The Partial Simplicial Category and Algebras for Monads

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Abstract

We construct explicitly the weights on the simplicial category so that the colimits and limits of 2-functors with those weights provide the Kleisli objects and the Eilenberg-Moore objects, respectively, in any 2-category. MS Classification 18A30, 18A40 (AMS 2010).

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1 Introduction

It is well know that monads in 2-categories correspond to 2-functors from the simplicial category Δ . It is also well known that the Kleisli and the Eilenberg-Moore objects can be build as weighted (co)limits, c.f. [St2]. In this paper we construct explicitly the weights W_r and W_l on Δ so that the W_r -weighted colimits provide the Kleisli objects and the W_l -weighted limits provide the Eilenberg-Moore objects in any 2-category \mathcal{D} .

2 Partial simplicial category Π

Let Δ be the usual (algebraists) simplicial category. The objects of Δ are finite linear orders denoted by $n = \langle n, \leq \rangle = \langle \{0, \ldots, n-1\}, \leq \rangle$, for $n \in \omega$. The morphisms of Δ are monotone functions. For $n \geq 1$ and $0 \leq i < n$, the morphism

$$\sigma_i^n: n+1 \longrightarrow n$$

is an epi that takes value i twice. For $n \ge 0$ and $0 \le i \le n$, the morphism

 $\delta_i^n:n\longrightarrow n+1$

is a mono that misses the value *i*. We usually omit the upper index when it can be read from the context. These morphisms satisfy the following simplicial identities. For $i \leq j$

$$\delta_i \delta_j = \delta_{j+1} \delta_i \qquad \qquad \sigma_j \sigma_i = \sigma_i \sigma_{j+1}$$

and

$$\sigma_j \delta_i = \begin{cases} \delta_i \sigma_{j-1} & \text{if } i < j \\ 1 & \text{if } i = j, j+1 \\ \delta_{i-1} \sigma_j & \text{if } i > j+1 \end{cases}$$

It is well known, c.f [CWM], that the morphisms σ_i and δ_i generates Δ subject to the above relations.

The partial simplicial category Π has the same objects as Δ but the morphisms in Π are partial monotone functions. Clearly, the morphism σ_i and δ_i of Δ are morphism in Π as well, and they satisfy the same simplicial identities. For $n \geq 0$ and $0 \leq i \leq n$, the morphism

$$\tau_i^n: n+1 \longrightarrow n$$

is an epi not defined on *i*. For $i \leq j$ we have

$$\tau_j \tau_i = \tau_i \tau_{j+1}$$

Moreover, the morphisms σ_i , δ_i , and τ_i satisfy the following identities in Π that, together with the above identities, will be called *partial simplicial identities*.

$$\tau_{j}\sigma_{i} = \begin{cases} \sigma_{i}\tau_{j+1} & \text{if } i < j \\ \tau_{j}\tau_{j+1} & \text{if } i = j \\ \sigma_{i+1}\tau_{j} & \text{if } i > j \end{cases} \qquad \tau_{j}\delta_{i} = \begin{cases} \delta_{i}\tau_{j+1} & \text{if } i < j \\ 1 & \text{if } i = j \\ \delta_{i-1}\tau_{j} & \text{if } i > j \end{cases}$$

We have

Lemma 2.1. Every morphism $f: n \to m$ in Π can be expressed in a canonical form as

$$f = \delta_{i_r} \dots \delta_{i_1} \sigma_{j_1} \dots \sigma_{j_s} \tau_{k_1} \dots \tau_{k_t} \tag{1}$$

with $i_1 < \ldots < i_r$, $j_1 < \ldots < j_s$, and $k_1 < \ldots < k_t$, m - n = r - s - t.

Proof. This can be easily seen directly or using the partial simplicial identities. \Box

Theorem 2.2. The category Π is generated by the morphisms σ_i , δ_i , and τ_i subject to the partial simplicial identities.

Proof. The partial simplicial identities hold in Π . Moreover, every morphism in Π can be written in a canonical form. Finally, two different canonical forms represent two different morphisms in Π .

Remark. The category Π is a strict monoidal category with the monoidal structure defined by the coproduct. Moreover, the inclusion functor $\Delta \to \Pi$ is a strict morphism of strict monoidal categories.

3 The categories Π_l and Π_r and the multiplication functors

As the category Δ is a strict monoidal category it can be considered as 2-category with one 0-cell *, and then the tensor becomes the composition of 1-cells. We denote this 2-category by Δ .

The left partial simplicial category Π_l is a subcategory of Π with the same objects as Π . A morphism $f: n \to m$ from Π is in Π_l iff for any $i \leq j \in n$, if f(j) is defined so is f(i). In other words the morphisms of Π are generated by the morphisms in Δ and the morphism $\tau_n^n: n+1 \to n$ for $n \in \omega$. Note that $\tau_n^n = id_n + \tau_0^0$. In Π_l Lemma 2.1 holds with an additional condition that $k_{i+1} = k_i + 1$ and $k_t = n - 1$.

We have a *left multiplication* 2-functor

$$W_l: \mathbf{\Delta} \to 2Cat$$

such that

$$W_l(*) = \Pi_l, \quad W_l(n) = n + (-), \quad W_l(f) = f + (-)$$

for $f: n \to m \in \Delta$. Thus W_l can be seen as an action of Δ on Π_l by tensoring on the left. Clearly, Π_l is closed with respect to such operations.

Dually, we have the right partial simplicial category Π_r , a subcategory of Π with the same objects as Π . A morphism $f: n \to m$ from Π is in Π_r iff for any $i \leq j \in n$, if f(i) is defined so is f(j). In other words the morphisms of Π are generated by the morphisms in Δ and the morphism $\tau_0^n: n+1 \to n$, for $n \in \omega$. Note that $\tau_0^n = \tau_0^0 + id_n$. In Π_l Lemma 2.1 holds with a additional condition that $k_{i+1} = k_i + 1$ and $k_1 = 0$.

We have a *right multiplication* 2-functor

$$W_r: \mathbf{\Delta} \to 2Cat$$

such that

$$W_r(*) = \Pi_r, \quad W_r(n) = (-) + n, \quad W_r(f) = (-) + f$$

for $f: n \to m \in \Delta$. Thus W_r can be seen as an action of Δ on Π_r by tensoring on the right. Clearly, Π_r is closed with respect to such operations.

4 Monads as 2-functors

The 2-functors $\mathbf{T} : \mathbf{\Delta} \to \mathcal{D}$ correspond bijectively to monads in the 2-category \mathcal{D} .

Suppose $(\mathcal{C}, T, \eta, \mu)$ is a monad in \mathcal{D} on \mathcal{C} . We define a 2-functor **T** as follows:

$$\mathbf{T}(*) = \mathcal{C}, \quad \mathbf{T}(0) = \mathbf{1}_{\mathcal{C}}, \quad \mathbf{T}(n) = T^n,$$

$$\mathbf{T}(\delta_{0}^{0}) = \eta, \quad \mathbf{T}(\delta_{i}^{n}) = T^{n-i}\eta_{T^{i}}, \quad \mathbf{T}(\sigma_{0}^{1}) = \mu, \quad \mathbf{T}(\sigma_{i}^{n}) = T^{n-i-1}\mu_{T^{i}},$$

The equations

$$\mathbf{T}(\sigma_i) \circ \mathbf{T}(\sigma_i) = \mathbf{T}(\sigma_i) \circ \mathbf{T}(\sigma_{i+1})$$

hold, as the consequence of the associativity of the multiplication $\mu \circ T\mu = \mu \circ \mu_T$. The equations

$$\mathbf{T}(\sigma_i) \circ \mathbf{T}(\delta_i) = 1 = \mathbf{T}(\sigma_{i+1}) \circ \mathbf{T}(\delta_i)$$

hold, as the consequence of the unit axiom $\mu \circ T\eta = 1 = \mu \circ \eta_T$. The remaining simplicial equations hold as a consequence of the Middle Exchange Law (MEL).

On the other hand, having a 2-functor $\mathbf{T} : \mathbf{\Delta} \to \mathcal{D}$, we get a monad $(\mathbf{T}(*), \mathbf{T}(1), \mathbf{T}(\delta_0^0), \mathbf{T}(\sigma_0^1)).$

Let 2Cat be the 3-category of 2-categories. As $2Cat(\Delta, D)$ is the 2-category of monads in D with strict morphisms, we can think of Δ as a 2-category representing monads with strict morphisms in 2-categories.

5 The 2-functor $Subeq_T$ and the Eilenberg-Moore objects

For a given monad $(\mathcal{C}, T, \eta, \mu)$ in a 2-category \mathcal{D} we define a 2-functor

$$Subeq_T: \mathcal{D}^{op} \longrightarrow \mathbf{Cat}$$

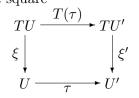
as follows. For a given 0-cell X in \mathcal{D} , the category $Subeq_T(X)$ has as objects pairs (U,ξ) such that $U: X \to \mathcal{C}$ is a 1-cell in $\mathcal{D}, \xi: TU \to U$ is a 2-cell in \mathcal{D} such that in the diagram

$$T^{2}U \xrightarrow[\mu_{U}]{} TU \xrightarrow{\xi} U$$

we have

$$\xi \circ \eta_U = 1_U, \qquad \xi \circ T(\xi) = \xi \circ \mu_U$$

In such case, we say that (U,ξ) subequalizes the monad T. A morphism $\tau : (U,\xi) \to (U',\xi')$ is a 2-cell $\tau : U \to U'$ such that the square



commutes. The 2-functor $Subeq_T$ is define on 1- and 2-cells in the obvious way, by composition.

Recall, from [St1], see [Z] for the notation, that the 2-category \mathcal{D} admits Eilenberg-Moore objects if the embedding ι

$$\mathcal{D} \xrightarrow{\iota} \mathbf{Mnd}(\mathcal{D})$$

has a right 2-adjoint $\iota \dashv EM$. $\mathbf{Mnd}(\mathcal{D})$ is the 2-category of monads in \mathcal{D} with lax morphisms of monads and transformations of lax morphisms. We have a 2-functor

$$\mathbf{Mnd}(\iota(-),T):\mathcal{D}^{op}\longrightarrow\mathbf{Cat}$$
(2)

sending 0-cell X in \mathcal{D} to the category $\mathbf{Mnd}(\iota(X), T)$ of lax morphisms from the identity monad on X to the monad T and transformations between such morphisms.

The following definition is a 'monad by monad' version of the previous one. We say that the monad T admits Eilenberg-Moore object iff the 2-functor $\mathbf{Mnd}(\iota(-), T)$ is representable.

A simple verification shows

Lemma 5.1. The 2-functors Subeq_T and $Mnd(\iota(-), T)$ are naturally isomorphic.

6 The 2-functor $Cone_{W_i}(\mathbf{T})$

Let $\mathbf{T} : \Delta \to \mathcal{D}$ be a 2-functor and T be the monad corresponding to \mathbf{T} . We define a 2-functor

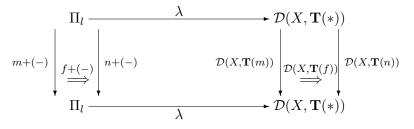
$$Cone_{W_l}(\mathbf{T}): \mathcal{D}^{op} \longrightarrow \mathbf{Cat}$$

of W_l -cones over **T**. We will show that the 2-functor $Cone_{W_l}(\mathbf{T})$ is isomorphic to the 2-functor $Subeq_T$.

Fix a 0-cell X in \mathcal{D} . An object of $Cone_{W_i}(\mathbf{T})(X)$ is a 2-natural transformation

$$\lambda: W_l \longrightarrow \mathcal{D}(X, \mathbf{T}(*))$$

with only one component the functor λ_* also denoted λ . The 2-naturality means that the square



commutes, for any $f: m \to n$ in Δ . Put $T = \mathbf{T}(1)$ and $\lambda(0) = U: X \to \mathbf{T}(*)$. We have with f as before

$$\lambda(m) = \lambda(m+0) = \mathbf{T}(m) \circ \lambda(0) = T^m U,$$

$$\lambda(f) = \lambda(f+0) = \mathbf{T}(f) \circ \lambda(0) = \mathbf{T}(f) \circ U$$

Moreover, putting $\lambda(\tau_0^0) = \xi : TU \to U$, we have

$$\lambda(\tau_n^n) = \lambda(id_n + \tau_0^0) = \mathbf{T}(id_n) \circ_0 \mathbf{T}(\tau_0^0) = \mathbf{T}(id_n) \circ_0 \xi =$$
$$= \mathbf{T}(id_1) \circ_0 \dots \circ_0 \mathbf{T}(id_1) \circ_0 \xi = id_T \circ_0 \dots \circ_0 id_T \circ_0 \xi = T^n(\xi)$$

Thus λ is uniquely determined by U and ξ . The equations

$$\tau_0^0 \circ \delta_0^1 = 1, \quad \tau_0^0 \circ \tau_1^1 = \tau_0^0 \circ \sigma_0^1$$

implies that (U,ξ) subequalizes the monad T. On the other hand, if (U,ξ) subequalizes T then we can define a 2-natural transformation $\lambda : W_l \to \mathcal{D}(X, \mathbf{T})$, as follows. The functor

$$\lambda = \lambda_* : \Pi_l \longrightarrow \mathcal{D}(X, \Pi(*))$$

is defined so that

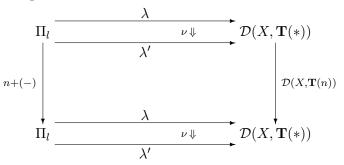
$$\lambda(0) = U, \ \lambda(\tau_0^0) = \xi$$

and for 2-naturality of λ we have

$$\lambda(n) = T^n U, \ \lambda(f) = \mathbf{T}(f)_U, \ \lambda(\tau_n^n) = T^n(\xi)$$

Then it is easy to verify that λ respect all the equations in Π_l .

A morphism in $Cone_{W_l}(\mathbf{T})(X)$ between two 2-natural transformations is a modification $\nu : \lambda \to \lambda'$ with one component ν_* , denoted also ν , that is a natural transformation so that, for any $n \in \omega$, the square



commutes. As we have

$$\nu_n = \nu_{(n+0)} = T^n(\nu_0)$$

the modification ν is uniquely determined by $\nu_0: U \to U' = \lambda'(0)$. The square

$$\begin{array}{c|c} \lambda(1) & \xrightarrow{\nu_1} & \lambda'(1) \\ \lambda(\tau_0^0) & & & \downarrow \lambda'(\tau_0^0) \\ \lambda(0) & \xrightarrow{\nu_0} & \lambda'(0) \end{array}$$

commutes as it is the naturality of $\nu_* : \lambda_* \to \lambda'_*$ on τ_0^0 .

On the other hand, any 2-cell $\nu_0: U \to U'$ in \mathcal{D} such that

$$\begin{array}{c|c} TU & \xrightarrow{T(\nu_0)} TU' \\ \xi & & & & \\ \xi & & & & \\ U & \xrightarrow{\nu_0} U' \end{array}$$

extends to a natural transformation from λ_* to λ'_* , i.e. a modification ν from λ to λ' .

The 2-functor $Cone_{W_l}(\mathbf{T})$ is defined on 1- and 2-cells in the obvious way.

Constructing this functor we have in fact proved

Lemma 6.1. The 2-functors $Subeq_T$ and $Cone_{W_l}(\mathbf{T})$ are naturally isomorphic.

7 The Eilenberg-Moore objects

Theorem 7.1. Let $(\mathcal{C}, T, \eta, \mu)$ be a monad in a 2-category \mathcal{D} and $\mathbf{T} : \mathbf{\Delta} \to \mathcal{D}$ the corresponding 2-functor. Then T admits Eilenberg-Moore object iff \mathbf{T} has a W_l -weighted limit. If it is the case then the Eilenberg-Moore object for T and the W_l -weighted limit of \mathbf{T} are isomorphic.

Proof. By Lemmas 5.1 and 6.1, the 2-functors $\mathbf{Mnd}(\iota(-), T)$ and $Cone_{W_l}(\mathbf{T})$ are naturally isomorphic. So if one is of representable so is the other and the representing objects are isomorphic. The representation of the first give rise to the Eilenberg-Moore object for T, and the representation of the second give rise to the W_l -weighted limit of \mathbf{T} . \Box

From the above theorem we get immediately

Corollary 7.2. Any 2-category \mathcal{D} admits Eilenberg-Moore object iff it has all W_l -weighted limit of 2-functors from Δ .

8 The Kleisli objects

Clearly, all the above considerations can be dualised. In this case we get results relating Kleisli objects and the W_r -weighted colimits of 2-functors from Δ .

We note for the record

Theorem 8.1. Let $(\mathcal{C}, T, \eta, \mu)$ be a monad in a 2-category \mathcal{D} and $\mathbf{T} : \Delta \to \mathcal{D}$ the corresponding 2-functor. Then T admits Kleisli object iff \mathbf{T} has a W_r -weighted colimit. If it is the case then the Kleisli object for T and the W_r -weighted colimit of \mathbf{T} are isomorphic.

Corollary 8.2. Any 2-category \mathcal{D} admits Kleisli object iff it has all W_r -weighted colimit of 2-functors from Δ .

9 Appendix: Weighted limits in 2-categories

We recall the definition of weighted limits in 2-categories in detail.

The 2-functor $\mathcal{D}(X, \mathbf{T})$

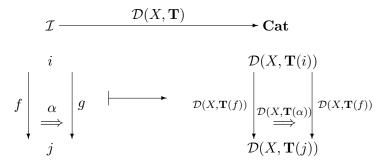
For two 2-functors between 2-categories as shown¹

¹There are some foundational problems that one should address. For example, it is desirable that the 2-category \mathcal{I} be small. But we will be ignoring this issues believing that the reader can fix all these problem on its own, the way she or he likes most.

$$\mathcal{I} \xrightarrow{W} Cat \qquad \qquad \mathcal{I} \xrightarrow{\mathbf{T}} \mathcal{D}$$

we are going to describe the W-weighted limit of \mathbf{T} .

For any 0-cell X in \mathcal{D} we can form a 2-functor



of 'homming into' \mathbf{T} .

The category $\mathcal{D}(X, \mathbf{T}(i))$ consists of 1- and 2-cells in \mathcal{D} from X to $\mathbf{T}(i)$. The functor

$$\mathcal{D}(X,\mathbf{T}(i)) \xrightarrow{\mathcal{D}(X,\mathbf{T}(f))} \mathcal{D}(X,\mathbf{T}(j))$$

is a whiskering along the 2-cell $\mathbf{T}(f)$:

$$X \xrightarrow[s]{r} \mathbf{T}(i) \xrightarrow[s]{\mathbf{T}(f) \circ r} \mathbf{T}(f) \xrightarrow[s]{\mathbf{T}(f) \circ s} \mathbf{T}(f) \xrightarrow[s]{\mathbf{T}(f) \circ s} \mathbf{T}(f)$$

The component of the natural transformation

$$\mathcal{D}(X,\mathbf{T}(f)) \xrightarrow{\mathcal{D}(X,\mathbf{T}(\alpha))} \mathcal{D}(X,\mathbf{T}(g))$$

at $r: X \to \mathbf{T}(i)$ is

$$\mathbf{T}(f) \circ r \xrightarrow{\mathbf{T}(\alpha)_r} \mathbf{T}(g) \circ r$$

The naturality of $\mathcal{D}(X, \mathbf{T}(f))$

$$\begin{array}{c|c} \mathbf{T}(f) \circ r & \xrightarrow{\mathbf{T}(\alpha)_r} \mathbf{T}(g) \circ r \\ \hline \mathbf{T}(f)(\gamma) & & & & \\ \mathbf{T}(f) \circ s & \xrightarrow{\mathbf{T}(\alpha)_s} \mathbf{T}(f) \circ r \end{array}$$

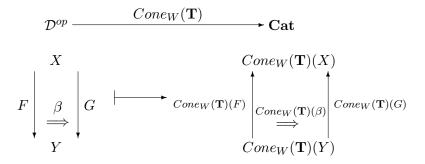
follows from MEL, where

$$X \xrightarrow[s]{r} \mathbf{T}(i) \xrightarrow[s]{\mathbf{T}(\alpha) \Downarrow} \mathbf{T}(j)$$

This ends the definition of the 2-functor $\mathcal{D}(X, \mathbf{T})$.

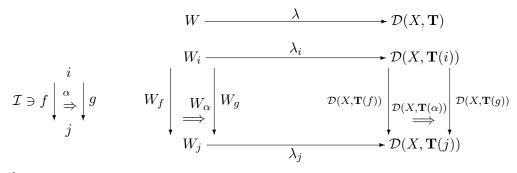
The 2-functor of weighted cones

Using the above 2-functor(s) we can form the 2-functor $Cone_W(\mathbf{T})$ of W-cones over \mathbf{T} .



Fix X in \mathcal{D} . The category $Cone_W(\mathbf{T})(X)$ consists of 2-natural transformations between 2-functors W and $\mathcal{D}(X, \mathbf{T})$ and modifications between them.

The objects in the category $Cone_W(\mathbf{T})(X)$ are 2-natural transformations



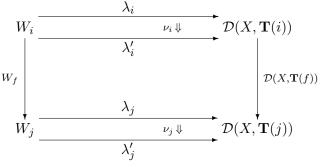
so that

$$\mathcal{D}(X, \mathbf{T}(\alpha)) \circ \lambda_i = \lambda_j \circ W_\alpha$$

The morphisms in the category $Cone_W(\mathbf{T})(X)$ are modifications $\nu : \lambda \to \lambda'$ or

$$W \xrightarrow{\lambda} \mathcal{D}(X, \mathbf{T})$$

such that, for $f: i \to j$ in \mathcal{I} , the square



commutes, in the sense that

$$\mathcal{D}(X,\mathbf{T}(f))\circ\nu_i=\nu_j\circ W_f$$

This ends the definition of the category $Cone_W(\mathbf{T})(X)$. The functor

$$Cone_W(\mathbf{T})(X) \xrightarrow{Cone_W(\mathbf{T})(F)} Cone_W(\mathbf{T})(Y)$$

sends the 2-natural transformation λ to the 2-natural transformation

$$\mathcal{T} \longrightarrow \mathcal{D}(Y, \mathbf{T}) \longrightarrow \mathcal{D}(X, \mathbf{T})$$

such that, for i in \mathcal{I} ,

$$W_i \xrightarrow{\lambda_i} \mathcal{D}(Y, \mathbf{T}(i)) \xrightarrow{\mathcal{D}(F, \mathbf{T}(i))} \mathcal{D}(X, \mathbf{T}(i))$$

is a functor such that, for $u: w \to w'$ in W_i , we have a diagram $\lambda_i(w)$

$$X \xrightarrow{F} Y \xrightarrow{\lambda_i(w)} \mathbf{T}(i)$$

and the following equations

$$\mathcal{D}(F, \mathbf{T}(i)) \circ \lambda_i(w) = \lambda_i(w) \circ F$$
$$\mathcal{D}(F, \mathbf{T}(i)) \circ \lambda_i(u) = \lambda_i(u)_F$$

hold. Moreover, the functor $Cone_W(\mathbf{T})(F)$ sends the modification ν

$$W \xrightarrow[\lambda']{\nu \Downarrow} \mathcal{D}(Y, \mathbf{T})$$

to the modification

$$W \xrightarrow{Cone_W(\mathbf{T})(F)(\lambda) = \lambda} \mathcal{D}(X, \mathbf{T})$$

$$W \xrightarrow{Cone_W(\mathbf{T})(F)(\nu) = \bar{\nu} \Downarrow} \mathcal{D}(X, \mathbf{T})$$

$$Cone_W(\mathbf{T})(F)(\lambda') = \bar{\lambda'}$$

$$\bar{\lambda}_i$$

such that, for i in \mathcal{I} ,

$$W \xrightarrow{\overline{\lambda}_i} \mathcal{D}(X, \mathbf{T}(i))$$

is a natural transformation such, that for $\overset{i}{w}$ in W_i ,

$$\bar{\lambda}_i(w) = \lambda_i(w) \circ F \xrightarrow{(\bar{\nu}_i)_w} = ((\nu_i)_w)_F \longrightarrow \lambda'_i(w) \circ F = \bar{\lambda}'_i(w)$$

is a morphism in $\mathcal{D}(X, \mathbf{T}(i))$.

The component, at the 2-natural transformation $\lambda : W \to \mathcal{D}(X, \mathbf{T})$, of the natural transformation

$$Cone_W(\mathbf{T})(F) \xrightarrow{Cone_W(\mathbf{T})(\beta)} Cone_W(\mathbf{T})(G)$$

is a modification $\mathcal{D}(\beta, \mathbf{T}) \circ \lambda$, i.e. the composition

$$W \xrightarrow{\lambda} \mathcal{D}(Y, \mathbf{T}) \xrightarrow{\mathcal{D}(F, \mathbf{T})} \mathcal{D}(X, \mathcal{T}) \xrightarrow{\mathcal{D}(X, \mathcal{T})} \mathcal{D}(X, \mathcal{T})$$

so that, at i in \mathcal{I} , it is the natural transformation $\mathcal{D}(\beta, \mathbf{T}(i)) \circ \lambda_i$

$$W_{i} \xrightarrow{\lambda_{i}} \mathcal{D}(Y, \mathbf{T}(i)) \xrightarrow{\mathcal{D}(F, \mathbf{T}(i))}{\mathcal{D}(G, \mathbf{T}(i))} \mathcal{D}(X, \mathbf{T}(i))$$

so that, for w in W_i , it is a morphism in $\mathcal{D}(X, \mathbf{T})$

$$\lambda_i(w) \circ F \xrightarrow{\lambda_i(w)(\beta)} \lambda_i(w) \circ G$$

from the diagram

$$X \xrightarrow[G]{} \stackrel{F}{\longrightarrow} Y \xrightarrow{} \lambda_i(w) \to \mathbf{T}(i)$$

The representation of the 2-functor $Cone_W(\mathbf{T})$

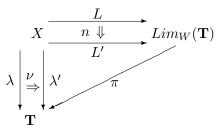
The representation of the functor $Cone_W(\mathbf{T})$ is the *W*-weighted limit of the 2-functor \mathbf{T} . Thus it is an object $Lim_W(\mathbf{T})$ together with a 2-natural isomorphism

$$\mathcal{D}(-, Lim_W(\mathbf{T})) \xrightarrow{\varrho} Cone_W(\mathbf{T})$$

The image of the identity on $Lim_W(\mathbf{T})$ is the limiting W-weighted cone

 $Lim_W(\mathbf{T}) \xrightarrow{\pi} \mathbf{T}$

in $Cone_W(\mathbf{T})(Lim_W(\mathbf{T}))$. For any 0-cell X we have a correspondence via π



or in another form, we have an isomorphism of categories

$$X \xrightarrow{L} Lim_W(\mathbf{T}) \qquad \text{in } \mathcal{D}$$

$$X \xrightarrow{\lambda} \\ X \xrightarrow{\nu \downarrow} \\ \lambda' \qquad X \qquad \text{in } Cone_W(\mathbf{T})(X)$$

natural in X.

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