworks for poslat powerset theories of S. E. Rodabaugh [18] (motivated by the classical notions of image and preimage operators on powersets) gave a stimulus to provide strict foundations for *catalg powerset operators* [25], highly relied upon in the previous items.

(5) The notion of attachment in a complete lattice introduced by C. Guido [11, 12] as a generalization of the standard set-theoretic membership relation "$\in$", motivated the concepts of *catalg attachment* [26] and *dual attachment pair* [8], bearing a strong relation to the hypergraph functor [14], which served as a way of interaction between fuzzy and crisp topological theories.

(6) The theory of (L, M)-fuzzy topological spaces of T. Kubiak, A. Šostak [15] and C. Guido [10] induced the concept of *lattice-valued catalg topology* [24], which provided a fuzzification of the already developed catalg theory, showing that the respective many-valued approach of S. E. Rodabaugh [19] (a *de facto* standard in the fuzzy community) was actually "crisp".

The talk is supposed to introduce the foundations of categorically-algebraic topology and then illustrate the new theory by several of the above-mentioned applications. Time permitting, we will outline some future developments of the catalg framework. Both category theory and algebra will be employed, relying more on the former. The necessary categorical background can be found in [1, 13, 16, 17]. For the notions of universal algebra, [3, 5, 9, 17] are recommended. For convenience of the listener, the talk will provide the basic categorical and algebraic preliminaries.

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