On positive face structures and positive-to-one computads

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Abstract

We introduce the notion of a positive face structure. The positive face structures to positive-to-one computads are like simple graphs, c.f. [MZ], to free ω-categories over ω-graphs. In particular, they allow to give an explicit combinatorial description of positive-to-one computads. Using this description we show, among other things, that positive-to-one computads form a presheaf category with the exponent category being the category of principal positive face structures. We also present the Harnik argument in this context showing that the ω-categories are monadic over positive-to-one computads with the 'free functor' being the inclusion $\mathbf{Comp}^{+/1} \to \omega Cat$.

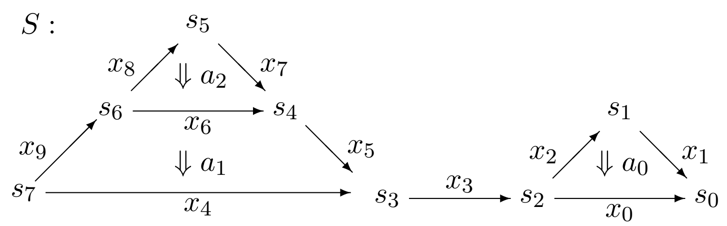
Contents

1 Introduction

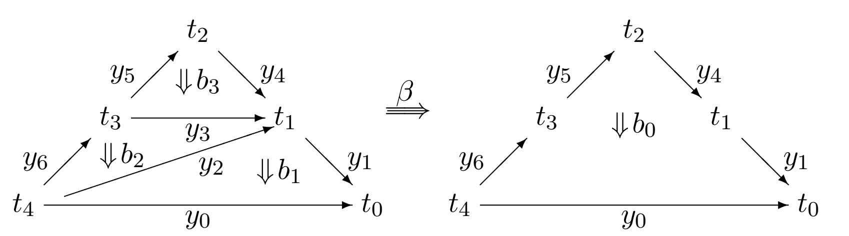
In this paper we present a combinatorial description of the category of the positive-to-one computads $\mathbf{Comp}^{+/1}$. We show, that this category is a presheaf category and we describe its exponent category in a very simple combinatorial way as the category of positive principal face structures $\mathbf{pFs}^{+/1}$, see section 3. However the proof of that requires some extended studies of the category of all positive face structures. Intuitively, the (isomorphism classes of) positive face structures correspond to the types of arbitrary cells in positive-to-one computads. The notion of a positive face structure in the main notion introduced in this paper. We describe in a combinatorial way, the embedding functor $\mathbf{e} : \mathbf{Comp}^{+/1} \to \omega Cat$ of the category of positive-to-one computads into the category into the ω-categories as the left Kan extension along a suitable functor $\mathbf{j}$, and its right adjoint as the restriction along $\mathbf{j}$. We end by adopting an argument due to V.Harnik to show that the right adjoint to $\mathbf{e}$ is monadic. This approach does not cover the problem of the cells with empty domains which is important for both Makkai's multitopic categories and Baez-Dolan opetopic categories. However it keeps something from the simplicity of the Joyal's θ-categories, i.e. the category of positive face structures or rather the category of positive computypes $\mathcal{C}types_\omega^{+/1}$, the full image of the former in ωCat is not much more complicated than the category of simple ω-categories, the dual of the category of disks, c.f. [J], [MZ], [Be]. In this sense this paper may be considered as a step towards a comparison of these two approaches.

Positive face structures

Positive face structures represent all possible shapes of cells in positive-to-one computads. A positive face structure S of dimension 2 can be pictured as a figure



and a positive face structure T of dimension 3 can be pictured as a figure



They have faces of various dimensions that fit together so that it make sense to compose them in a unique way. By S_n we denote faces of dimension n in S. Each faces a has a face $\gamma(a)$ as its codomain and a *non-empty set* of faces $\delta(a)$ as its domain. In S above we have for a_1

$$\gamma(a_1) = x_4 \quad \text{and} \quad \delta(a_1) = \{x_5, x_6, x_9\}$$

and in T we have for β

$$\gamma(\beta) = b_0 \quad \text{and} \quad \delta(\beta) = \{b_1, b_2, b_3\}$$

This is all the data we need. Moreover, these (necessarily finite) data satisfy four conditions (see Section 3 for details). Below we explain them in an intuitive way.

Globularity. This is the main condition. It relate the sets that are obtained by duble application of γ and δ. They are

$$\gamma\gamma(a) = \gamma\delta(a) - \delta\delta(a), \qquad \delta\gamma(a) = \delta\delta(a) - \gamma\delta(a).$$

Let us look how it works for two faces a_1 and β. In case of the face a_1 we have

$$\gamma\delta(a_1) = \{s_3, s_4, s_6\}, \qquad \delta\delta(a_1) = \{s_4, s_6, s_7\}$$

$$\gamma\gamma(a_1) = s_3, \qquad \delta\gamma(a_1) = \{s_7\}$$

So we have indeed

$$\delta\delta(a_1) - \gamma\delta(a_1) = \{s_4, s_6, s_7\} - \{s_3, s_4, s_6\} = \{s_7\} = \delta\gamma(a_1)$$

$$\gamma\delta(a_1) - \delta\delta(a_1) = \{s_3, s_4, s_6\} - \{s_4, s_6, s_7\} = \{s_3\} = \{\gamma\gamma(a_1)\}$$

Similarly for the face β we have

$$\gamma\gamma(\beta) = y_0, \qquad \delta\gamma(\delta) = \{y_1, y_4, y_5, y_6\}$$

$$\gamma\delta(\beta) = \{y_0, y_2, y_3\}, \qquad \delta\delta(\beta) = \{y_1, y_2, y_3, y_4, y_5, y_6\}$$

and hence

$$\gamma\delta(\beta) - \delta\delta(\beta) = \{y_0, y_2, y_3\} - \{y_1, y_2, y_3, y_4, y_5, y_6\} = \{y_0\} = \{\gamma\gamma(\beta)\}$$

$$\delta\delta(\beta) - \gamma\delta(\beta) = \{y_1, y_2, y_3, y_4, y_5, y_6\} - \{y_0, y_2, y_3\} = \{y_1, y_4, y_5, y_6\} = \delta\gamma(\beta)$$

As we see in both cases a_1 and β the first actual formula is a bit more baroque (due to the curely brackets around $\gamma\gamma(a_1)$, $\gamma\gamma(\beta)$) that in the globularity condition stated above. However in the following we omit curely bracket on purpose to have a simpler notation hoping that it will contribute to simplicity without messing things up.

Using δ's and γ's we can define two binary relations $<^+$ and $<^-$ on faces of the same dimension which are transitive closures of the relations $\lhd^+$ and $\lhd^-$, respectively. $a \lhd^+ b$ holds iff there is a face α such that $a \in \delta(\alpha)$ and $\gamma(\alpha) = b$, and $a \lhd^- b$ holds iff $\gamma(a) \in \delta(b)$. We call $<^+$ the *upper order* and $<^-$ the *lower order*. The following three conditions refer to these relations.

Strictness. In each dimension, the relation $<^+$ is a strict order. The relation $<^+$ on 0-dimensional faces is required to be a linear order.

Disjointness. This condition says that no two faces can be comparable with respect to both orders $<^+$ and $<^-$.

Pencil linearity. This final condition says that the sets of cells with common codomain (γ*-pencil*) and the sets of cells that have the same distinguished cell in the domain (δ*-pencil*) are linearly ordered by $<^+$.

The morphism of positive face structures are functions that preserves dimensions and operations γ and δ. The size of a positive face structure S is defined as an infinite sequence of natural numbers $size(S) = \{size(S)_k\}_{k\in\omega} = \{S_k - \delta(S_{k+1})\}_{k\in\omega}$ (almost all equal 0). We order the sequences lexicographically with higher dimensions being more important. The induction on the size of face structures is a convenient way of reasoning about positive face structures. Dimension of a face structure S is the index of the largest non-zero number in the sequence $size(S)$. If for $k \leq dim(S)$ ($k < dim(S)$), $size(S)_k = 1$ then S is *principal* (*normal*). The normal positive face structures plays role of the pasting diagram in [HMP] and the principal positive face structures plays role of the (positive) multitopes. Note that, contrary to [HMP], we do not consider either the empty-domain multitopes or the pasting diagrams. The precise connection between these two approaches will be described elsewhere. On positive face structures we define operations of the domain, codomain, and special pushouts which plays the role of composition. With these operations (isomorphisms classes of) the positive face structures form the terminal positive-to-one computad, and at the same time the monoidal globular category in the sense of Batanin.

Categories and functors

We shall define the following categories

$$\begin{array}{ccc} \mathbf{pFs}^{+/1} & & \mathcal{S} \\ \Big\downarrow{\scriptstyle \mathbf{i}} & & \Big\downarrow{\scriptstyle \mathbf{k}} \\ \mathbf{Fs}^{+/1} & \xrightarrow{\ \mathbf{j}\ } & \mathcal{C}types^{+/1}_{\omega} \\ {\scriptstyle (-)^*}\Big\downarrow & & \Big\downarrow \\ \mathbf{Comp}^{+/1} & \xrightarrow[\ \mathbf{e}\ ]{} & \omega Cat \end{array}$$

where $\mathbf{pFs}^{+/1}$ is the category of principal positive face structures, $\mathbf{Fs}^{+/1}$ is the category of positive face structures, $\mathcal{S}$ is the category of simple categories c.f. [MZ], $(-)^*$ is the embedding functor of positive face structures into positive-to-one computads, $\mathbf{e}$ is the inclusion functor, $\mathcal{C}types^{+/1}_{\omega}$ is the full image of the composition functor $(-)^*; \mathbf{e}$, with the non-full embedding $\mathbf{j}$.

Having these functor we can form the following diagram

$$\begin{array}{ccc} \mathbf{Comp}^{+/1} & \xrightarrow{\ \mathbf{e}\ } & \omega Cat \\ \widehat{(-)}\Big\downarrow\ \Big\uparrow\widetilde{(-)} & & \widehat{(-)}\Big\downarrow\ \Big\uparrow\widetilde{(-)} \\ sPb((\mathbf{Fs}^{+/1})^{op}, Set) & \underset{\mathbf{j}^*}{\overset{Lan_{\mathbf{j}}}{\rightleftarrows}} & sPb((\mathcal{C}types^{+/1}_{\omega})^{op}, Set) \\ \mathbf{i}^*\Big\downarrow\ \Big\uparrow Ran_{\mathbf{i}} & & \mathbf{k}^*\Big\downarrow\ \Big\uparrow Ran_{\mathbf{k}} \\ Set^{(\mathbf{pFs}^{+/1})^{op}} & & sPb(\mathcal{S}^{op}, Set) \end{array}$$

in which all the vertical arrows comes in pairs and they are adjoint equivalences of categories. The unexplained categories in the diagram above are: $sPb((\mathbf{Fs}^{+/1})^{op}, Set)$ - the category of the special pullback preserving morphisms from $(\mathbf{Fs}^{+/1})^{op}$ to Set and natural transformations, $sPb(\mathcal{S}^{op}, Set)$ - the category of the special pullback preserving morphisms from $\mathcal{S}^{op}$ to Set and natural transformations, $sPb((\mathcal{C}types^{+/1}_{\omega})^{op}, Set)$ - the category of the special pullbacks preserving morphisms from $(\mathcal{C}types^{+/1}_{\omega})^{op}$ to Set and natural transformations.

The functors $\widehat{(-)} : \mathbf{Comp}^{+/1} \rightarrow sPb((\mathbf{Fs}^{+/1})^{op}, Set)$ and $\widehat{(-)} : \omega Cat \rightarrow sPb((\mathcal{C}types^{+/1}_{\omega})^{op}, Set)$ are defined similarly, due to the embeddings $(-)^*$ in the previous diagram. For a computad Q and an ω-category C, $\widehat{Q}$ and $\widehat{C}$ are presheaves so that for a positive face structure S we have

$$\widehat{Q} = \mathbf{Comp}^{+/1}(S^*, Q) \qquad\qquad \widehat{C} = \omega Cat(S^*, C)$$

The adjoint functors $\widetilde{(-)}$ that produce ω-categories are slightly more complicated. They are defined in Sections 13 and 15. The other functors are standard. The functors $\mathbf{i}^*$, $\mathbf{j}^*$, $\mathbf{k}^*$ are inverse image functors. $Ran_{\mathbf{i}}$ and $Ran_{\mathbf{k}}$ are the right Kan extensions along $\mathbf{i}$, $\mathbf{k}$, respectively and $Lan_{\mathbf{j}}$ is the left Kan extension along $\mathbf{j}$.

Since we have $\mathbf{e}; \widehat{(-)} = \widehat{(-)}; Lan_{\mathbf{j}}$, and $\widehat{(-)}$'s are equivalences of categories, the functor $Lan_{\mathbf{j}}$ is like $\mathbf{e}$ but moved into a more manageable context. In fact we have a very neat description of this functor.

The content

Since the paper is quite long I describe below the content of each section to help the reading. Sections 2 and 3 introduce the notion of a positive hyper-graph and positive face structure. Section 4 is concerned with establishing what kind of inclusions hold between iterated applications of γ's and δ's. Section 5 contains many technical statements concerning positive face structure. All of them are there because they are needed afterwards. Section 6 describes the embedding $(-)^* : \mathbf{Fs}^{+/1} \rightarrow \omega Cat$ i.e. it's main goal is to define an ω-category S^* for any positive face structure S. Section 7 describes rather technical but useful properties of normal positive face structures. In section 8 we study a way we can decompose positive face structures if they are at all decomposable. Any positive face structure is either principal or decomposable. This provides a way of proving the properties of positive face structures by induction on the size. Using this in section 9 we show that the ω-category S^* and in fact the whole functor $(-)^*$ end up in $\mathbf{Comp}^{+/1}$. The next two short sections 10, 11 describe just what is in their titles: factorization in $\mathcal{C}types^{+/1}_{\omega}$, and the terminal positive-to-one computad in terms of positive face structures. Section 12 gives an explicite description of all the cells in a given positive-to-one computad with the help of positive face structures. In other words, it describe in concrete terms the functor $\overline{(-)} : \mathbf{Comma}^{+/1}_n \longrightarrow \mathbf{Comp}^{+/1}_n$. Section 13 establishes the equivalence of categories between $\mathbf{Comp}^{+/1}$ and the category of presheaves over $\mathbf{pFs}^{+/1}$. In Section 14 the principal pullbacks are introduced and the V.Harnik's argument in the present context is presented. Section 15 describes a full nerve functor

$$\widehat{(-)} : \omega Cat \longrightarrow Set^{(\mathcal{C}types^{+/1}_{\omega})^{op}}$$

and identifies its essential image as the special pullbacks preserving functors. Section 16 describes the inclusion functor as the left Kan extension

$$Lan_{\mathbf{j}} : sPb((\mathbf{Fs}^{+/1})^{op}, Set) \longrightarrow sPb((\mathcal{C}types^{+/1}_{\omega})^{op}, Set)$$

with the formulas involving just coproduct (and no other colimits). This gives as a corollary the fact that $\mathbf{e} : \mathbf{Comp}^{+/1} \rightarrow \omega Cat$ preserves connected limits. Then it is shown that the right adjoint to $Lan_{\mathbf{j}}$

$$\mathbf{j}^* : sPb((\mathcal{C}types^{+/1}_{\omega})^{op}, Set) \longrightarrow sPb((\mathbf{Fs}^{+/1})^{op}, Set)$$

(and hence the right adjoint to $\mathbf{e} : \mathbf{Comp}^{+/1} \to \omega Cat$) is monadic. In Appendix we recall the definition of the category of positive-to-one computads.

Acknowledgments

I am very grateful to Victor Harnik and Mihaly Makkai for the conversations concerning matters of this paper.

Notation

In the paper we will use both directions of compositions of morphisms in categories. But each time we will write which way we compose the morphism. So, for the composition of a pair of morphism $x \xrightarrow{f} y \xrightarrow{g} z$ we can write either $f;g$ or $g \circ f$. ω is the set of natural numbers.

2 Positive hypergraphs

A *positive hypergraph* S is a family $\{S_k\}_{k\in\omega}$ of finite sets of faces, a family of functions $\{\gamma_k : S_{k+1} \to S_k\}_{k\in\omega}$, and a family of total relations $\{\delta_k : S_{k+1} \to S_k\}_{0\leq k<n}$. Moreover $\delta_0 : S_1 \to S_0$ is a function and only finitely many among sets $\{S_k\}_{k\in\omega}$ are non-empty. As it is always clear from the context we shall never use the indices of the functions γ and δ.

A *morphism of positive hypergraphs* $f : S \longrightarrow T$ is a family of functions $f_k : S_k \longrightarrow T_k$, for $k \in \omega$, such that the diagrams

$$\begin{array}{ccc} S_{k+1} & \xrightarrow{f_{k+1}} & T_{k+1} \\ \gamma \downarrow & & \downarrow \gamma \\ S_k & \xrightarrow[f_k]{} & T_k \end{array} \qquad\qquad \begin{array}{ccc} S_{k+1} & \xrightarrow{f_{k+1}} & T_{k+1} \\ \delta \downarrow & & \downarrow \delta \\ S_k & \xrightarrow[f_k]{} & T_k \end{array}$$

commute, for $k \in \omega$. The commutation of the left hand square is the commutation of the diagram of sets an functions but in case of the right hand square we mean more than commutation of a diagram of relations, i.e. we demand that for any $a \in S_{\geq 1}$, $f_a : \delta(a) \longrightarrow \delta(f(a))$ be a bijection, where f_a is the restriction of f to $\delta(a)$. The category of positive hypergraphs is denoted by $\mathbf{Hg}^{+1}$.

When convenient and does not lead to confusions, for $a \in S_k$ we sometime treat $\gamma(a)$ as an element of S_{k-1} and sometimes as a subset $\{\gamma(a)\}$ of S_{k-1}.

Before we go on, we need some notation. Let S be a positive hypergraph.

1. The dimension of S is $max\{k \in \omega : S_k \neq \emptyset\}$, and it is denoted by $dim(S)$.

2. The sets of faces of different dimensions are assumed to be disjoint (i.e. $S_k \cap S_l = \emptyset$, for $k \neq l$). S is also used to mean the set of all faces of S, i.e. $\bigcup_{k=0}^{n} S_k$; the notation $A \subseteq S$ mean that A is a set of some faces of S; $A_k = A \cap S_k$, for $k \in \omega$.

3. If $a \in S_k$ then the face a has dimension k and we write $dim(a) = k$.

4. For $a \in S_{\geq 1}$ the set $\theta(a) = \delta(a) \cup \gamma(a)$ is *the set of codimension* 1 *faces* in a.

5. $S_{\geq k} = \bigcup_{i\geq k} S_i$, $\quad S_{\leq k} = \bigcup_{i\leq k} S_i$. The set $S_{\leq k}$ is closed under δ and γ so it is a sub-hypergraph of S, called k-truncation of S.

6. The image of $A \subseteq S$ under δ and γ will be denoted by

$$\delta(A) = \bigcup_{a \in A} \delta(a), \qquad \gamma(A) = \{\gamma(a) \,:\, a \in A\},$$

respectively. In particular $\delta\delta(a) = \bigcup_{x \in \delta(a)} \delta(x)$, $\gamma\delta(a) = \{\gamma(x) : x \in \delta(a)\}$.

7. $\iota(a) = \delta\delta(a) \cap \gamma\delta(a)$ is *the set of internal faces* of the face $a \in S_{\geq 2}$.

8. On each set S_k we introduce two binary relations $<^{S_k,-}$ and $<^{S_k,+}$, called *lower* and *upper order*, respectively. We usually omit k and even S in the superscript.

   (a) $<^{S_0,-}$ is the empty relation. For $k > 0$, $<^{S_k,-}$ is the transitive closure of the relation $\lhd^{S_k,-}$ on S_k, such that $a \lhd^{S_k,-} b$ iff $\gamma(a) \in \delta(b)$. We write $a \perp^- b$ iff either $a <^- b$ or $b <^- a$, and we write $a \leq^- b$ iff either $a = b$ or $a <^- b$.

   (b) $<^{S_k,+}$ is the transitive closure of the relation $\lhd^{S_k,+}$ on S_k, such that $a \lhd^{S_k,+} b$ iff there is $\alpha \in S_{k+1}$, such that $a \in \delta(\alpha)$ and $\gamma(\alpha) = b$. We write $a \perp^+ b$ iff either $a <^+ b$ or $b <^+ a$, and we write $a \leq^+ b$ iff either $a = b$ or $a <^+ b$.

   (c) $a \not\perp b$ if both conditions $a \not\perp^+ b$ and $a \not\perp^- b$ hold.

9. Let $a, b \in S_k$. A *lower path* $a_0, \ldots, a_m$ *from* a *to* b in S is a sequence of faces $a_0, \ldots, a_m \in S_k$ such that $a = a_0$, $b = a_m$ and for $\gamma(a_{i-1}) \in \delta(a_i)$, $i = 1, \ldots, m$.

10. Let $x, y \in S_k$. An *upper path* $x, a_0, \ldots, a_m, y$ *from* x *to* y in S is a sequence of faces $a_0, \ldots, a_m \in S_{k+1}$ such that $x \in \delta(a_0)$, $y = \gamma(a_m)$ and $\gamma(a_{i-1}) \in \delta(a_i)$, for $i = 1, \ldots, m$.

11. The iterations of γ and δ will be denoted in two different ways. By γ^k and δ^k we mean k applications of γ and δ, respectively. By $\gamma^{(k)}$ and $\delta^{(k)}$ we mean the application as many times γ and δ, respectively, to get faces of dimension k. For example in $a \in S_5$ then $\delta^3(a) = \delta\delta\delta(a) \subseteq S_2$ and $\delta^{(3)}(a) = \delta\delta(a) \subseteq S_3$.

12. For $l \leq k$, $a, b \in S_k$ we define $a <_l b$ iff $\gamma^{(l)}(a) <^- \gamma^{(l)}(b)$.

13. A face a is *unary* iff $\delta(a)$ is a singleton.

We have

Lemma 2.1 *If S is an hypergraph and $k \in \omega$, then $<^{S_{k+1},-}$ is a strict partial order iff $<^{S_k,+}$ is a strict partial order.*

3 Positive face structures

To simplify the notation, we treat both δ and γ as functions acting on faces as well as on sets of faces, which means that sometimes we confuse elements with singletons. Clearly, both δ and γ when considered as functions on sets are monotone.

A positive hypergraph S is a *positive face structure* if it is non-empty, i.e. $S_0 \neq \emptyset$ and

1. *Globularity:* for $a \in S_{\geq 2}$:

$$\gamma\gamma(a) = \gamma\delta(a) - \delta\delta(a), \qquad \delta\gamma(a) = \delta\delta(a) - \gamma\delta(a);$$

2. *Strictness*: for $k \in \omega$, the relation $<^{S_k,+}$ is a strict order; $<^{S_0,+}$ is linear;

3. *Disjointness*: for $k > 0$,
$$\perp^{S_k,-} \cap \perp^{S_k,+} = \emptyset$$

4. *Pencil linearity*: for any $k > 0$ and $x \in S_{k-1}$, the sets
$$\{a \in S_k \mid x = \gamma(a)\} \quad \text{and} \quad \{a \in S_k \mid x \in \delta(a)\}$$
are linearly ordered by $<^{S_k,+}$.

S is a *weak positive face structure* if S is a globular, strict, disjoint, positive hypergraph (i.e. pencil linearity is not required to hold).

The *category of (weak) positive face structures* is the full subcategory of $\mathbf{Hg}^{+1}$ whose objects are the (weak) positive face structures and is denoted by denoted by $\mathbf{Fs}^{+/1}$ ($\mathbf{wFs}^{+/1}$).

Remarks.

1. The reason why we call the first condition 'globularity' is that it will imply the usual globularity condition in the ω-categories generated by positive face structures.

2. For each $k \in \omega$, the k-truncation of a weak positive face structure S is again a weak positive face structure $S_{\leq k}$. In particular, any k-truncation of a positive face structure S is a weak positive face structure $S_{\leq k}$, but it does not necessarily satisfy local linearity condition.

3. Note that if we were to assume that each positive face structure has a single cell of dimension -1 then linearity of $<^{S_0,+}$ would become a special case of pencil linearity.

4. The fact that, for $x \in S_{k-1}$, the set $\{a \in S_k \mid x = \gamma(a)\}$ is linearly ordered is sometimes referred to as γ-*linearity* of $<^{S_k,+}$, and the fact that the set $\{a \in S_k \mid \ x \in \delta(a)\}$ is linearly ordered is sometimes referred to as δ-*linearity* of $<^{S_k,+}$.

5. If S has dimension n, as hypergraph, then we say that S is an n-*face structure*.

6. The *size of positive face structure* S is the sequence natural numbers $size(S) = \{|S_n - \delta(S_{n+1})|\}_{n\in\omega}$, with almost all being equal 0. We have an order $<$ on such sequences, so that $\{x_n\}_{n\in\omega} < \{y_n\}_{n\in\omega}$ iff there is $k \in \omega$ such that $x_k < y_k$ and for all $l > k$, $x_l = y_l$. This order is well founded and many facts about positive face structures will be proven by induction on the size.

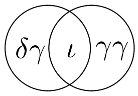
7. Let S be a positive face structure. S is k-principal iff $size(S)_l = 1$, for $l \leq k$. S is principal iff S is $dim(S)$-principal. S is normal iff S is $(dim(S)-1)$-principal. S is k-normal iff S is normal and $dim(S) = k$. By $\mathbf{pFs}^{+/1}$ ($\mathbf{nFs}^{+/1}$) we denote full subcategories of $\mathbf{Fs}^{+/1}$ whose objects are principal (normal) positive face structures.

4 Atlas for γ and δ

We have an easy

Lemma 4.1 *Let S be a positive face structure, $a \in S_n$, $n > 1$. Then*

1. *the sets $\delta\gamma(a)$, $\iota(a)$, and $\gamma\gamma(a)$ are disjoint;*



2. $\delta\delta(a) = \delta\gamma(a) \cup \iota(a)$;

3. $\gamma\delta(a) = \gamma\gamma(a) \cup \iota(a)$.

Proof. These are immediate consequences of globularity. □
Moreover

Lemma 4.2 *Let S be a positive face structure, $a \in S_n$, $n > 2$. Then we have*

1. $\delta\gamma\gamma(a) \subseteq \delta\gamma\delta(a) \subseteq \delta\gamma\gamma(a) \cup \iota\gamma(a) = \delta\delta\gamma(a) = \delta\delta\delta(a)$;

2. $\gamma\gamma\gamma(a) \subseteq \gamma\gamma\delta(a) \subseteq \gamma\gamma\gamma(a) \cup \iota\gamma(a) = \gamma\delta\gamma(a) = \gamma\delta\delta(a)$.

Proof. From globularity we have $\gamma\gamma(\alpha) \subseteq \gamma\delta(\alpha)$. Thus by monotonicity of δ and γ we get

$$\gamma\gamma\gamma(\alpha) \subseteq \gamma\gamma\delta(\alpha) \quad \text{and} \quad \delta\gamma\gamma(\alpha) \subseteq \delta\gamma\delta(\alpha) \quad \text{and} \quad \gamma\gamma\delta(\alpha) \subseteq \gamma\delta\delta(\alpha).$$

Similarly, as we have from globularity: $\delta\gamma(\alpha) \subseteq \delta\delta(\alpha)$ it follows by monotonicity of δ and γ:

$$\gamma\delta\gamma(\alpha) \subseteq \gamma\delta\delta(\alpha) \quad \text{and} \quad \delta\delta\gamma(\alpha) \subseteq \delta\delta\delta(\alpha) \quad \text{and} \quad \delta\gamma\delta(\alpha) \subseteq \delta\delta\delta(\alpha).$$

The equalities

$$\delta\gamma\gamma(a) \cup \iota\gamma(a) = \delta\delta\gamma(a) \quad \text{and} \quad \gamma\gamma\gamma(a) \cup \iota\gamma(a) = \gamma\delta\gamma(a)$$

follow easily from Lemma 4.1.

Thus it remains to show that:

1. $\delta\delta\gamma(a) \supseteq \delta\delta\delta(a)$,

2. $\gamma\delta\gamma(a) \supseteq \gamma\delta\delta(a)$.

Both inclusions can be proven similarly. We shall show the first only.

Suppose contrary, that there is $u \in \delta\delta\delta(a) - \delta\delta\gamma(a)$. Let $x \in \delta(a)$ be $<^-$-minimal element in $\delta(a)$ such that there is $s \in \delta(x)$ with $u \in \delta(s)$. If $s \in \delta\gamma(a)$ then $u \in \delta\delta\gamma(a)$ contrary to the supposition. Thus $s \not\in \delta\gamma(a)$. Since $\delta\gamma(a) = \delta\delta(a) - \gamma\delta(a)$ it follows that $s \in \gamma\delta(a)$. Hence there is $x' \in \delta(a)$ with $\gamma(x') = s$. In particular $x' <^- x$. Moreover

$$u \in \delta(s) = \delta\gamma(x') \subseteq \delta\delta(x').$$

Then there is $s' \in \delta(x')$ so that $u \in \delta(s')$. This contradicts the $<^-$-minimality of x. □

From Lemma 4.2 we get

Corollary 4.3 *Let S be a positive face structure, $a \in S_n$, $n > 2$, $k < n$. Then, with ξ^l and ξ'^l being two fixed strings of γ's and δ's of length l, we have*

1. $\gamma^k(a) \subseteq \gamma\xi^{k-1}(a)$;

2. $\delta\xi^{k-1}(a) \subseteq \delta^k(a)$;

3. $\delta^k(a) \cap \gamma^k(a) = \emptyset$;

4. $\xi^k(a) \subseteq \gamma^k(a) \cup \delta^k(a)$;

5. $\delta^2\xi^{k-2}(a) = \delta^2\xi'^{k-2}(a)$, (e.g. $\delta^k(a) = \delta^2\gamma^{k-2}(a)$);

6. $\gamma\delta\xi^{k-2}(a) = \gamma\delta\xi'^{k-2}(a)$, (e.g. $\gamma\delta^{k-1}(a) = \gamma\delta\gamma^{k-2}(a)$);

7. $\xi^{k-2}\delta\gamma(a) = \xi^{k-2}\delta^2(a)$, for $k > 2$;

8. $\delta^k(a) = \delta\gamma^{k-1}(a) \cup \iota\gamma^{k-2}(a)$, for $k > 1$.

5 Combinatorial properties of positive face structures

Local properties

Proposition 5.1 *Let S be a positive face structure, $k > 0$ and $\alpha \in S_k$, $a_1, a_2 \in \delta(\alpha)$, $a_1 \neq a_2$. Then we have*

1. $a_1 \not\perp^+ a_2$;

2. $\delta(a_1) \cap \delta(a_2) = \emptyset$ *and* $\gamma(a_1) \neq \gamma(a_2)$.

Proof. Ad 1. Suppose contrary that there are $a_1, a_2 \in \delta(\alpha)$ such that $a_1 <^+ a_2$. So we have an upper path

$$a_1, \beta_1, \ldots, \beta_r, a_2$$

and hence a lower path

$$\beta_1, \ldots, \beta_r, \alpha.$$

In particular $\beta_1 <^- \alpha$. As $a_1 \in \delta(\beta_1) \cap \delta(\alpha)$ by δ-linearity we have $\beta_1 \perp^+ \alpha$. But then $(\alpha, \beta_1) \in \perp^+ \cap \perp^- \neq \emptyset$ i.e. S does not satisfy the disjointness. This shows 1.

Ad 2. This is an immediate consequence of 1. If $a_1, a_2 \in \delta(\alpha)$ and either $\gamma(a_1) = \gamma(a_2)$ or $\delta(a_1) \cap \delta(a_2) \neq \emptyset$ then by pencil linearity we get that $a_1 \perp^+ a_2$, contradicting 1. □

After proving the above proposition we can introduce more notation. Let S be a positive face structure, $n \in \omega$.

1. For a face $\alpha \in S_{n+2}$, we shall denote by $\rho(\alpha) \in \delta(\alpha)$ be the only face in $\delta(\alpha)$, such that $\gamma(\rho(\alpha)) = \gamma\gamma(\alpha)$.

2. $X \subseteq S_{n+1}$, $a, b \in S_n$ and $a, \alpha_1, \ldots, \alpha_k, b$ be an upper path in S. We say that it is *a path in X* (or X-path) if $\{\alpha_1, \ldots, \alpha_k\} \subseteq X$.

Lemma 5.2 *Let S be a positive face structure, $n \in \omega$, $\alpha \in S_{n+2}$, $a, b \in S_{n+1}$, $y \in \delta\delta(\alpha)$. Then*

1. *there is a unique upper $\delta(\alpha)$-path from y to $\gamma\gamma(\alpha)$;*

2. *there is a unique $x \in \delta\gamma(\alpha)$ and an upper $\delta(\alpha)$-path from x to y such that $\gamma(x) = \gamma(y)$;*

3. *if $t \in \delta(y)$ there is an $x \in \delta\gamma(\alpha)$ and an upper $\delta(\alpha)$-path from x to y and $t \in \delta(x)$;*

4. *If $a <^+ b$ then $\gamma(a) \leq^+ \gamma(b)$.*

Proof. Ad 1. Uniqueness follows from Lemma 5.1. To show the existence, let us suppose contrary that there is no $\delta(\alpha)$-path from y to $\gamma\gamma(\alpha)$. We shall construct an infinite upper $\delta(\alpha)$-path from y

$$y, a_1, a_2, \ldots$$

As $y \in \delta\delta(\alpha)$ there is $a_1 \in \delta(\alpha)$ such that $y \in \delta(a_1)$. So now suppose that we have already constructed $a_1, \ldots, a_k$. By assumption $\gamma(a_k) \neq \gamma\gamma(\alpha)$ so, by globularity, $\gamma(a_k) \in \delta\delta(\alpha)$. Hence there is $a_{k+1} \in \delta(\alpha)$ such that $\gamma(a_k) \in \delta(a_{k+1})$. This ends the construction of the path.

As in positive face structure there are no infinite paths, this is a contradiction and if fact there is a $\delta(\alpha)$-path from y to $\gamma\gamma(\alpha)$.

Ad 2. Suppose not, that there is no $x \in \delta\gamma(\alpha)$ as claimed. We shall construct an infinite descending lower $\delta(\alpha)$-path

$$\ldots \lhd^- a_1 \lhd^- a_0$$

such that $\gamma(a_0) = y$, $\gamma\gamma(a_n) = \gamma(y) = t$, for $n \in \omega$.

By assumption $y \not\in \delta\gamma(\alpha) = \delta\delta(\alpha) - \gamma\delta(\alpha)$. So $y \in \gamma\delta(\alpha)$. Hence there is $a_0 \in \delta(\alpha)$ such, that $\gamma(a_0) = y$. Now, suppose that the lower $\delta(\alpha)$-path

$$a_k \lhd^- a_{k-1} \lhd^- \ldots \lhd^- a_0$$

has been already constructed. By globularity, we can pick $z \in \delta(a_k)$, such that $\gamma(z) = t$. By assumption $z \not\in \delta\gamma(\alpha) = \delta\delta(\alpha) - \gamma\delta(\alpha)$. So $z \in \gamma\delta(\alpha)$. Hence there is $a_{k+1} \in \delta(\alpha)$ such, that $\gamma(a_{k+1}) = z \in \delta(a_k)$. Clearly, $\gamma\gamma(a_{k+1}) = t$. This ends the construction of the path. But by strictness such a path has to be finite, so there is x as needed.

Ad 3. This case is similar. We put it for completeness.

Suppose not, that there is no $x \in \delta\gamma(\alpha)$ as above. We shall construct an infinite descending lower $\delta(\alpha)$-path

$$\ldots \lhd^- a_1 \lhd^- a_0$$

such that $\gamma(a_0) = y$, $t \in \delta\gamma(a_n)$, for $n \in \omega$.

By assumption $y \not\in \delta\gamma(\alpha) = \delta\delta(\alpha) - \gamma\delta(\alpha)$. So $y \in \gamma\delta(\alpha)$. Hence there is $a_0 \in \delta(\alpha)$ such, that $\gamma(a_0) = y$. Now, suppose that the lower $\delta(\alpha)$-path

$$a_k \lhd^- a_{k-1} \lhd^- \ldots \lhd^- a_0$$

has been already constructed. By globularity, we can pick $z \in \delta(a_k)$, such that $t \in \delta(z)$. By assumption $z \not\in \delta\gamma(\alpha) = \delta\delta(\alpha) - \gamma\delta(\alpha)$. So $z \in \gamma\delta(\alpha)$. Hence there is $a_{k+1} \in \delta(\alpha)$ such, that $\gamma(a_{k+1}) = z \in \delta(a_k)$. Clearly, $t \in \delta\gamma(a_{k+1})$. This ends the construction of the path. But by strictness such a path has to be finite, so there is x as needed.

Ad 4. The essential case is when $a \lhd^+ b$. This follows from 1. Then use the induction. $\Box$

Lemma 5.3 *Let S be a positive face structure, $n > 1$, $\alpha \in S_{n+1}$, and $a, b \in S_n$ such that $a <^+ b$. Then*

1. $\iota\delta(\alpha) = \iota\gamma(\alpha)$ *;*
2. $\iota(a) \subseteq \iota(b)$*;*
3. $\iota(a) \cup \gamma\gamma(a) \subseteq \iota(b) \cup \gamma\gamma(b)$*;*

4. $\delta\delta(a) \subseteq \delta\delta(b)$;

5. $\theta\theta(a) \subseteq \theta\theta(b)$.

Proof. Ad 1. $\iota\delta(\alpha) \subseteq \iota\gamma(\alpha)$:

Fix $a \in \delta(\alpha)$ and $t \in \iota(a)$. Thus there are $x, y \in \delta(a)$ such, that $\gamma(x) = t \in \delta(y)$. By Lemma 5.2 2,3 there are $x', y' \in \delta\gamma(\alpha)$ such, that $x' \leq^+ x$, $y' \leq^+ y$ and $\gamma(x') = t \in \delta(y')$. Thus $t \in \iota\gamma(\alpha)$ and the first inclusion is proved.

$\iota\delta(\alpha) \supseteq \iota\gamma(\alpha)$:

Fix $t \in \iota\gamma(\alpha)$. In particular, there are $x, y \in \delta\gamma(\alpha)$, so that $\gamma(x) = t \in \delta(y)$. Suppose that $t \notin \iota\delta(\alpha)$. We shall build an infinite $\delta(\alpha)$-path

$$a_1 \lhd^- a_2 \ldots$$

such that $\gamma\gamma(a_i) = t$ for $i \in \omega$.

Since $\delta\gamma(\alpha) \subseteq \delta\delta(\alpha)$, there is $a_1 \in \delta(\alpha)$ such, that $x \in \delta(a_1)$. Since $t \notin \iota\delta(\alpha)$, it follows that $\gamma\gamma(a_1) = t$. Suppose now that we have already constructed the path

$$a_1 \lhd^- a_2 \ldots a_k$$

as above. We have $\gamma\gamma(a_k) = t \lhd^+ \gamma(y) \leq^+ \gamma\gamma\gamma(\alpha)$. So $\gamma(a_k) \neq \gamma\gamma(\alpha)$ and $\gamma(a_k) \in \delta\delta(\alpha)$. Then there is $a_{k+1} \in \delta(\alpha)$ such, that $\gamma(a_k) \in \delta(a_{k+1})$. Again, as $t \notin \iota\delta(\alpha)$, it follows that $\gamma\gamma(a_{k+1}) = t$. This ends the construction of the path. Since, by strictness, such a path cannot exists we get the other inclusion.

Ad 2. Since the inclusion is transitive, it is enough to consider the case $a \lhd^+ b$, i.e. there is an $\alpha \in S_{n+1}$ such, that $a \in \delta(\alpha)$ and $b = \gamma(\alpha)$. Then by 1. we have

$$\iota(a) \subseteq \iota\delta(\alpha) = \iota\gamma(\alpha) = \iota(b)$$

Ad 3. As above it is enough to consider the case $a \lhd^+ b$, i.e. that there is $\alpha \in S_{n+1}$ such that $a \in \delta(\alpha)$ and $\gamma(\alpha) = b$. By 2. we need to show that $\gamma\gamma(a) \subseteq \iota(b) \cup \gamma\gamma(b)$. We have

$$\gamma\gamma(a) \in \gamma\gamma\delta(\alpha) \subseteq \iota\gamma(\alpha) \cup \gamma\gamma\gamma(\alpha) = \iota(b) \cup \gamma\gamma(b).$$

Ad 4. Again it is enough to consider the case $a \lhd^+ b$, i.e. that there is $\alpha \in S_{n+1}$ such that $a \in \delta(\alpha)$ and $\gamma(\alpha) = b$. We have

$$\delta\delta(a) \subseteq \delta\delta\delta(\alpha) = \delta\delta\gamma(\alpha) = \delta\delta(b).$$

Ad 5. This follows from 3. and 4. □

Global properties

Lemma 5.4 *Let S be a positive face structure, $n \in \omega$, $a, b \in S_n$, $a <^+ b$. Then, there is an upper $S_{n+1} - \gamma(S_{n+2})$-path from a to b.*

Proof. Let $a, \alpha_1, \ldots, \alpha_k, b$ be an upper path is S. By Lemma 5.2 we can replace each face α_i in this path which is not in $S - \gamma(S)$ by a sequence of faces which are $<^+$-smaller. Just take $\Gamma \in S_{n+2}$, such that $\gamma(\Gamma) = \alpha_i$ and take instead of α_i a path in $\delta(\Gamma)$ from $\gamma(\alpha_{i-1})$ (if $i = 0$ then from a) to $\gamma(\alpha_i)$. Repeated application of this procedure will eventually yield the required path. □

Lemma 5.5 *Let S be a positive face structure, $n > 0$, $a \in S_n$, $\alpha \in S_{n+1}$, and either $\gamma(a) \in \iota(\alpha)$ or $\delta(a) \cap \iota(\alpha) \neq \emptyset$. Then $a <^+ \gamma(\alpha)$. Moreover, if $\alpha \in S - \gamma(S)$ then there is a unique $a' \in \delta(\alpha)$ such that $a \leq^+ a'$.*

Proof. If $a \in \delta(\alpha)$ there is nothing to prove. So we assume that $a \notin \delta(\alpha)$.

We begin with the second part of Lemma, i.e. we assume that $\alpha \in S_{n+1} - \gamma(S_{n+2})$.

Let $\gamma(a) \in \iota(\alpha)$. Thus there are $b, c \in \delta(\alpha)$ such that $\gamma(a) = \gamma(b) \in \delta(c)$. In particular $a <^- c$. By γ-linearity either $b <^+ a$ or $a <^+ b$. Suppose that $b <^+ a$. Then we have an $(S - \gamma(S))$-upper path $b, \beta_0, \ldots, \beta_r, a$. As $b \in \alpha \cap \beta_0$ and $\alpha, \beta_0 \in S - \gamma(S)$, we have $\alpha = \beta_0$. But then $c \in \delta(\alpha) = \delta(\beta_0)$ and hence $c <^+ \gamma(\beta_0) \leq^+ a$. So we get $a <^- c$ and $c <^+ a$ contradicting the disjointness of $\perp^+$ and $\perp^-$. Thus we can put $a' = b$ and we have $a <^+ a'$. The uniqueness of a' follows from the fact that $\gamma(a) = \gamma(a')$.

The case $\delta(a) \cap \iota(\alpha) \neq \emptyset$ is similar and we put it for completeness. Thus there are $b, c \in \delta(\alpha)$ such that $\gamma(b) \in \delta(a) \cap \delta(c)$. In particular $b <^- a$. By δ-linearity either $c <^+ a$ or $a <^+ c$. Suppose that $c <^+ a$. Then we have an $(S - \gamma(S))$-upper path $c, \beta_0, \ldots, \beta_r, a$. As $c \in \alpha \cap \beta_0$ and $\alpha, \beta_0 \in S - \gamma(S)$, we have $\alpha = \beta_0$. But then $b \in \delta(\alpha) = \delta(\beta_0)$ and hence $b <^+ \gamma(\beta_0) \leq^+ a$. So we get $b <^- a$ and $b <^+ a$ contradicting the disjointness of $\perp^+$ and $\perp^-$. Thus we can put $a' = c$ and we have $a <^+ a'$. The uniqueness of a' follows from the fact that $\gamma(b) \in \delta(a')$ and $a' \in \delta(\alpha)$ and Lemma 5.1.

The first part of the Lemma follows from the above, Lemma 5.2.4 and the following Claim.

Claim. If $\alpha \in S_{n+1}$ and $x \in \iota(\alpha)$ then there is an $\alpha' \in S_{n+1}$ such that $\alpha' \leq^+ \alpha$, $x \in \iota(\alpha')$ and $\alpha' \notin \gamma(S_{n+2})$.

Proof of the Claim. Suppose that Claim is not true. To get a contradiction, we shall build an infinite descending $\gamma(S_{n+2})$-path

$$\ldots \lhd^+ \alpha_1 \lhd^+ \alpha_0 = \alpha$$

such that $x \in \iota(\alpha_i)$, for $i \in \omega$.

We put $\alpha_0 = \alpha$. Suppose that we have already constructed $\alpha_0, \ldots, \alpha_k \in \gamma(S_{n+2})$. Hence there is $\beta \in S_{n+2}$ such, that $\gamma(\beta) = \alpha_k$. Since $\iota\delta(\beta) = \iota\gamma(\beta) = \iota(\alpha_k)$, there is $\alpha_{k+1} \in \delta(\beta)$ such, that $x \in \iota(\alpha_{k+1})$. This ends the construction of the infinite path and the proof of the Claim and the Lemma. $\Box$

Corollary 5.6 *Let S be a positive face structure. If $a \in S - \delta(S)$ then $\gamma(a) \in S - \iota(S)$ and $\delta(a) \subseteq S - \iota(S)$.*

Proof. Let $a \in S_n$ and $\alpha \in S_{n+2}$. If either $\gamma(a) \in \iota(\alpha)$ or $\delta(a) \cap \iota(\alpha) \neq \emptyset$ then by Lemma 5.5 $a <^+ \gamma(\alpha)$. Thus $a \in \delta(S)$. $\Box$

A lower path $b_0, \ldots, b_m$ is a *maximal path* if $\delta(b_0) \subseteq \delta(S) - \gamma(S)$ and $\gamma(b_m) \in \gamma(S) - \delta(S)$, i.e. if it can't be extended either way.

Lemma 5.7 (Path Lemma) *Let $k \geq 0$, $B = (a_0, \ldots, a_k)$ be a maximal S_n-path in a positive face structure S, $b \in S_n$, $0 \leq s \leq k$, $a_s <^+ b$. Then there are $0 \leq l \leq s \leq p \leq k$ such, that*

1. *$a_i <^+ b$ for $i = l, \ldots, p$;*
2. *$\gamma(a_p) = \gamma(b)$;*
3. *either $l = 0$ and $\delta(a_0) \subseteq \delta(b)$ or $l > 0$ and $\gamma(a_{l-1}) \in \delta(b)$;*
4. *$\gamma(a_i) \in \iota(S)$, for $l \leq i < p$.*

Proof. Let $0 \leq l \leq p \leq k$ be such that $a_i <^+ b$ for $l \leq i \leq p$ and either $l = 0$ or $a_{l-1} \not<^+ b$ and either $p = k$ or $a_{k+1} \not<^+ b$. We shall show that l and p have the properties stated in the Lemma. From the very definition the property 1 holds.

We shall show 2. Take an upper $(S - \gamma(S))$-path from a_p to b: $a_p, \beta_0, \ldots, \beta_r, b$. If $\gamma(a_p) = \gamma\gamma(\beta_i)$, for $i = 0, \ldots, r$ then $\gamma(a_p) = \gamma\gamma(\beta_r) = \gamma(b)$ and we are done. So suppose contrary and let

$$i_0 = min\{i : \gamma(a_p) \neq \gamma\gamma(\beta_i)\}$$

Then there are $a, c \in \delta(\beta_{i_0})$ such that $\gamma(a_p) = \gamma(a) \in \delta(c)$ (NB. $a = a_p$ if $i_0 = 0$ and $a = \gamma(\beta_{i_0-1})$ otherwise). In particular $\gamma(a_p) \in \iota(\beta_{i_0})$. As $\gamma(a_p) \in \delta(S)$, we have $p < k$. Thus $\gamma(a_p) \in \delta(a_{p+1}) \cap \iota(\beta_{i_0})$, and by Lemma 5.5 $a_{p+1} <^+ c <^+ b$. But this contradicts the choice of p. So the property 2. holds.

Now we shall show 3. Take an upper $(S-\gamma(S))$-path from a_l to b: $a_l, \beta_0, \ldots, \beta_r, b$. We have two cases: $l = 0$ and $l > 0$.

If $l = 0$ then there is no face $a \in S$ such that $\gamma(a) \in \delta(a_l)$. As $\delta(a_l) \subseteq \delta\delta(\beta_0)$ we must have $\delta(a_l) \in \delta\gamma(\beta_i)$, for $i = 0, \ldots, r$. Hence $\delta(a_l) \subseteq \delta\gamma(\beta_r) = \delta(b)$ and 3. holds in this case.

Now suppose that $l > 0$. If $\gamma(a_{l-1}) \in \delta\gamma(\beta_i)$, for $i = 0, \ldots, r$, then $\gamma(a_{l-1}) \in \delta\gamma(\beta_r) = \delta(b)$ and 3. holds again. So suppose contrary , and let

$$i_1 = min\{i : \gamma(a_{l-1}) \not\in \delta\gamma(\beta_i)\}$$

Then there are $a, c \in \delta(\beta_{i_1})$ such that $\gamma(a_{l-1}) = \gamma(a) \in \delta(c)$ (NB: $c = a_l$ if $i_1 = 0$ and $c = \gamma(\beta_{i_1-1})$ otherwise). In particular $\gamma(a_{l-1}) \in \iota(\beta_{i_1})$, and by Lemma 5.5 we have $a_{l-1} <^+ a <^+ b$ contrary to the choice of l. Thus 3. holds in this case as well.

Finally, we shall show 4. Let $l \leq j \leq p$ and $a_j, \beta_0, \ldots, \beta_r, b$ be an upper $(S-\delta(S))$-path from a_j to b. As $a_j <^- a_p$ and $a_p <^+ b$ we have $\gamma(a_j) \neq \gamma(b)$. So we can put

$$i_2 = min\{i : \gamma(a_j) \neq \gamma\gamma(\beta_i)\}$$

But then $\gamma(a_j) \in \gamma\delta(\beta_{i_2}) - \gamma\gamma(\beta_{i_2}) = \iota(\beta_{i_2})$. Therefore $\gamma(a_j) \in \iota(S)$ and 4. holds. □

Lemma 5.8 *Let S be a positive face structure, $n \in \omega$, $x, y \in S_n$, $x <^+ y$. Then if $x, y \not\in \iota(S_{n+2})$, then there is an upper path from x to y contained in $S_{n+1} - \delta(S_{n+2})$.*

Proof. Assume $x, y \in (S - \iota(S))$ and $x <^+ y$. Let

$$x, b_0, \ldots, b_k, y$$

be an upper path from x to y with the longest possible initial segment $b_0, \ldots, b_l$ in $S - \delta(S)$.

We need to show that $k = l$. So suppose contrary, that $l < k$. By Corollary 5.6 $\gamma(b_l) not \in \iota(S)$. Let α be a face such that $b_{l+1} \in \iota(\alpha)$. In particular $b_{l+1} <^+ \gamma(\alpha)$. As $\gamma(b_l) \not\in \iota(S)$ we have $\gamma(b_l) \in \delta\gamma(\alpha)$ $(= \delta\delta(\alpha) - \iota(\alpha))$. So $\gamma(b_l) \in \delta(b_{l+1} \cap \delta\gamma(\alpha)$.

Let a be the $<^+$-largest element of the set $\{b \in S : \gamma(b_l) \in \delta(b)\}$. Then $b_{l+1} \leq a$. By a similar argument as above for b_{l+1} we get that $a \not\in \delta(S)$. By Lemma 5.7 there is p such that $l + 1 \leq p \leq k$ such that $\gamma(b_p) = \gamma(a)$ (Note that the facts that $y \not\in \iota(S)$ and Lemma 5.7.4 are needed here.). Thus we have an upper path from x to y, $x, b_0, \ldots, b_l, a, b_{p+1}, \ldots, b_k, y$ with a longer initial segment in $S - \delta(S)$. But this means in fact that in fact $l = k$, as required. □

Order

Lemma 5.9 *Let S be a positive face structure, $n \in \omega$, $a, b \in S_n$. Then we have*

1. *If $a <^+ b$ then for any $x \in \delta(a)$ there is $y \in \delta(b)$ such that $y \leq^+ x$;*
2. *If $a <^+ b$ and $\gamma(a) = \gamma(b)$ then for any $y \in \delta(b)$ there is $x \in \delta(a)$ such that $y \leq^+ x$;*
3. *If $\gamma(a) = \gamma(b)$ then either $a = b$ or $a \perp^+ b$;*
4. *If $\gamma(a) <^+ \gamma(b)$ then either $a <^+ b$ or $a <^- b$;*
5. *If $a <^+ b$ then $\gamma(a) \leq^+ \gamma(b)$;*
6. *If $a <^- b$ then $\gamma(a) <^+ \gamma(b)$;*
7. *If $\gamma(a) \perp^- \gamma(b)$ then $a \not<^- b$ and $a \not<^+ b$.*

Proof. Ad 1. Let $a <^+ b$ and $x \in \delta(a)$. We have two cases: either $x \in \gamma(S)$ or $x \notin \gamma(S)$.

In the first case there is $a' \in$, such that $\gamma(a') = x$. Let $a_0, \ldots, a_k$ be a maximal path containing a', a, say $a_{s-1} = a'$ and $a_s = a$, where $0 < s \leq k$. As $a_s <^+ b$, by Lemma 5.7 there is $l \leq s$ and $y \in \delta(a_l) \cap \delta(b)$. Clearly, $y \leq^+ x$.

In the second case consider an upper path from a to b: $a, \beta_0, \ldots, \beta_r, b$. We have $x \in \delta(a) \subseteq \delta\delta(\beta_0)$. As $x \notin \gamma(S)$ so $x \notin \gamma\delta(\beta_0)$, and hence $x \in \delta\delta(\beta_0) - \gamma\delta(\beta_0) = \delta\gamma(\beta_0)$. Thus we can define

$$r' = max\{i : x \in \delta\gamma(\beta_i)\}$$

If we had $r' < r$ then again we would have $x \in \delta\delta(\beta_{r'+1}) - \gamma\delta(\beta_{r'+1}) = \delta\gamma(\beta_{r'+1})$, contrary to the choice of r'. So $r' = r$ and $x \in \delta\gamma(\beta_r) = \delta(b)$. Thus we can put $y = x$.

Ad 2. Fix $a <^+ b$ such that $\gamma(a) = \gamma(b)$ and $y \in \delta(b)$. We need to find $x \in \delta(a)$ with $y \leq^=$. Take a maximal $(S - \gamma(S))$-path $a_0, \ldots, a_k$ passing through y, i.e. there is $0 \leq j \leq k$ such that $y \in \delta(a_j)$ and if $y \in \gamma(S)$ then moreover $j > 0$ and $\gamma(a_{j-1}) = y$. Since $a_j \notin \gamma(S)$ by δ-linearity $a_j <^+ b$. Thus by Lemma 5.7 there is $j \leq p \leq k$ such that $\gamma(a_p) = \gamma(b) = \gamma(a)$. Since $a_p \notin \gamma(S)$ by γ-linearity we have $a_p \leq^+ a$. If $a_p = a$ then we can take as the face x either y if $p = 0$ or $\gamma(a_{p-1})$ if $p > 0$. So assume now that $a_p <^+ a$. Again by Lemma 5.7 there is $0 \leq l \leq p$ such that either $l = 0$ and $\delta(a_0) \subseteq \delta(a)$ or $l > 0$ and $\gamma(a_{l-1}) \in \delta(a)$. As a_l is the first face in the path $a_0, \ldots, a_k$ such that $a_l <^+ a$ and a_j is the first face in the path $a_0, \ldots, a_k$ such that $a_l <^+ b$ and moreover $a <^+ b$ it follows that $j \leq l$. Thus in this case we can take as the face x either y if $l = 0$ or $\gamma(a_{l-1})$ if $l > 0$.

Ad 3. This is an immediate consequence of γ-linearity.

Ad 4. Suppose $\gamma(a) <^+ \gamma(b)$. So there is an upper path

$$\gamma(a), c_1, \ldots, c_k, \gamma(b)$$

with $k > 0$. We put $c_0 = a$. We have $\gamma(c_k) = \gamma(b)$ so by γ-linearity $c_k \perp^+ b$. So we have two cases: either $b <^+ c_k$ or $c_k <^+ b$.

If $b <^+ c_k$ then by Lemma 5.7 for any maximal path that contains b and the face c_k we get that $c_{k-1} <^- b$. Thus we have $a <^- b$.

If $c_k <^+ b$ then by Lemma 5.7 for any maximal path that extends $c_0, c_1, \ldots, c_k$ and face b we get that either there is $0 \leq i < k$ such that $\gamma(c_i) \in \delta(b)$ and then $a <^- b$ or else $a = c_0 <^+ b$.

Ad 5. This is repeated from Lemma 5.2.

Ad 6. Suppose $a <^- b$. Then there is a lower path

$$a = a_0, a_1, \dots, a_k = b$$

with $k > 0$. Then we have an upper path

$$\gamma(a) = \gamma(a_0), a_1, \dots, a_k, \gamma(a_k) = \gamma(b).$$

Hence $\gamma(a) <^+ \gamma(b)$.

Ad 7. Easily follows from 5 and 6. □

Proposition 5.10 *Let S be a positive face structure, $a, b \in S_n$, $a \neq b$. Let $\{a_i\}_{0\leq i\leq n}$, $\{b_i\}_{0\leq i\leq n}$ be two sequences of codomains of a and b, respectively, so that*

$$a_i = \gamma^{(i)}(a) \qquad\qquad b_i = \gamma^{(i)}(b)$$

(i.e. $dim(a_i) = i$), for $i = 0, \dots, n$. Then there are two numbers $0 \leq l \leq k \leq n$ such that either

1. *$a_i = b_i$ for $i < l$,*
2. *$a_i <^+ b_i$ for $l \leq i \leq k$,*
3. *$a_i <^- b_i$ for $k + 1 = i \leq n$,*
4. *$a_i \not\perp b_i$ for $k + 2 \leq i \leq n$,*

or the roles of a and b are interchanged.

Proof. The above conditions we can present more visually as:

$$a_0 = b_0, \dots, a_{l-1} = b_{l-1}, \quad a_l <^+ b_l, \dots a_k <^+ b_k,$$

$$a_{k+1} <^- b_{k+1}, \quad a_{k+2} \not\perp b_{k+2}, \dots, a_n \not\perp b_n.$$

These conditions we will verify from the bottom up. Note that by strictness $<^{S_0,+}$ is a linear order. So either $a_0 = b_0$ or $a_0 \perp^+ b_0$. In the later case $l = 0$. As $a \neq b$ then there is $i \leq n$ such that $a_i \neq b_i$. Let l be minimal such, i.e. $l = min\{i : a_i \neq b_i\}$. By Lemma 5.9 3., $a_l \perp^+ b_l$. So assume that $a_l <^+ b_l$. We put $k = max\{i \leq n : a_i <^+ b_i\}$. If $k = n$ we are done. If $k < n$ then by Lemma 5.9 4., we have $a_{k+1} <^- b_{k+1}$. Then if $k + 1 < n$, by Lemma 5.9 5. 6. 7., $a_i \not\perp b_i$ for $k + 2 \leq i \leq n$. This ends the proof. □

From the above Proposition we get immediately

Corollary 5.11 *Let S be a positive face structure, $a, b \in S_n$, $a \neq b$. Then either $a \perp^+ b$ or there is a unique $0 \leq l \leq k$ such that $a \perp^-_l b$, but not both.*

The above Corollary allows us to define an order $<^S$ (also denoted $<$) on all cells of S as follows. For $a, b \in S_n$,

$$a <^S b \quad \text{iff} \quad a <^+ b \ \text{ or } \ \exists_l \ a <^-_l b.$$

Corollary 5.12 *For any positive face structure S, and $n \in \omega$, the relation $<^S$ restricted to S_n is a linear order.*

Proof. We need to verify that $<^S$ is transitive.

Let $a, b, c \in S_n$. There are some cases to consider.

If $a <^+ b <^+ c$ then clearly $a <^+ c$.

If $a <^+ b <^-_l c$ then we have $\gamma^{(l)}(a) <^+ \gamma^{(l)}(b) <^- \gamma^{(l)}(c)$. By Lemma 5.2 4., and transitivity of $<^-$ we have $\gamma^{(l)}(a) <^- \gamma^{(l)}(c)$. Hence $a <^-_l c$.

If $a <^-_l b <^+ c$ then, by use Lemma 5.7, either the maximal lower path from that contains both $\gamma^{(l)}(a)$ and $\gamma^{(l)}(b)$ passes between $\gamma^{(l)}(a)$ and $\gamma^{(l)}(b)$ through a face in $\delta\gamma^{(l+1)}(c)$ and $a <^-_l c$ or it does not passes through a face $\delta\gamma^{(l+1)}(c)$ and then $\gamma^{(l)}(a) <^+ \gamma^{(l)}(c)$. Thus by Proposition 5.10 either there is $l \leq l' \leq n$ such that $\gamma^{(l')}(a) <^+ \gamma^{(l')}(c)$ and hence $a <^-_{l'} c$ or $a <^+ c$.

Finally suppose that $a <^-_k b <^-_l c$.

If $k = l$ then clearly $a <_l c$.

If $k > l$ then $\gamma^{(l)}(a) <^+ \gamma^{(l)}(b) <^- \gamma^{(l)}(c)$ and, by the previous argument, $\gamma^{(l)}(a) <^- \gamma^{(l)}(c)$, i.e. $a <^-_l c$.

Finally, assume that $k < l$, i.e. $\gamma^{(k)}(a) <^- \gamma^{(k)}(b) <^+ \gamma^{(k)}(c)$. Then, by Path Lemma, either $\gamma^{(k)}(a) <^- \gamma^{(k)}(c)$ or $\gamma^{(k)}(a) <^+ \gamma^{(k)}(c)$. In the latter case, by Proposition 5.10, either $a <^+ b$ or there is k', such that $k < k' \leq n$ and $\gamma^{(k')}(a) <^+ \gamma^{(k')}(c)$. In any case we have $a < c$, as required. $\Box$

From the proof of the above corollary we get

Lemma 5.13 *Let S be a positive face structure, $a \in S_n$. Then the set*

$$\{b \in S_n \,:\, a \leq^+ b\}$$

is linearly ordered by $\leq^+$.

Proof. Suppose $a \leq^+ b, b'$. If we where to have $b <^-_l b'$ for some $l \leq n$ then, by Corollary 5.12 we would have $a <^-_l b'$ which is a contradiction. $\Box$

Corollary 5.14 *Any morphism of positive face structures is one-to-one. Moreover any automorphism of positive face structures is an identity.*

Proof. By Corollary 5.12, the (strict, linear in each dimension) order $<^S$ is defined internally using relations $<^-$ and $<^+$ that are preserved by any morphism. Hence $<^S$ must be preserved by any morphism, as well. From this observation the Corollary follows. $\Box$

Lemma 5.15 *Let S be a positive face structure, $a, b \in S_n$. Then*

1. *if $\iota(a) \cap \iota(b) \neq \emptyset$ then $a \perp^+ b$;*
2. *if $\emptyset \neq \iota(a) \subseteq \iota(b) \neq \iota(a)$ then $a <^+ b$;*
3. *if $a \perp^- b$ then $\iota(a) \cap \iota(b) = \emptyset$.*

Proof. 2. is an easy consequence of 1. and Lemma 5.3. 3. is an easy consequence of 1. and Disjointness. We shall show 1.

Assume that $u \in \iota(a) \cap \iota(b)$. Thus there are $x, y \in \delta(a)$ and $x', y' \in \delta(b)$ such that $\gamma(x) = \gamma(x') = u \in \delta(y) \cap \delta(y')$. If $x = x'$ then by Local linearity $a \perp^+ b$, as required. So assume that $x \neq x'$. Again by Local linearity $x \perp^+ x'$, say $x' \perp^+ x$. Thus there is a $T - \gamma(T)$-path $x', a_1, \ldots, a_k, x$. As, for $i = 1, \ldots, k$, $\gamma\gamma(a_i) = u$ and $\gamma\gamma(b) \notin \iota(b) \ni u$, we have that $\gamma(a_i) \neq \gamma(b)$ and $a_i \neq b$. Once again by Local linearity $a_0 \perp^+ b$ and by Path Lemma $a_i < b$, for $i = 1, \ldots, k$ with $\gamma(a_k) \neq \gamma(b)$. As $\gamma(a_k) = x \in \delta(a)$, again by Path Lemma $a <^+ b$, as well. $\Box$

Proposition 5.16 *Let S be a positive face structure, $a, b \in S_k$, $\alpha \in S_{k+1}$, so that α is a $<^+$-minimal element in S_{k+1}, and $a \in \delta(\alpha)$, $b = \gamma(\alpha)$. Then b is the $<^+$-successor of a.*

Proof. Assume that α is a $<^+$-minimal element in S_{k+1}. Suppose that there is $c \in S_k$ such that $a <^+ c <^+ b$. Thus we have an upper path

$$a, \beta_1, \ldots, \beta_i, c, \beta_{i+1}, \ldots, \beta_l, b.$$

Hence $\beta_1 <^- \beta_l$. Moreover $a \in \delta(\beta_1) \cap \delta(\alpha)$ and $\gamma(\beta_l) = b = \gamma(\alpha)$. Thus both β_1 and β_l are $<^+$-comparable with α. Since α is $<^+$-minimal we have $\alpha <^+ \beta_1, \beta_l$. By Lemma 5.13, $\beta_1 \perp^+ \beta_l$. But then we have $(\beta_1, \beta_l) \in \perp^+ \cap \perp^- \neq \emptyset$, contradicting strictness. $\Box$

Proposition 5.17 *Let T be a positive face structure and $X \subseteq T$ a subhypergraph of T. Then X is a positive face structure iff the relation $<^{X,+}$ is the restriction of $<^{T,+}$ to X.*

Proof. Assume that X is a subhypergraph of a positive face structure T. Then X satisfies axioms of globularity, disjointness, and strictness of the relations $<^{X_k,+}$ for $k > 0$.

Clearly, if $<^{X_k,+} = <^{T_k,+} \cap (X_k)^2$ then the relation $<^{X_0,+}$ is linear, the relations $<^{X_k,+}$, for $k > 0$, satisfy pencil linearity, i.e. X is a positive face structure.

Now we assume that the subhypergraph X of positive face structure T is a positive face structure. We shall show that for $k \in \omega$, $a, b \in X_k$ we have $a <^{X_k,+} b$ iff $a <^{T_k,+} b$. Since X is a subhypergraph $a <^{X_k,+} b$ implies $a <^{T_k,+} b$. Thus it is enough to show that if $a <^{T_k,+} b$ then $a \perp^{X_k,+} b$. We shall prove this by induction on k. For $k = 0$ it is obvious, since $<^{X_0,+}$ is linear. So assume that for faces $x, y \in X_l$, with $l < k$ we already know that $x <^{X_l,+} y$ iff $x <^{T_l,+} y$. Fix $a, b \in T_k$ such that $a <^{T_k,+} b$. Then by Lemma 5.9.2 $\gamma(a) \leq^{T_{k-1},+} \gamma(b)$ and hence by inductive hypothesis $\gamma(a) \leq^{X_{k-1},+} \gamma(b)$. Thus we have an upper X-path $a = a_r, \gamma(a), a_{r-1} \ldots, a_1, \gamma(b)$, with $r \geq 1$. As $a <^{T_k,+} b$, by Path Lemma $a_i <^{T_k,+} b$ for $i = 1, \ldots, r$. Again by induction on r we shall show that $a_i <^{X_k,+} b$, for $i = 1, \ldots, r$. As $\gamma(a_1) = \gamma(b)$ by pencil linearity we have $a_1 \perp^{X_k,+} b$. So $a_1 <^{X_k,+} b$. Suppose that $a_i <^{X_k,+} b$, for $i \leq l < r$. Let $a_l, \alpha_s, \ldots, \alpha_1, b$ be an upper T-path. As $a_{l+1} <^{T_k,+} b$, we cannot have $\gamma(a_{l+1}) \in \delta(b)$. Therefore, for some $j \geq 1$, $\gamma(a_{l+1}) \in \iota(\alpha_j)$. So by Lemma 5.5, $a_{l+1} <^{X_k,+} \gamma(\alpha_j) \leq^{X_k,+} b$, as required. $\Box$

Lemma 5.18 *Let T be a positive face structure, $a, b, \alpha \in T$. If $a \in \delta(\alpha)$ and $a <^+ b <^+ \gamma(\alpha)$ then $b \in \iota(T)$.*

Proof. Assume that $a, b, \alpha \in T$ are as in the assumption of the Lemma. Thus we have an upper path $a, \alpha_0, \ldots, \alpha_r, b$. As $a \in \delta(\alpha) \cap \delta(\alpha_0)$, by pencil linearity we have $\alpha \perp^+ \alpha_0$. If $\alpha <^+ \alpha_0 <^- \alpha_r$ then $\gamma(\alpha) \leq^+ \gamma(\alpha_r) = b$ contradicting our assumption. Thus $\alpha_0 <^+ \alpha$. Then by Path Lemma, since $b = \gamma(\alpha_r) <^+ \gamma(\alpha)$, we have $\alpha_r <^+ \alpha$ and $b \in \iota(T)$, as required. $\Box$

Some equations

Proposition 5.19 *Let S be a positive face structure $0 < k \in \omega$. Then*

1. $\iota(S_{k+1}) = \iota(S_{k+1} - \delta(S_{k+2}))$;
2. $\delta(S_k) = \delta(S_k - \gamma(S_{k+1}))$;

3. $\gamma(S_k) = \gamma(S_k - \gamma(S_{k+1}))$;

4. $\delta(S_k) = \delta(S_k - \iota(S_{k+2}))$;

5. $\delta(S_k) = \delta(S_k - \delta(S_{k+1})) \cup \iota(S_{k+1})$.

Proof. In all the above equations the inclusion $\supseteq$ is obvious. So in each case we need to check the inclusion $\subseteq$ only.

Ad 1. Let $s \in \iota(S_{k+1})$, i.e. there is $a \in S_{k+1}$ such that $s \in \iota(a)$. By strictness, there is $b \in S_{k+1}$ such that $a <^+ b$ and $b \not\in \delta(S_{k+1})$. By Lemma 5.3, we have

$$s \in \iota(a) \subseteq \iota(b) \subseteq \iota(S_{k+1} - \delta(S_{k+2}))$$

as required.

Ad 2. Let $x \in \delta(S_k)$. Let $a \in S_k$ be the $<^+$-minimal element in S_k such that $x \in \delta(a)$. We shall show that $a \in S_k - \gamma(S_{k+1})$. Suppose contrary that there is an $\alpha \in S_{k+1}$ such that $a = \gamma(\alpha)$. The by globularity

$$x \in \delta(a) = \delta\gamma(\alpha) = \delta\delta(\alpha) - \gamma\delta(\alpha).$$

So there is $b \in \delta(\alpha)$ such that $x \in \delta(b)$. As $b <^+ a$ this contradicts the minimality of a.

Ad 3. This is similar to the previous one but simpler.

Ad 4. Since $\iota(S_{k+2}) \subseteq \gamma(S_{k+1})$ 4. follows from 2.

Ad 5. Let $x \in \delta(S_k)$. Let $a \in S_k$ be the $<^+$-maximal element in S_k such that $x \in \delta(a)$. If $a \not\in \delta(S_{k+1})$ then $x \in \delta(S_k - \delta(S_{k+1}))$ as required. So assume that $a \in \delta(S_{k+1})$, i.e. there is $\alpha \in S_{k+1}$ such that $a \in \delta(\alpha)$. Thus $x \in \delta\delta(\alpha)$. As $a <^+ \gamma(\alpha)$, by choice of a we have $x \not\in \delta\gamma(\alpha)$ $(= \delta\delta(\alpha) - \gamma\delta(\alpha))$. So $x \in \gamma\delta(\alpha)$ and hence $x \in \iota(\alpha) \subseteq \iota(S_{k+1})$, as required. $\Box$

6 The ω-categories generated by the positive face structures

Let T^* (T^*_n) be the set of all face substructures of the face structure T (of dimension at most n). We introduce operations

$$\mathbf{d}^{(n)}, \mathbf{c}^{(n)} : T^* \longrightarrow T^*_n$$

of the n-th *domain* and the n-th *codomain*. For S in $(T^*)_{\geq n}$ the faces n-th *domain* $\mathbf{d}^{(n)}S$ are:

1. $(\mathbf{d}^{(n)}S)_k = \emptyset$, for $k > n$,

2. $(\mathbf{d}^{(n)}S)_n = S_n - \gamma(S_{n+1})$,

3. $(\mathbf{d}^{(n)}S)_k = S_k$, for $k < n$.

and faces n-th *codomain* $\mathbf{c}^{(n)}S$ are:

1. $(\mathbf{c}^{(n)}S)_k = \emptyset$, for $k > n$,

2. $(\mathbf{c}^{(n)}S)_n = S_n - \delta(S_{n+1})$,

3. $(\mathbf{c}^{(n)}S)_{n-1} = S_{n-1} - \iota(S_{n+1})$,

4. $(\mathbf{c}^{(n)}S)_k = S_k$, for $k < n-1$.

If $k \in \omega$, $dim(S) = k+1$, we write $\mathbf{d}S$ for $\mathbf{d}^{(k)}(S)$, and $\mathbf{c}S$ for $\mathbf{c}^{(k)}(S)$.
We have

Lemma 6.1 *Let S be a positive face structure. Then*

1. *if $dim(S) \geq k$ then $\mathbf{d}^{(k)}(S)$, $\mathbf{c}^{(k)}(S)$ are positive face structures of dimension k;*
2. *if $dim(S) > k$ then $\mathbf{dd}^{(k+1)}(S) = \mathbf{d}^{(k)}(S)$, $\mathbf{dd}^{(k+1)}(S) = \mathbf{d}^{(k)}(S)$;*
3. *if $dim(S) > 1$ then $\mathbf{dd}S = \mathbf{dc}S$, $\mathbf{cd}S = \mathbf{cc}S$.*
4. *For any $k \in \omega$ and $\alpha \in S_k$, the least sub-hypergraph of S containing the face α is again a positive face structure of dimension k; it is denoted by $[\alpha]$. Moreover, if $k > 0$, then*
$$\mathbf{c}[\alpha] = [\gamma(\alpha)], \qquad \mathbf{d}[\alpha] = [\delta(\alpha)]$$
where $[\delta(\alpha)]$ is the least sub-hypergraph of S containing the set of face $\delta(\alpha)$.

Proof. Ad 1. Clearly, $\mathbf{d}^{(k)}S$ is a sub-hypergraph S and $\mathbf{c}^{(k)}S$ is a sub-hypergraph S by Corollary 5.6. Any sub-hypergraph T of a positive face structure S satisfies the conditions of globularity, strictness (possibly without $<^{T_0,+}$ being linear), and disjointness.

By Lemma 5.4, for $a, b \in \mathbf{d}^{(k)}S_l$ we have $a <^{S_l,+} b$ iff $a <^{\mathbf{d}^{(k)}S_l,+} b$. Moreover, by Lemma 5.8, for $a, b \in \mathbf{c}^{(k)}S_l$ we have $a <^{S_l,+} b$ iff $a <^{\mathbf{c}^{(k)}S_l,+} b$. Hence by Lemma 5.17 both $\mathbf{d}^{(k)}S$ and $\mathbf{c}^{(k)}S$ are positive face structures.

Ad 2. Fix a positive face structure S and $k \in \omega$ such that $dim(S) > k$. Then the faces of $\mathbf{c}^{(k+1)}(S)$, $\mathbf{cc}^{(k+1)}(S)$, and $\mathbf{c}^{(k)}(S)$ are as in the table

dim	$\mathbf{c}^{(k+1)}(S)$	$\mathbf{cc}^{(k+1)}(S)$	$\mathbf{c}^{(k)}(S)$
$k+1$	$S_{k+1} - \delta(S_{k+2})$	$\emptyset$	$\emptyset$
k	$S_k - \iota(S_{k+2})$	$(S_k - \iota(S_{k+2})) - \delta(S_{k+1} - \delta(S_{k+2}))$	$S_k - \delta(S_{k+1})$
$k-1$	S_{k-1}	$S_{k-1} - \iota(S_{k+1} - \delta(S_{k+2}))$	$S_{k-1} - \iota(S_{k+1})$
l	S_l	S_l	S_l

where $l < k-1$. Moreover the faces of $\mathbf{d}^{(k+1)}(S)$, $\mathbf{dd}^{(k+1)}(S)$, and $\mathbf{d}^{(k)}(S)$ are as in the table

dim	$\mathbf{d}^{(k+1)}(S)$	$\mathbf{dd}^{(k+1)}(S)$	$\mathbf{d}^{(k)}(S)$
$k+1$	$S_{k+1} - \gamma(S_{k+2})$	$\emptyset$	$\emptyset$
k	S_k	$S_k - \gamma(S_{k+1} - \gamma(S_{k+2}))$	$S_k - \gamma(S_{k+1})$
l	S_l	S_l	S_l

where $l < k$. Thus the equalities in question all follow from Lemma 5.19.

Ad 3. Let $dim(S) = n > 1$. Note that both $(\mathbf{dd}S)_{n-2}$ and $(\mathbf{dc}S)_{n-2}$ are the sets of all $<^+$-minimal elements in S_{n-2}, i.e. they are equal and the equation $\mathbf{dd}S = \mathbf{dc}S$ holds.

To see that $\mathbf{cd}S = \mathbf{cc}S$ holds, note first that both $(\mathbf{cd}S)_{n-2}$ and $(\mathbf{cc}S)_{n-2}$ are the sets of all $<^+$-maximal elements in S_{n-2}. Moreover

$$(\mathbf{cd}S)_{n-3} = S_{n-3} - \iota(S_{n-1} - \gamma(S_n)),$$

$$(\mathbf{cc}S)_{n-3} = S_{n-3} - \iota(S_{n-1} - \delta(S_n)).$$

Now the equality $\mathbf{cd}S = \mathbf{cc}S$ follows from the following equalities

$$\iota(S_{n-1} - \gamma(S_n)) = \iota(S_{n-1}) = \iota(S_{n-1} - \delta(S_n)).$$

Both equalities follow from Lemma 5.3. We shall show the first equality only.

Suppose contrary, that there is $x \in \iota(S_{n-1})$ such that $x \notin \iota(S_{n-1} - \gamma(S_n))$. Let $a \in S_{n-1}$ be a $<^+$-minimal face such that $x \in \iota(a)$. Since $x \notin \iota(S_{n-1} - \gamma(S_n))$, there is $\alpha \in S_n$ such that $a = \gamma(\alpha)$. By Lemma 5.3 we have

$$\iota\delta(\alpha) = \iota\gamma(\alpha) = \iota(a).$$

Therefore, there is $a' \in \delta(\alpha)$ such that $x \in \iota(a')$. Clearly $a' \lhd^+ a$, and hence a is not $<^+$-minimal contrary to the supposition. This ends the proof of the first equality above.

Ad 4. Fix $\alpha \in S_k$. We need to show that $[\alpha]$ is a positive face structure. The globularity, strictness (except for linearity of $<^{[\alpha]_0,+}$), and disjointness are clear.

The linearity of $<^{[\alpha]_0,+}$. If $k \leq 2$ it is obvious. Put $a = \gamma^{(k+2)}(\alpha)$. Using Corollary 4.3, we have

$$[\alpha]_0 = \delta^{(k)}(\alpha) \cup \gamma^{(k)}(\alpha) =$$
$$= \delta\delta(\gamma^{(n-2)}(\alpha)) \cup \gamma\gamma(\gamma^{(n-2)}(\alpha)) = \delta\delta(a) \cup \gamma\gamma(a)$$

Thus it is enough to assume that $k = 2$. But in this case, as we mentioned, the linearity of $<^{[\alpha]_0,+}$ is obvious.

The γ-linearity of $[\alpha]$. The proof proceeds by induction on $k = dim(\alpha)$. For $k \leq 2$, the γ-linearity is obvious. So assume that $k > 2$ and that for $l < k$ and $a \in S_l$.

First we shall show that $\mathbf{c}([\alpha]) = [\gamma(\alpha)]$. We have

$$\mathbf{c}([\alpha])_{k-1} = (\gamma(\alpha) \cup \delta(\alpha)) - \delta(\alpha) = \gamma(\alpha) = [\gamma(\alpha)]_{k-1}$$

$$\mathbf{c}([\alpha])_{k-2} = (\gamma\gamma(\alpha) \cup \delta\delta(\alpha)) - \iota(\alpha) = \delta\gamma(\alpha) \cup \gamma\gamma(\alpha) = [\gamma(\alpha)]_{k-2}$$

and for $l < k-2$

$$\mathbf{c}([\alpha])_l = \gamma^{(l)}(\alpha) \cup \delta\gamma^{(l)}(\alpha)) = \gamma^{(l)}(\alpha) \cup \delta^{(l)}\gamma(\alpha) = [\gamma(\alpha)]_l$$

Note that the definition of $\mathbf{c}(H)$ make sense for any positive hypergraph H and in the above argument we haven't use the fact (which we don't know yet) that $[\alpha]$ is a positive face structure.

Thus, for $l < k-2$, $[\alpha]_l = [\gamma(\alpha)]_l$. By induction, $[\gamma(\alpha)]$ is a positive face structure, and hence $[\alpha]_l$ is γ-linear for $l < k-2$. Clearly $[\alpha]_l$ is γ-linear for $l = k-1, k$. Thus it remains to show the γ-linearity of $k-2$-cells in $[\alpha]$.

Fix $t \in [\alpha]_{k-3}$, and let

$$\Gamma_t = \{x \in [\alpha]_{k-2} : \gamma(x) = t\}.$$

We need to show that Γ_t is linearly ordered by $<^+$. We can assume that $t \in \gamma([\alpha]_{n-2}) = \gamma\delta\delta(\alpha) = \gamma\delta\gamma(\alpha)$ (otherwise $\Gamma_t = \emptyset$ is clearly linearly ordered by $<^+$). By Proposition 5.1 there is a unique $x_t \in \delta\gamma(\alpha)$ such that $\gamma(x_t) = t$. From Lemma 5.2.2 we get easily the following Claim.

Claim 1. For every $x \in \Gamma_t$ there is a unique upper $\delta(\alpha)$-path from x_t to x.

Now fix $x, x' \in \Gamma_t$. By the Claim 1, we have the unique upper $\delta(\alpha)$-path

$$x_t, a_0, \ldots, a_l, x, \quad x_t, a'_0, \ldots, a'_l, x'.$$

Suppose $l \leq l'$. By Proposition 5.1, for $i \leq l$, $a_i = a'_i$. Hence either $l = l'$ and $x = x'$ or $l < l'$ and

$$x, a_{l+1}, \ldots, a_{l'}, x'$$

is a $\delta(\alpha)$-upper path. Hence either $x = x'$ or $x \perp^+ x'$ and $[\alpha]_{k-2}$ satisfy the γ-linearity, as required.

The proof is of the δ-linearity of $[\alpha]$ is very similar to the one above. For the same reasons the only non-trivial thing to check is the condition for $(k-2)$-faces. We pick $t \in \delta\delta(\alpha)$ and consider the set

$$\Delta_t = \{x \in [\alpha]_{k-2} : t \in \delta(x)\}.$$

Then we have a unique $y_t \in \delta\gamma(\alpha)$ such that $t \in \delta(y_t)$. From Lemma 5.2.3 we get the following Claim.

Claim 2. For every $y \in \Delta_t$ there is a unique upper $\delta(\alpha)$-path from y_t to y.

The δ-linearity of the $(k-2)$-faces in $[\alpha]$ can be proven from Claim 2 similarly as the γ-linearity from Claim 1.

It remains to verify the equalities

$$\mathbf{c}[\alpha] = [\gamma(\alpha)], \qquad \mathbf{d}[\alpha] = [\delta(\alpha)].$$

The first one we already checked on the way. To see that the second equality also hold we calculate

$$\mathbf{d}[\alpha]_{k-1} = (\gamma(\alpha) \cup \delta(\alpha)) - \gamma(\alpha) = \delta(\alpha) = [\delta(\alpha)]_{k-1}$$

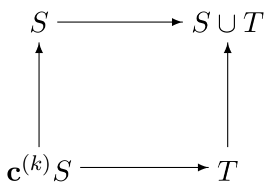
$$\mathbf{d}[\alpha]_{k-2} = (\gamma\gamma(\alpha) \cup \delta\delta(\alpha)) = \gamma\delta(\alpha) \cup \delta\delta(\alpha) = [\delta(\alpha)]_{k-2}$$

and for $l < k-2$

$$\mathbf{d}[\alpha]_l = \gamma^{(l)}(\alpha) \cup \delta^{(l)}(\alpha)) = \gamma^{(l)}\delta(\alpha) \cup \delta^{(l)}(\alpha) = [\delta(\alpha)]_l$$

So the second equality holds as well. $\Box$

Lemma 6.2 *Let S and T be positive face structures such that $\mathbf{c}^{(k)}S \subseteq \mathbf{d}^{(k)}T$. Then the pushout $S+_kT$ in $\mathbf{Fs}^{+/1}$ of S and T over $\mathbf{c}^{(k)}S$ exists. Moreover, if $\mathbf{c}^{(k)}S = S\cap T$ then the diagram of inclusions in $\mathbf{Fs}^{+/1}$*



is the pushout.

Proof. Assume that $\mathbf{c}^{(k)}S = S \cap T \subseteq \mathbf{d}^{(k)}T$. Let $S \cup T$ be the obvious sum of S and T as positive hypergraphs. The fact that $S\cup T$ is a pushout in $\mathbf{Hg}^{+1}$ is obvious. Thus the only thing we need to verify that $S \cup T$ is a positive face structure.

First we write in details the condition $\mathbf{c}^{(k)}S = S \cap T \subseteq \mathbf{d}^{(k)}T$:

1. $S_l \cap T_l = \emptyset$, for $l > k$,
2. $S_k - \delta(S_{k+1}) \subseteq T_k - \gamma(T_{k+1})$,
3. $S_{k-1} - \iota(S_{k+1}) \subseteq T_{k-1}$,
4. $S_l \subseteq T_l$, for $l < k-1$.

Now we describe the orders $<^{+}$ in $S \cup T$:

$$<^{(S\cup T)_l,+} = \begin{cases} <^{S_l,+} + <^{T_l,+} & \text{for } l > k, \\ <^{S_l,+} +_{(S_k-\delta(S_{k+1}))} <^{T_l,+} & \text{for } l = k, \\ <^{S_l,+} +_{(S_{k-1}-\iota(S_{k+1}))} <^{T_l,+} & \text{for } l = k-1, \\ <^{T_l,+} & \text{for } l <= k-1. \end{cases}$$

We shall comment on these formulas. For $l > k$ the formulas say that the order $<^{+}$ in $(S \cup T)_l$ is the disjoint sum of the orders in S_l and T_l. This is obvious.

For $l < k-1$ the order $<^{+}$ in $(S\cap T)_l$ is just the order $<^{T_l,+}$. The only case that requires an explanation is $l = k-2$. So suppose that $a, b \in T_{k-2}$ and $a <^{(S\cup T)_{k-2},+} b$. So we have an upper path

$$a, \alpha_1, \ldots, \alpha_m, b$$

such that $\alpha_i \in (S \cup T)_{k-1} = \iota(S_{k+1}) \cup T_{k-1}$. By Lemma 5.4, we can assume that if $\alpha_i \in S_{k-1}$ then $\alpha_i \notin \gamma(S_k)$. But then $\alpha_i \notin \iota(S_{k+1})$. So in fact $\alpha_i \in T_{k-1}$, as required.

The most involved are the formulas for $<^{(S\cap T)_l,+}$ for $l = k$ and $l = k-1$. In both cases the comparison in $S \cup T$ involves orders both from S and T. In the former case we have that, for $a, b \in (S \cup T)_k$, we have

$$a <^{(S\cup T)_k,+} b \text{ iff}$$

$$\begin{cases} \text{either } a, b \in T_k & \text{and } a <^{T_k,+} b, \\ \text{or } a, b \in S_k & \text{and } a <^{S_k,+} b, \\ \text{or } a \in \delta(S_{k+1}),\ b \in T_k & \text{and } \exists_{a'\in S_k-\delta(S_{k+1})} a <^{S_k,+} a' \text{ and } a' \leq^{T_k,+} b. \end{cases}$$

The orders $<^{S_k,+}$ and $\leq^{T_k,+}$ are glued together along the set $S_k - \delta(S_{k+1})$ which is the set of $<^{S_k,+}$-maximal elements in S_k and at the same time it is contained in the set of $<^{T_k,+}$-minimal elements $T_k - \gamma(T_{k+1})$. This is obvious when we realize that $\delta(S_{k+1}) \cap \gamma(T_{k+1}) = \emptyset$.

In the later case we have for $x, y \in (S \cup T)_{k-1}$ we have

$$x <^{(S\cup T)_{k-1},+} y \text{ iff}$$

$$\begin{cases} \text{either } x, y \in S_{k-1} & \text{and } x <^{S_{k-1},+} y, \\ \text{or } x, y \in T_{k-1} & \text{and } x <^{T_{k-1},+} y, \\ \text{or } x \in \iota(S_{k+1}),\ y \in T_k & \text{and } \exists_{x'\in S_{k-1}-\iota(S_{k+1})} x <^{S_k,+} x' \text{ and } x' \leq^{T_k,+} y, \\ \text{or } x \in T_k,\ y \in \iota(S_{k+1}) & \text{and } \exists_{x'\in S_{k-1}-\iota(S_{k+1})} x <^{T_k,+} x' \text{ and } x' \leq^{S_k,+} y. \end{cases}$$

The order $<^{S_{k-1},+}$ is 'plugged into' the order $\leq^{T_{k-1},+}$ along the set $S_k - \iota(S_{k+1})$.

To show that these formulas hold true we argue by cases. Assume that $x, y \in (S \cup T)_{k-1}$ and that $x <^{(S\cup T)_{k-1},+} y$ i.e. there is an upper path

$$x, a_1, \ldots, a_m, y$$

with $a_i \in (S \cup T)_k$, for $i = 1, \ldots, m$.

First suppose that $x, y \in S_{k-1}$ and that the set $\{a_i\}_i \not\subseteq S_k$. Let $a_{i_0}, a_{i_0+1}, \ldots, a_{i_1}$ be a maximal subsequence of consecutive elements of the path $a_1, \ldots, a_m$ such that $\{a_i\}_{i_0\leq i\leq i_1} \subseteq T_k$. Thus it is an upper path in T_k from $\bar{x}$ to $\bar{y} = \gamma(a_{i_1})$, where

$$\bar{x} = \begin{cases} x & \text{if } i_0 = 1, \\ \gamma(a_{i_0-1}) & \text{otherwise.} \end{cases}$$

Note that from maximality of the path $a_{i_0}, \ldots, a_{i_1}$ follows that both $\bar{x}, \bar{y} \in S_{k-1} - \iota(S_{k+1})$. As we have $\bar{x} <^{T_{k-1},+} \bar{y}$ from Corollary 5.11 we have $\bar{x} \not<^{T_l,-} \bar{y}$, for all

$l < k-1$. Clearly $\perp^{S_l,-} \subseteq \perp^{T_l,-}$. Thus $\bar{x} \not\perp^{S_l,-} \bar{y}$, for all $l < k-1$, as well. But then again by Corollary 5.11 we have that $\bar{x} \perp^{S_{k-1},+} \bar{y}$. If we were to have $\bar{y} <^{S_{k-1},+} \bar{x}$ then, as $\bar{x}, \bar{y} \in S_{k-1} - \iota(S_{k+1})$, we would have $\bar{y} <^{T_{k-1},+} \bar{x}$. But this would contradict the strictness of $<^{T_{k-1},+}$. So we must have $\bar{x} <^{S_{k-1},+} \bar{y}$. In this way we can replace the upper path $a_1, \ldots, a_m$ in $(S \cup T)_k$ from x to y by an upper path from x to y in S_k.

Next, suppose that $x, y \in T_{k-1}$ and that the set $\{a_i\}_i \not\subseteq T_k$. Let $a_{i_0}, a_{i_0+1}, \ldots, a_{i_1}$ be a maximal subsequence of consecutive elements of the path $a_1, \ldots, a_m$ such that $\{a_i\}_{i_0 \le i \le i_1} \subseteq S_k$. Thus it is an upper path in S_k from $\bar{x}$ to $\bar{y} = \gamma(a_{i_1})$, where

$$\bar{x} = \begin{cases} x & \text{if } i_0 = 1, \\ \gamma(a_{i_0-1}) & \text{otherwise.} \end{cases}$$

Note that from maximality of the sequence $a_{i_0}, \ldots, a_{i_1}$ follows that both $\bar{x}, \bar{y} \in S_{k-1} - \iota(S_{k+1}) \subseteq T_{k-1}$. Thus by Lemma 5.8 there is an upper path from $\bar{x}$ to $\bar{y}$ in $S_{k-1} - \delta(S_k) \subseteq T_{k-1}$. In this way we can replace the upper path $a_1, \ldots, a_m$ in $(S \cup T)_k$ from x to y by an upper path from x to y in T_k.

Thus we have justified the first two cases of the above formula. The following two cases are easy consequences these two. This end the description of the orders in $S \cup T$.

From these descriptions follows immediately that $<^{(S \cup T),+}$ is strict for all l. It remains to show the pencil linearity. Both γ- and δ-linearity of l-cells, for $l < k-1$ or $l > k$, are obvious.

To see the γ-linearity of k-cells assume $a \in S_k$ and $b \in T_k$, such that $\gamma(a) = \gamma(b)$. Let $\bar{a} \in S_k$ be the $<^{S_k,+}$-maximal k-cells, such that $\gamma(a) = \gamma(\bar{a})$. Then $\bar{a} \in \mathbf{c}^{(k)}(S)_k \subseteq \mathbf{d}^{(k)}(T)_k$. So $\bar{a} \in T_k$ is $<^{T_k,+}$-minimal k-cells, such that $\gamma(\bar{a}) = \gamma(b)$. Thus

$$a \le^{S_k,+} \bar{a} \le^{T_k,+} b.$$

Thus the γ-linearity of k-cells holds. The proof of δ-linearity of k-cells is similar.

Finally, we need to establish the γ- and δ- linearity of $(k-1)$-cells in $S \cup T$.

In order to prove the γ-linearity, let $x \in \iota(S_{k+1})$ and $y \in T_{k-1}$ such that $\gamma(x) = \gamma(y)$. We need to show that $x \perp^{(S \cup T)_{k-1},+} y$.

Let $\alpha_0 \in S_{k+1}$ such that $x \in \iota(\alpha_0)$, $a \in \delta(\alpha_0)$ such that $x = \gamma(a)$ and let $\alpha_0, \ldots, \alpha_l$ be a lower path in S_{k+1} such that $\gamma(\alpha_l) \in T_k$. Since $x \in \iota(\alpha_0)$, then $x \in \gamma\delta(\alpha_0)$ and, by Lemma 4.2

$$\gamma(x) \in \gamma\gamma\delta(\alpha_0) \subseteq \iota\gamma(\alpha_0).$$

As $\gamma(\alpha_0) \le^+ \gamma(\alpha_l)$, by Lemma 5.3, we have $\gamma(x) \in \iota\gamma(\alpha_l) \cup \gamma\gamma\gamma(\alpha_l)$. Thus we have two cases:

1. $\gamma(x) \in \iota\gamma(\alpha_l)$,
2. $\gamma(x) = \gamma\gamma\gamma(\alpha_l)$.

Case 1: $\gamma(x) \in \iota\gamma(\alpha_l)$.

By Lemma 5.2.2, there is a unique $z \in \delta\gamma(\alpha_l)$ such that $\gamma(z) = \gamma(x)$ and $z <^+ x$. As $\gamma(\alpha_l) \in T_k$, so $z \in T_{k-1}$. If $y <^{T_{k-1},+} z$ then indeed $y <^{(S \cup T)_{k-1},+} z$, as required. By γ-linearity in T_{k-1}, it is enough to show that it is impossible to have $z <^{T_{k-1},+} y$.

Suppose contrary, that there is an upper path $z, b_0, \ldots, b_r, y$ in T. Since $\gamma(\alpha_l)$ is $<^+$-minimal in T (as $\alpha_l \in S$) and $z \in \delta\gamma(\alpha_l) \cap \delta(b_0)$, by δ-linearity in T_k we have $\gamma(\alpha_l) <^+ b_0$. By Lemma 5.3, we have

$$\gamma(x) \in \iota\gamma(\alpha_l) \subseteq \iota(b_0) \subseteq \iota(b_r)$$

But $\gamma(b_r) = y$ so $\gamma\gamma(b_r) = \gamma(y) = \gamma(x)$. In particular $\gamma(x) \notin \iota(b_r)$ and we get a contradiction.

Case 2: $\gamma(x) = \gamma\gamma\gamma(\alpha_l)$.

By Lemma 5.2.2 there is $z \in \delta\gamma(\alpha_l)$ such that $\gamma(x) = \gamma(z)(= \gamma\gamma\gamma(\alpha_l))$, so that we have

$$z <^{S_{k-1},+} x <^{S_{k-1},+} \gamma\gamma(\alpha_l).$$

As $\gamma(\alpha_l) \in T_k$ and it is $<^+$-minimal in T_k, by Proposition 5.16, there is no face $y \in T_{k-1}$ so that

$$z <^{T_{k-1},+} y <^{T_{k-1},+} \gamma\gamma(\alpha_l).$$

So if $y \in T_{k-1}$ and $\gamma(y) = \gamma(x)$ then either

$$y \leq^{T_{k-1},+} z <^{S_{k-1},+} x \quad \text{or} \quad x <^{S_{k-1},+} \gamma\gamma(\alpha_l) \leq^{S_{k-1},+} y.$$

In either case $x \perp^{(S\cup T)_{k-1},+} y$, as required. This ends the proof of γ-linearity of $(k-1)$-faces in $(S \cup T)$.

Finally, we prove the δ-linearity of $(k-1)$-faces in $S \cup T$. Let $x \in \iota(S_{k+1})$ and $y \in T_{k-1}$, $t \in T_{k-2}$ such that $t \in \delta(x) \cap \delta(y)$. We need to show that $x \perp^{(S\cup T)_{k-1},+} y$.

Let $\alpha \in S_{k+1}$ such that $x \in \iota(\alpha_0)$, $a \in \delta(\alpha_0)$ such that $x = \gamma(a)$, and let $\alpha_0, \ldots, \alpha_l$ be a lower path in S_{k+1} such that $\gamma(\alpha_l) \in T_k$. As $x \in \iota(\alpha_0)$, using Lemma 4.2 we have

$$t \in \delta(x) \subseteq \delta\gamma\delta(\alpha_0) \subseteq \delta\gamma\gamma(\alpha_0) \cup \iota\gamma(\alpha_0).$$

As $\gamma(\alpha_0) <^+ \gamma(\alpha_l)$, by Lemma 5.3, we have two cases:

1. $t \in \iota\gamma(\alpha_l)$,
2. $t \in \delta\gamma\gamma(\alpha_l)$.

Case 1: $t \in \iota\gamma(\alpha_l)$.

By Lemma 5.2.3, there is a unique $z \in \delta\gamma(\alpha_l)$ such that $t \in \delta(z)$ and $z <^+ x$. As $\gamma(\alpha_l) \in T_k$, so $z \in T_{k-1}$. If $y <^{T_{k-1},+} z$ then indeed $y <^{(S\cup T)_{k-1},+} z$, as required. By δ-linearity in T_k, it is enough to show that it is impossible to have $z <^{T_{k-1},+} y$.

Suppose contrary, that there is an upper path in T

$$z, b_0, \ldots, b_r, y.$$

Since $\gamma(\alpha_l)$ is $<^+$-minimal in T_k and $z \in \delta\gamma(\alpha_l) \cap \delta(b_0)$, by δ-linearity of k-faces in T we have $\gamma(\alpha_l) <^+ b_0$. By Lemma 5.3, we have

$$t \in \iota\gamma(\alpha_l) \subseteq \iota(b_0) \subseteq \ldots \subseteq \iota(b_r).$$

But $\gamma(b_r) = y$, so $t \in \delta(y) \subseteq \delta\gamma(b_r)$. In particular $t \notin \iota(b_r)$ and we get a contradiction.

Case 2: $t \in \delta\gamma\gamma(\alpha_l)$.

By Lemma 5.2.3 there is $z \in \delta\gamma\gamma(\alpha_l)$ such that $t \in \delta(z)$ and we have

$$z <^{S_{k-1},+} x <^{S_{k-1},+} \gamma\gamma(\alpha_l).$$

As $\gamma(\alpha_l) \in T_k$, and it is $<^+$-minimal face in T_k, by Lemma 5.16, there is no face $y \in T_{k-1}$ such that

$$z <^{T_{k-1},+} y <^{T_{k-1},+} \gamma\gamma(\alpha_l).$$

So if $y \in T_{k-1}$ and $t \in \delta(y)$ then either

$$y \leq^{T_{k-1},+} z <^{S_{k-1},+} x \quad \text{or} \quad x <^{S_{k-1},+} \gamma\gamma(\alpha_l) \leq^{S_{k-1},+} y.$$

In either case $x \perp^{(S\cup T)_{k-1},+} y$, as required. This ends the proof of δ-linearity of $(k-1)$-faces in $(S \cup T)$ and the whole proof that $S \cup T$ is a positive face structure. $\Box$

Let S and T be positive face structures such that $\mathbf{c}^{(k)}S = \mathbf{d}^{(k)}T$. Then the pushout

$$\begin{array}{ccc} S & \longrightarrow & S +_k T \\ \uparrow & & \uparrow \\ \mathbf{c}^{(k)}S & \longrightarrow & T \end{array}$$

is called *special pushouts*

Now we shall describe an ω-category S^* generated by the face structure S.

The set S^*_n of n-cell of S^* is the set of all positive face sub-structures of S of dimension at most n, for $n \in \omega$. The *domain* and *codomain* operations in S^* are restricted operations

$$\mathbf{d}^{(k)}, \mathbf{c}^{(k)} : S^*_n \longrightarrow S^*_k$$

of the k-th *domain* and the k-th *codomain*, for $k \leq n$. The *identity* operation

$$\mathbf{i}^{(n)} : S^*_k \longrightarrow S^*_n$$

is an inclusion and the composition map

$$\mathbf{m}_{n,k,n} : S^*_n \times_{S^*_k} S^*_n \longrightarrow S^*_n$$

is the sum, i.e. if X, Y are sub-face structures of S of dimension at most n such that $\mathbf{c}^{(k)}X = \mathbf{d}^{(k)}Y$ then

$$\mathbf{m}_{n,k,n}(X, Y) = X +_k Y = X \cup Y.$$

Corollary 6.3 *Let S be a weak positive face structure. Then S^* is an ω-category. In fact, we have a functor*

$$(-)^* : \mathbf{Fs}^{+/1} \longrightarrow \omega Cat$$

Proof. The fact that the operation on S^* defined above satisfy the laws of ω-category is obvious. The image $f(X)$ of a sub-face structure X of a positive face structure S under a morphism $f : S \to T$ is a sub-face structure of T. The association $X \mapsto f(X)$ is easily seen to be an ω-functor. $\Box$

Let S be a positive face structure. We have a functor

$$\Sigma^S : \mathbf{pFs}^{+/1} \downarrow S \longrightarrow \mathbf{Fs}^{+/1}$$

such that

$$\Sigma^S(f : B \to S) = B$$

and a cocone

$$\sigma^S : \Sigma^S \longrightarrow S$$

such that

$$\sigma^S_{(f:B\to S)} = f : \Sigma^S(f : B \to S) = B \longrightarrow S$$

We have

Lemma 6.4 *The cocone $\sigma^S : \Sigma^S \stackrel{\cdot}{\longrightarrow} S$ is a colimiting cocone in $\mathbf{Fs}^{+/1}$. Such colimiting cones are called* special colimits*. Any special limit in $\mathbf{Fs}^{+/1}$ can be obtained via some special pushouts and vice versa any special pushout can be obtained from special limits. In particular, a functor from $\mathbf{Fs}^{+/1}$ preserves special limits if and only if it preserves special pushouts.* $\Box$

7 Normal positive face structures

Let S be a k-normal positive face structure S. By $\mathbf{p}_l^S$ we denote the unique element of the set $S_l - \delta(S_{l+1})$, for $l < k$. Moreover, as we shall show below $\mathbf{p}_{k-1} \in \gamma(S_k)$ and hence the set $\{x \in S_k : \gamma(x) = \mathbf{p}_{k-1}\}$ is not empty. We denote by $\mathbf{p}_k$ the $<^+$-maximal element of this set. We shall omit the superscript S if it does not lead to a confusion.

Lemma 7.1 *Let S be a $(k-1)$-principal positive face structure of dimension at least k, $k > 0$. Then*

1. $S_l = \delta^{(l)}(S_k) \cup \gamma^{(l)}(S_k) = \delta^{(l)}(S_k) \cup \{\mathbf{p}_l\}$, *for* $l < k$.
2. $\delta(S_{l+1}) = \delta^{(l)}(S_k)$, *for* $l < k$.
3. $\mathbf{p}_k$ *is* $<^-$*-largest element in* $S_k - \delta(S_{k+1})$.
4. $\gamma(\mathbf{p}_l) = \mathbf{p}_{l-1}$, *for* $0 < l \leq k$.
5. $\delta(\mathbf{p}_l) = \delta(S_l) - \gamma(S_l)$, *for* $0 < l < k$.
6. $S_l = \delta^{(l)}(\mathbf{p}_{k-1}) \cup \gamma^{(l)}(\mathbf{p}_{k-1})$, *for* $l < k-2$.

Proof. Ad 1. If H is a hypergraph of dimension greater than l and $\gamma(H_{l+1}) \subseteq \delta(H_{l+1})$ then there is an infinite lower path in H_{l+1}, i.e. $<^{H_l,+}$ is not strict. Thus, if S is a positive face structure of dimension greater than l, we have $\delta(S_{l+1}) \subsetneq S_l$. A positive face structure is normal iff this difference

$$S_l - \delta(S_{l+1})$$

is minimal possible (i.e. one-element set), for $l < k$. Thus, by the above, we must have

$$S_l = \delta(S_{l+1}) \cup \gamma(S_{l+1}) \tag{1}$$

The first equation of the statement 1. we shall show by the downward induction on l. Suppose that we have $S_{l+1} = \delta^{(l)}(S_k) \cup \gamma^{(l)}(S_k)$ (for $l = k-2$ it is true by the above). Then

$$\begin{aligned} S_l = \delta^{(l)}(S_k) \cup \gamma^{(l)}(S_k) = \\ = \delta(\delta^{(l+1)}(S_k) \cup \gamma^{(l+1)}(S_k)) \cup \gamma(\delta^{(l+1)}(S_k) \cup \gamma^{(l+1)}(S_k)) = \\ = \delta\delta^{(l+1)}(S_k) \cup \delta\gamma^{(l+1)}(S_k) \cup \gamma\delta^{(l+1)}(S_k) \cup \gamma\gamma^{(l+1)}(S_k) = \\ = \delta^{(l)}(S_k) \cup \delta\gamma^{(l)}(S_k) \cup \gamma\delta^{(l)}(S_k) \cup \gamma^{(l)}(S_k) = \\ = \delta^{(l)}(S_k) \cup \gamma^{(l)}(S_k) \end{aligned}$$

where the last equation follows from Corollary 4.3.

The second equation of 1. is obvious for $l = k-1$. So assume that $l < k-1$. We have

$$\begin{aligned} \{\mathbf{p}_l\} = S_l - \delta(S_{l+1}) = \\ = S_l - \delta(\delta^{(l+1)}(S_k) \cup \gamma^{(l+1)}(S_k)) = \\ = S_l - (\delta^{(l)}(S_k) \cup \delta\gamma^{(l+1)}(S_k)) = \\ S_l - \delta^{(l)}(S_k). \end{aligned}$$

Thus

$$S_l = \delta^{(l)}(S_k) \cup \{\mathbf{p}_l\}$$

as required.

Ad 2. Let $l < k$ Then using 1. we have

$$\delta^{(l)}(S_k) \subseteq \delta(S_{l+1}) \subsetneq \delta^{(l)}(S_k) \cup \{\mathbf{p}_l\}$$

Hence

$$\delta^{(l)}(S_k) = \delta(S_{l+1}).$$

Ad 3. First we shall show that $\mathbf{p}_k \in S_k - \delta(S_{k+1})$. Suppose contrary, that there is $\alpha \in S_{k+1}$ such that $\mathbf{p}_k \in \delta(\alpha)$. Then $\gamma(\mathbf{p}_k) \in \gamma\delta(\alpha) = \gamma\gamma(\alpha) \cup \iota(\alpha)$. If $\gamma(\mathbf{p}_k) = \gamma\gamma(\alpha)$ then $\mathbf{p}_k <^+ \gamma(\alpha)$ i.e. $\gamma(\alpha)$ is $<^+$-smaller element than $\mathbf{p}_k$ such that $\gamma(\gamma(\alpha)) = \mathbf{p}_{k-1}$. This contradicts the choice of $\mathbf{p}_k$. If $\gamma(\mathbf{p}_k) = \iota(\alpha)$ then there is $a \in \delta(\alpha)$ such that $\gamma(\mathbf{p}_k) \in \delta(a)$. But this means that $\mathbf{p}_{k-1} = \gamma(\mathbf{p}_k) \in \delta(S_k)$ contradicting the choice of $\mathbf{p}_{k-1} \in S_{k-1} - \delta(S_k)$. This shows that $\mathbf{p}_k \in S_k - \delta(S_{k+1})$.

We need to prove that any maximal lower $(S_k - \delta(S_{k+1}))$-path ends at $\mathbf{p}_k$. By strictness, it is enough to show that if $x \in S_k - \delta(S_{k+1})$ and $x \neq \mathbf{p}_k$ then there is $x' \in S_k - \delta(S_{k+1})$ such that $\gamma(x) \in \delta(x')$. So fix $x \in S_k - \delta(S_{k+1})$. If we were to have $\gamma(x) \in \iota(\beta)$ for some $\beta \in S_{k+1}$, then by Lemma 5.5 we would have $x <^+ \gamma(\beta)$. In particular, $x \in \delta(S_{k+1})$, contrary to the assumption. Therefore $\gamma(x) \in S_{k-1} - \iota(S_{k+1})$. As $x, \mathbf{p}_k \in S_k - \delta(S_{k+1})$, by γ-linearity we have $\gamma(x) \neq \gamma(\mathbf{p}_k) = \mathbf{p}_{k-1}$. Hence by 1. the set

$$\Delta_{\gamma(x)} = \{y \in S_k : \gamma(x) \in \delta(y)\}$$

is not empty. Let x' be the $<^+$-maximal element of this set. It remains to show that $x' \not\in \delta(S_{k+1})$. Suppose contrary, that there is $\alpha \in S_{k+1}$ such that $x' \in \delta(\alpha)$. As $\gamma(x) \not\in \iota(S_{k+1})$ and $\gamma(x) \in \delta(x')$ so $\gamma(x) \not\in \iota(\alpha)$ and $\gamma(x) \neq \gamma\gamma(\alpha)$. Thus $\gamma(x) \in \delta\gamma(\alpha)$. But this means that $x' <^+ \gamma(\alpha)$ and $\gamma(\alpha) \in \Delta_{\gamma(x)}$. This contradicts the choice of x'. This ends the proof of 3.

Ad 4. $\gamma(\mathbf{p}_k) = \mathbf{p}_{k-1}$ by definition. Fix $0 < l < k$. As $S_l = \delta(S_{l+1}) \cup \{\mathbf{p}_l\}$, $\mathbf{p}_l$ is $<^+$-greatest element in S_l. Assume that $\gamma(\mathbf{p}_l) \neq \mathbf{p}_{l-1}$. Thus $\gamma(\mathbf{p}_l) <^+ \mathbf{p}_{l-1}$. Let $x \in S_l$. Then $x \leq \mathbf{p}_l$ and, by Lemma 5.9, $\gamma(x) \leq^+ \gamma(\mathbf{p}_l) <^+ \mathbf{p}_{l-1}$. Thus $\mathbf{p}_{l-1} \not\in \gamma(S_l)$. So $\gamma(S_l) \subseteq \delta(S_l)$. But this is impossible in a positive face structure as we noticed in the proof of 1. This ends 4.

Ad 5. Fix $l < k$. First we shall show that

$$\delta(\mathbf{p}_l) \cap \gamma(S_l) = \emptyset \tag{2}$$

Let $z \in \gamma(S_l)$, i.e. there is $a \in S_l$ such that $\gamma(a) = z$. By 1. $a \leq^+ \mathbf{p}_l$. By Lemma 5.7, there are $x \in \delta(\mathbf{p}_l)$ and $y \in \delta(a)$ such that $x \leq^+ y$. Hence $x <^+ \gamma(a) = z$. By Proposition 5.1, since $x \in \delta(\mathbf{p}_l)$ it follows that $z \not\in \delta(\mathbf{p}_l)$. This shows (2).

By Lemma 5.19, we have

$$\delta(S_l) = \delta(S_l - \delta(S_{l+1})) \cup \iota(S_{l+1}) \tag{3}$$

Since $\delta(\mathbf{p}_l) = \delta(S_l - \delta(S_{l+1}))$ and $\iota(S_{l+1}) \subseteq \gamma(S_l)$ we have by (2)

$$\delta(S_l - \delta(S_{l+1})) \cap \iota(S_{l+1}) = \emptyset \tag{4}$$

Next we shall show that

$$\iota(S_{l+1}) = \gamma(S_l) \cap \delta(S_l) \tag{5}$$

The inclusion $\subseteq$ is obvious. Let $x \in \gamma(S_l) \cap \delta(S_l)$. Hence there are $a, b \in S_l$ such that $\gamma(a) = x \in \delta(b)$. We can assume that a is $<^+$-maximal with this property. As $a <^- b$, neither a nor b is equal to the $<^+$-greatest element $\mathbf{p}_l \in S_l$. Therefore there is $\alpha \in S_{l+1}$ such that $a \in \delta(\alpha)$. If we were to have $x = \gamma(a) = \gamma\gamma(\alpha)$ then $\gamma(\alpha)$

would be a $<^+$-greater element than a with $\gamma(\gamma(\alpha)) = x$. So $\gamma(a) \neq \gamma\gamma(\alpha)$. Clearly, $x \in \gamma\delta(\alpha)$. By globularity, $x \in \delta\delta(\alpha)$, as well. Thus $x \in \iota(\alpha)$, and (5) is shown.

Using (2), (3), (4), and (5) we have

$$\delta(\mathbf{p}_l) = \delta(S_l - \delta(S_{l+1})) =$$

$$= \delta(S_l) - \iota(S_{l+1}) =$$

$$= \delta(S_l) - (\gamma(S_l) \cap \delta(S_l)) =$$

$$= \delta(S_l) - \gamma(S_l)$$

as required.

Ad 6. By 1. and 2. it is enough to show

$$\delta^{(l)}(S_{k-1}) = \delta^{(l)}(\mathbf{p}_{k-1}),$$

for $l < k-2$. The inclusion $\supseteq$ is obvious.

Pick $x \in S_{k-1}$ We have an upper path $x, a_1, \ldots, a_r, \mathbf{p}_{k-1}$. By Corollary 4.3, as $\gamma(a_i) \in \delta(a_{i+1})$, we have

$$\delta^{(l)}(a_i) = \delta^{(l)}\gamma(a_i) \subseteq \delta^{(l)}(\delta(a_{i+1})) = \delta^{(l)}(a_{i+1})$$

for $i = 0, \ldots, r-1$. Then, by transitivity of $\subseteq$ and again Corollary 4.3 we get

$$\delta^{(l)}(x) \subseteq \delta^{(l)}(a_1) \subseteq \delta^{(l)}(a_r) \subseteq \delta^{(l)}(\gamma(a_r)) = \delta^{(l)}(\mathbf{p}_{k-1}).$$

This ends the proof of the inclusion $\subseteq$ and 6. □

Lemma 7.2 *Let S be a positive face structure of dimension at least k. Then*

1. *S is $(k-1)$-principal iff $\mathbf{d}^{(k)}(S)$ is normal iff $\mathbf{c}^{(k-1)}(S)$ is principal,*
2. *if S is normal, so is $\mathbf{d}(S)$,*
3. *if S is principal, so is $\mathbf{c}(S)$.*

Proof. The whole Lemma is an easy consequence of Lemma 5.19. We shall show 1. leaving 2. and 3. for the reader.

First note that all three conditions in 1. imply that, $|S_l - \delta(S_{l+1}| = 1$ for $l < k-2$. In addition to this they say:

1. S is $(k-1)$-principal iff $|S_l - \delta(S_{l+1})| = 1$ for $l = k-2, k-1$.
2. $\mathbf{d}^{(k)}(S)$ is normal iff
   (a) $|S_{k-1} - \delta(S_k - \gamma(S_{k+1}))| = 1$, and
   (b) $|S_{k-2} - \delta(S_{k-1})| = 1$.
3. $\mathbf{c}^{(k-1)}(S)$ is principal iff
   (a) $|(S_{k-1} - \iota(S_{k+1})) - \delta(S_k - \delta(S_{k+1}))| = 1$, and
   (b) $|S_{k-2} - \delta(S_{k-1} - \iota(S_{k+1}))| = 1$.

So the equivalence of these conditions follows directly from Lemma 5.19. □

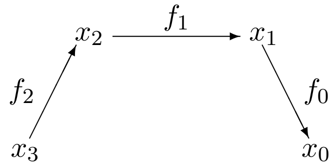
Let N be a n-normal positive face structure. We define a $(n+1)$-hypergraph $N^\bullet$, that contains two additional faces: $\mathbf{p}^{N^\bullet}_{n+1}$ of dimension $n+1$, and $\mathbf{p}^{N^\bullet}_{n}$ of dimension n. We shall drop superscripts if it does not lead to confusions. We also put

$$\delta(\mathbf{p}_{n+1}) = N_n, \quad \gamma(\mathbf{p}_{n+1}) = \mathbf{p}_n,$$

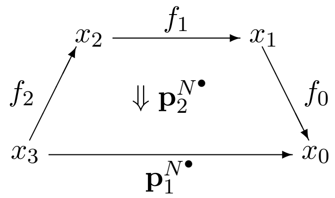
$$\delta(\mathbf{p}_n) = \delta(N_n) - \gamma(N_n), \quad \gamma(\mathbf{p}_n) = \mathbf{p}_{n-1} (= \gamma(N_n) - \delta(N_n)).$$

As N is normal the $\gamma(N_n) - \delta(N_n)$ has one element so $\gamma(\mathbf{p}_n)$ is well defined. This determines $N^\bullet$ uniquely. $N^\bullet$ is called *a simple extension of* N.

Example. For a normal positive face structure N like this



the hyper-graph $N^\bullet$ looks like this



We have

Proposition 7.3 *Let N be a normal positive face structure of dimension n. Then*

1. *$N^\bullet$ is a principal positive face structure of dimension $n+1$.*
2. *We have $\mathbf{d}(N^\bullet) \cong N$, $\mathbf{c}(N^\bullet) \cong (\mathbf{d}N)^\bullet$.*
3. *If N is a principal, then $N \cong (\mathbf{d}N)^\bullet$.*
4. *If T is a positive sub-face structure of $N^\bullet$ then either $T = N^\bullet$ or $T = \mathbf{c}(N^\bullet)$ or $T \subseteq N$.*

Proof. Ad 1. We shall check globularity of the new added cells. The other conditions are simple.

For $\mathbf{p}_{n+1}$, we have:

$$\gamma\gamma(\mathbf{p}_{n+1}) = \gamma(\mathbf{p}_n) =$$
$$= \gamma(N_n) - \delta(N_n) = \gamma\delta(\mathbf{p}_{n+1}) - \delta\delta(\mathbf{p}_{n+1})$$

and

$$\delta\gamma(\mathbf{p}_{n+1}) = \delta(\mathbf{p}_n) =$$
$$= \delta(N_n) - \gamma(N_n) = \delta\delta(\mathbf{p}_{n+1}) - \gamma\delta(\mathbf{p}_{n+1}).$$

So globularity holds for $\mathbf{p}_{n+1}$.

For $\mathbf{p}_n$, using Lemmas 7.1, 5.19 and normality of N, we have:

$$\gamma\gamma(\mathbf{p}_n) = \gamma(\mathbf{p}_{n-1}) = \mathbf{p}_{n-2} =$$

$$= \gamma(N_{n-1}) - \delta(N_{n-1}) =$$

$$= \gamma(N_{n-1} - \gamma(N_n)) - \delta(N_{n-1} - \gamma(N_n)) =$$

$$= \gamma(\delta(N_n) - \gamma(N_n)) - \delta(\delta(N_n) - \gamma(N_n)) =$$

$$= \gamma\delta(\mathbf{p}_n) - \delta\delta(\mathbf{p}_n)$$

and similarly

$$\delta\gamma(\mathbf{p}_n) = \delta(\mathbf{p}_{n-1}) =$$
$$= \delta(N_n) - \gamma(N_n) =$$
$$= \delta(\delta(N_n) - \gamma(N_n)) - \gamma(\delta(N_n) - \gamma(N_n)) =$$
$$= \delta\delta(\mathbf{p}_n) - \gamma\delta(\mathbf{p}_n)$$

So globularity for $\mathbf{p}_n$ holds, as well.

Ad 2. The first isomorphism is obvious.

The faces of $(N^\bullet)$, $\mathbf{c}(N^\bullet)$, $\mathbf{d}N$, and $(\mathbf{d}N)^\bullet$ are as in the tables

dim	$(N^\bullet)$	$\mathbf{c}(N^\bullet)$
$n+1$	$\{\mathbf{p}_{n+1}^{N^\bullet}\}$	$\emptyset$
n	$N_n \cup \{\mathbf{p}_n^{N^\bullet}\}$	$\{\mathbf{p}_n^{N^\bullet}\}$
$n-1$	N_{n-1}	$N_{n-1} - (\gamma(N_n) \cap \delta(N_n))$
$n-2$	N_{n-2}	N_{n-2}

and

dim	$\mathbf{d}N$	$(\mathbf{d}N)^\bullet$
$n+1$	$\emptyset$	$\emptyset$
n	$\emptyset$	$\{\mathbf{p}_n^{(\mathbf{d}N)^\bullet}\}$
$n-1$	$N_{n-1} - \gamma(N_n)$	$(N_{n-1} - \gamma(N_n)) \cup \{\mathbf{p}_{n-1}^{(\mathbf{d}N)^\bullet}\}$
$n-2$	N_{n-2}	N_{n-2}

We define the isomorphism $f : \mathbf{c}(N^\bullet) \longrightarrow (\mathbf{d}N)^\bullet$ as follows

$$f_n(\mathbf{p}_{n+1}^{N^\bullet}) = \mathbf{p}_{n+1}^{(\mathbf{d}N)^\bullet},$$

$$f_{n-1}(x) = \begin{cases} \mathbf{p}_{n-1}^{(\mathbf{d}N)^\bullet} & \text{if } x = \gamma(\mathbf{p}_n^{N^\bullet}), \\ x & \text{otherwise.} \end{cases}$$

and $f_l = 1_{N_l}$ for $l < n-1$. Clearly, all f_i's are bijective. The preservation of the domains and codomains is left for the reader.

3. is left as an exercise.

Ad 4. If $\mathbf{p}_{n+1} \in T_{n+1}$ then $T = N^\bullet$. If $\mathbf{p}_n \not\in T_n$ then $T \subseteq N$.

Suppose that $\mathbf{p}_{n+1} \not\in T_{n+1}$ but $\mathbf{p}_n \in T_n$. Since $N^\bullet = [\mathbf{p}_{n+1}]$, by Lemma 6.1 it is enough to show that $T = [\mathbf{p}_n]$. Clearly $[\mathbf{p}_n] \subseteq T$. As $[\mathbf{p}_n]_l = N_l$, for $l < n-1$ we have $[\mathbf{p}_n]_l = T_l$, for $l < n-1$, as well.

Fix $x \in N_n$. As $x \in \delta(\mathbf{p}_{n+1})$ and $\gamma(\mathbf{p}_{n+1}) = \mathbf{p}_n$, we have $x <^{N^\bullet,+} \mathbf{p}_n$. So by Corollary 5.11 $x \not<_l^{N^\bullet,-} \mathbf{p}_n$, for any $l \leq n$. Thus we cannot have $x \perp_l^{T,-} \mathbf{p}_n$, for any $l \leq n$, as well. As T is a positive face structure, again by Corollary 5.11, $x \not\in T$. Since x was an arbitrary element of N_n, we have $T_n = \{\mathbf{p}_n\} = [\mathbf{p}_n]_n$.

It remains to show that $T_{n-1} = [\mathbf{p}_n]_{n-1}$. Suppose that $x \in N_{n-1} - (\delta(\mathbf{p}_n) \cup \gamma(\mathbf{p}_n))$. Then $x <^{N,+} \gamma(\mathbf{p}_n)$ and hence $x \not<_l^{N^\bullet,-} \gamma(\mathbf{p}_n)$, for $l \leq n$. So x and $\gamma(\mathbf{p}_n)$ cannot be $<_l^{T,-}$ comparable, for $l \leq n$. Since, as we have shown, $N_n \cap T_n = \emptyset$, it follows that x and $\gamma(\mathbf{p}_n)$ cannot be $<^{T,+}$ comparable. So by Lemma 5.11, $x \not\in T_{n-1}$, i.e. $T_{n-1} = \delta(\mathbf{p}_n) \cup \gamma(\mathbf{p}_n) = [\mathbf{p}_n]_{n-1}$. □

8 Decomposition of positive face structures

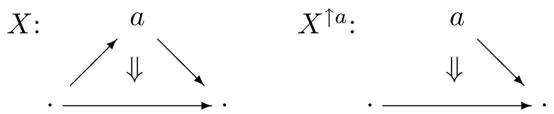
Let T be a positive face structure, $X \subseteq T$ a subhypergraph of T, $k \in \omega$, $a \in (T_k - \iota(T_{k+2}))$. We define two subhypergraphs of T, $X^{\downarrow a}$ and $X^{\uparrow a}$, as follows:

$$X^{\downarrow a}_l = \begin{cases} \{\alpha \in X_l : \gamma^{(k)}(\alpha) \leq^+ a\} & \text{for } l > k, \\ \{b \in X_k : b \leq^+ a \text{ or } \ b \not\in \gamma(X_{k+1})\} & \text{for } l = k \\ X_l & \text{for } l < k. \end{cases}$$

$$X^{\uparrow a}_l = \begin{cases} \{\alpha \in X_l : \gamma^{(k)}(\alpha) \not\leq^+ a\} & \text{for } l > k, \\ \{b \in X_k : b \not<^+ a \text{ or } b \not\in \delta(X_{k+1})\} & \text{for } l = k \\ X_{k-1} - \iota(X^{\downarrow a}_{k+1}) & \text{for } l = k-1 \\ X_l & \text{for } l < k-1. \end{cases}$$

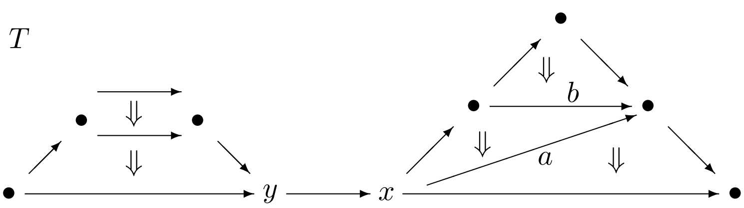
Intuitively, if X is a positive face substructure, $X^{\downarrow a}$ is the least positive face substructure of X that contains faces 'smaller or equal' a and can be k-pre-composed with the 'rest' to get X. $X^{\uparrow a}$ is this 'rest' or in other words it is the largest positive face substructure of X that can be k-post-composed with $X^{\downarrow a}$ to get X (or largest positive face substructure of X that do not contains faces 'smaller' than a).

Examples. If X is a hypergraph $a \in T$ then $X^{\downarrow a}$ is a hypergraph as well. However, this is not the case with $X^{\uparrow a}$, if $a \in \iota(T)$, as we can see below:

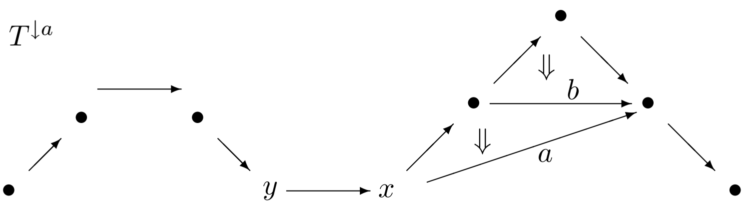


Here $X = T$. The faces in the domain of the 2-dimensional face are not in $X^{\uparrow a}$, i.e. $X^{\uparrow a}$ is not closed under δ.

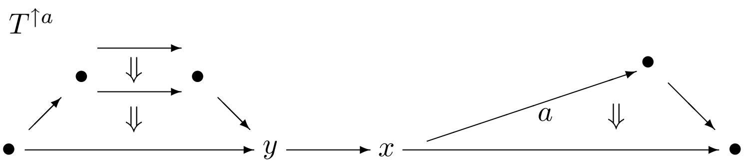
To see some real decompositions let fix a positive face structure T as follows:



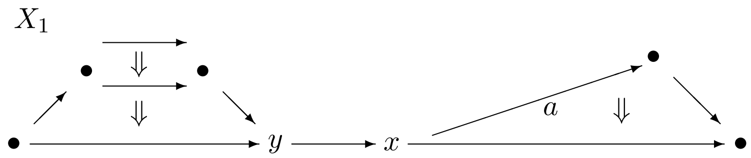
Clearly $x, y, a, b \in T - \iota(T)$. Then



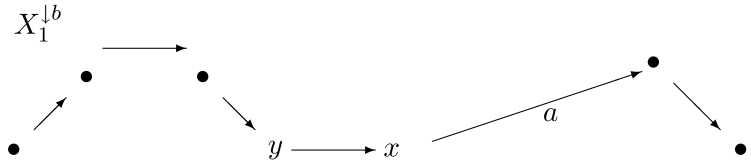
and



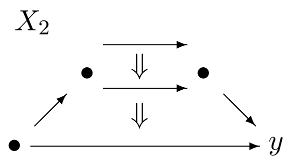
Moreover with



we have $X_1^{\uparrow b} = X_1$ and



i.e. $X_1^{\downarrow b} = \mathbf{d}^{(1)}(X_1)$. For



we have $X_2^{\downarrow x} = X_2$ and $X_2^{\uparrow x} = \{y\}$.

We have

Lemma 8.1 *Let T be a positive face structure, $X \subseteq T$ a subhypergraph of T, $a \in (T - \iota(T))$, $a \in X_k$. Then*

1. *$X^{\downarrow a}$ and $X^{\uparrow a}$ are positive face structures;*
2. *$\mathbf{c}^{(k)}(X^{\downarrow a}) = \mathbf{d}^{(k)}(X^{\uparrow a}) = X^{\downarrow a} \cap X^{\uparrow a}$;*
3. *$\mathbf{d}^{(k)}(X^{\downarrow a}) = \mathbf{d}^{(k)}(X)$, $\mathbf{c}^{(k)}(X^{\uparrow a}) = \mathbf{c}^{(k)}(X)$;*
4. *$X = X^{\downarrow a} +_k X^{\uparrow a} = X^{\downarrow a} \cup X^{\uparrow a}$.*

Proof. Ad 1. The verification that both $X^{\downarrow a}$ and $X^{\uparrow a}$ are closed under γ and δ is routine.

For any k, if $x, y \in X_k^{\downarrow a}$ then $x <^{+,X} y$ iff $x <^{+,X^{\downarrow a}} y$. Similarly, for any k, if $x, y \in X_k^{\uparrow a}$ then $x <^{+,X} y$ iff $x <^{+,X^{\uparrow a}} y$. Thus by Lemma 5.17 both $X^{\downarrow a}$ and $X^{\uparrow a}$ positive face structures.

Ad 2. Let as spell in details both sides of the equation.

$\mathbf{c}^{(k)}(X^{\downarrow a})$ is:

1. $\mathbf{c}^{(k)}(X^{\downarrow a})_l = \emptyset$, for $l > k$;
2. $\mathbf{c}^{(k)}(X^{\downarrow a})_k =$
   $(\{b \in X_k : b \leq^+ a\} \cup (X_k - \gamma(X_{k+1}))) - \delta(\{\alpha \in X_{k+1} : \gamma(\alpha) \leq^+ a\})$;
3. $\mathbf{c}^{(k)}(X^{\downarrow a})_{k-1} = X_{k-1} - \iota(X_{k+1}^{\downarrow a})$;
4. $\mathbf{c}^{(k)}(X^{\downarrow a})_l = X_l$, for $l < k - 1$.

and $\mathbf{d}^{(k)}(X^{\uparrow a})$ is:

1. $\mathbf{d}^{(k)}(X^{\uparrow a})_l = \emptyset$, for $l > k$;
2. $\mathbf{d}^{(k)}(X^{\uparrow a})_k =$
   $\{b \in X_k : b \not<^+ a \text{ or } b \notin \delta(X_{k+1})\} - \gamma(X_{k+1} - \{\alpha \in X_{k+1} : \gamma(\alpha) \leq^+ a\})$;
3. $\mathbf{d}^{(k)}(X^{\uparrow a})_{k-1} = X_{k-1} - \iota(X_{k+1}^{\downarrow a})$;

4. $\mathbf{d}^{(k)}(X^{\uparrow a})_l = X_l$, for $l < k-1$.

Thus to show that $\mathbf{c}^{(k)}(X^{\downarrow a}) = \mathbf{d}^{(k)}(X^{\uparrow a})$ we need to verify that $\mathbf{c}^{(k)}(X^{\downarrow a})_k = \mathbf{d}^{(k)}(X^{\uparrow a})_k$. As both sets are contained in X_k, we can compare their complements. We have

$$X_k - \mathbf{c}^{(k)}(X^{\downarrow a})_k = \{b \in \delta(X_{k+1}) : b <^+ a\} \cup \gamma(X_{k+1} - \{\alpha \in X_{k+1} : \gamma(\alpha) \not\leq^+ a\})$$

and

$$X_k - \mathbf{d}^{(k)}(X^{\uparrow a})_k = \{b \in \gamma(X_{k+1}) : b \not\leq^+ a\} \cup \delta(\{\alpha \in X_{k+1}\gamma(\alpha) \leq^+ a\}).$$

But it easy to see that

$$\{b \in \delta(X_{k+1}) : b <^+ a\} = \delta(\{\alpha \in X_{k+1}\gamma(\alpha) \leq^+ a\})$$

and

$$\gamma(X_{k+1} - \{\alpha \in X_{k+1} : \gamma(\alpha) \not\leq^+ a\}) = \{b \in \gamma(X_{k+1}) : b \not\leq^+ a\}.$$

The second equality uses the fact that $a \not\in \iota(T)$. Thus $\mathbf{c}^{(k)}(X^{\downarrow a})_k = \mathbf{d}^{(k)}(X^{\uparrow a})_k$, as required.

Ad 3. To see that $\mathbf{c}^{(k)}(X^{\uparrow a}) = \mathbf{c}^{(k)}(X)$ it is enough to note that $\iota(X_{k+1}) = \iota(X^{\downarrow a}_{k+1}) \cup \iota(X^{\uparrow a}_{k+1})$. The equation $\mathbf{d}^{(k)}(X^{\downarrow a}) = \mathbf{d}^{(k)}(X)$ is even simpler.

Ad 4. Obvious. $\Box$

Corollary 8.2 *Let T be a positive face structure, $k \in \omega$, $a \in (T_k - \iota(T_{k+2}))$. Then the square*

$$\begin{array}{ccc} T^{\downarrow a} & \longrightarrow & T \\ \uparrow & & \uparrow \\ \mathbf{c}^{(k)}(T^{\downarrow a}) & \longrightarrow & T^{\uparrow a} \end{array}$$

is a special pushout in $\mathbf{Fs}^{+/1}$.

Proof. Follows immediately from Lemmas 6.2 and 8.1. $\Box$

We need some notions and notations. Let X, T be a positive face structures $X \subseteq T$, $a \in (T - \iota(T))$. The decomposition $X = X^{\downarrow a} \cup X^{\uparrow a}$ is said to be *proper* iff $size(X^{\downarrow a}), size(X^{\uparrow a}) < size(X)$. If the decomposition $X = X^{\downarrow a} \cup X^{\uparrow a}$ is proper then a is said to be a *saddle face* of X. $Sd(X)$ is the set of saddle faces of X; $Sd(X)_k = Sd(X) \cap X_k$.

Lemma 8.3 *Let X, S, T be positive face structures, $X \subseteq T$, $l \in \omega$. Then*

1. *if $a \in (T_l - \iota(T))$ then $a \in Sd(X)$ iff there are $\alpha, \beta \in X_{l+1}$ such that $\gamma(\alpha) \leq^+ a$ and $\gamma(\beta) \not\leq^+ a$;*

2. *if $\mathbf{c}^{(k)}(S) = \mathbf{d}^{(k)}(T)$ then*

$$size(S +_k T)_l = \begin{cases} size(S)_l + size(T)_l & \text{if } l > k, \\ size(T)_l & \text{if } l \leq k; \end{cases}$$

3. *$size(S)_k \geq 1$ iff $k \leq dim(S)$;*

4. *if $a \in Sd(S)_k$ then $size(S)_{k+1} \geq 2$;*

5. *S is principal iff $Sd(S)$ is empty.*

Proof. We shall show 5. The rest is easy.

If there is $a \in Sd(S)_k$ then by 2., 3. and Lemma 8.1 we have that $size(S)_{k+1} = size(S^{\downarrow a})_{k+1} + size(S^{\uparrow a})_{k+1} \geq 1 + 1 > 1$. So in that case S is not principal.

For the converse, assume that S is not principal. Fix $k \in \omega$ such, that $size(S)_{k+1} > 1$. Thus there are $a, b \in S_{k+1}$, that $a \neq b$. Suppose $\gamma(a) \in \iota(\alpha)$ for some $\alpha \in S_{k+2}$. Then by Lemma 5.5, $a <^+ \gamma(\alpha)$ contrary to the assumption on a. Hence $a \in S - \iota(S)$ and for similar reasons $b \in S - \iota(S)$. We have $a \not\perp^+ b$ and, by pencil linearity, $\gamma(a) \neq \gamma(b)$. Then either $\gamma(a) \not<^+ \gamma(b)$ and then $\gamma(b) \in Sd(S)_k$ or $\gamma(b) \not<^+ \gamma(a)$ and then $\gamma(a) \in Sd(S)_k$. In either case $Sd(S)$ is not empty, as required. $\Box$

Lemma 8.4 *Let T, X be positive face structures, $X \subseteq T$, and $a, x \in X - \iota(X)$, $k = dim(x) < dim(a) = m$.*

1. *We have the following equations of positive face structures:*
$$X^{\downarrow x \downarrow a} = X^{\downarrow a \downarrow x}, \quad X^{\downarrow x \uparrow a} = X^{\uparrow a \downarrow x}, \quad X^{\uparrow x \downarrow a} = X^{\downarrow a \uparrow x}, \quad X^{\uparrow x \uparrow a} = X^{\uparrow a \uparrow x},$$
i.e. 'the decompositions of different dimension commute'.
2. *If $x \in Sd(X)$ then $x \in Sd(X^{\downarrow a}) \cap Sd(X^{\uparrow a})$.*
3. *Moreover, we have the following equations concerning domains and codomains*
$$\mathbf{c}^{(k)}(X^{\downarrow x \downarrow a}) = \mathbf{c}^{(k)}(X^{\downarrow x \uparrow a}) = \mathbf{d}^{(k)}(X^{\uparrow x \downarrow a}) = \mathbf{d}^{(k)}(X^{\uparrow x \uparrow a})$$
$$\mathbf{c}^{(m)}(X^{\downarrow x \downarrow a}) = \mathbf{d}^{(m)}(X^{\downarrow x \uparrow a}), \quad \mathbf{c}^{(m)}(X^{\uparrow x \downarrow a}) = \mathbf{d}^{(m)}(X^{\uparrow x \uparrow a}).$$
4. *Finally, we have the following equations concerning compositions*
$$X^{\downarrow x \downarrow a} +_m X^{\downarrow x \uparrow a} = X^{\downarrow x}, \quad X^{\uparrow x \downarrow a} +_m X^{\uparrow x \uparrow a} = X^{\uparrow x}$$
$$X^{\downarrow x \downarrow a} +_k X^{\uparrow x \downarrow a} = X^{\downarrow a}, \quad X^{\downarrow x \uparrow a} +_k X^{\uparrow x \uparrow a} = X^{\uparrow a}.$$

Proof. Simple check. $\Box$

Lemma 8.5 *Let T, X be positive face structures, $X \subseteq T$, and $a, b \in X - \iota(X)$, $dim(a) = dim(b) = m$.*

1. *We have the following equations of positive face structures:*
$$X^{\downarrow a \downarrow b} = X^{\downarrow b \downarrow a}, \quad X^{\uparrow a \uparrow b} = X^{\uparrow b \uparrow a},$$
i.e. 'the decompositions in the same dimension and the same directions commute'.
2. *Assume $a <^+ b$. Then we have the following farther equations of positive face structures:*
$$X^{\downarrow a} = X^{\downarrow a \downarrow b}, \quad X^{\uparrow b} = X^{\uparrow a \uparrow b}, \quad X^{\downarrow b \uparrow a} = X^{\uparrow a \downarrow b}.$$
Moreover, if $a, b \in Sd(X)$ then $a \in Sd(X^{\downarrow b})$ and $b \in Sd(X^{\uparrow a})$.
3. *Assume $a <^-_l b$, for some $l < m$. Then $X^{\uparrow b \downarrow a}$, $X^{\uparrow a \downarrow b}$, are positive face structures, and*
$$X^{\downarrow a} +_m X^{\uparrow a \downarrow b} = X^{\downarrow b} +_m X^{\uparrow b \downarrow a}$$
Moreover, if $a, b \in Sd(X)$ then either there is k such that $l - 1 \leq k < m$ and $\gamma^{(k)}(a) \in Sd(X)$ or $a \in Sd(X^{\uparrow b})$ and $b \in Sd(X^{\uparrow a})$.

Proof. Simple check. □

Lemma 8.6 *Let T, X be positive face structures, $X \subseteq T$, $dim(X) = n$, $l < n-1$, $a \in Sd(X)_l$. Then*

1. $a \in Sd(\mathbf{c}X) \cap Sd(\mathbf{d}X)$;
2. $\mathbf{d}(X^{\downarrow a}) = (\mathbf{d}X)^{\downarrow a}$;
3. $\mathbf{d}(X^{\uparrow a}) = (\mathbf{d}X)^{\uparrow a}$;
4. $\mathbf{c}(X^{\downarrow a}) = (\mathbf{c}X)^{\downarrow a}$;
5. $\mathbf{c}(X^{\uparrow a}) = (\mathbf{c}X)^{\uparrow a}$.

Proof. The proof is again by a long and simple check. We shall check part of 5. We should consider separately cases: $l = n-2$, $l = n-3$, and $l < n-3$, but we shall check the case $l = n-3$ only. The other cases can be also shown by similar, but easier, check.

$(\mathbf{c}X)^{\uparrow a}$ is:

1. $(\mathbf{c}X)^{\uparrow a}_l = \emptyset$, for $l \geq n$;
2. $(\mathbf{c}X)^{\uparrow a}_{n-1} = \{x \in X_{n-1} : \gamma^{(n-3)}(x) \not\leq^+ a,\ \ x \notin \delta(X_n)\}$;
3. $(\mathbf{c}X)^{\uparrow a}_{n-2} = \{x \in X_{n-2} : \gamma(x) \not\leq^+ a,\ \ x \notin \iota(X_n)\}$;
4. $(\mathbf{c}X)^{\uparrow a}_{n-3} = \{x \in X_{n-3} : x \not<^+ a \text{ or } x \notin \delta(X_{n-2} - \iota(X_n))\}$;
5. $(\mathbf{c}X)^{\uparrow a}_{n-4} = X_{n-4} - \iota(\{x \in X_{n-2} : x \notin \iota(X_n),\ \gamma(x) \leq^+ a\})$;
6. $X^{\downarrow a}_l = X_l$, for $l < n-4$.

and $\mathbf{c}(X^{\uparrow a})$ is:

1. $\mathbf{c}(X^{\uparrow a})_l = \emptyset$, for $l \geq n$;
2. $\mathbf{c}(X^{\uparrow a})_{n-1} = \{x \in X_{n-1} : \gamma^{(n-3)}(x) \not\leq^+ a\} - \delta(\{z \in X_n : \gamma^{(n-3)}(z) \not\leq^+ a\})$;
3. $\mathbf{c}(X^{\uparrow a})_{n-2} = \{x \in X_{n-2} : \gamma(x) \not\leq^+ a\} - \iota(\{z \in X_n : \gamma^{(n-3)}(z) \not\leq^+ a\})$;
4. $\mathbf{c}(X^{\uparrow a})_{n-3} = \{x \in X_{n-3} : x \not<^+ a \text{ or } x \notin \delta(X_{n-2})\}$;
5. $\mathbf{c}(X^{\uparrow a})_{n-4} = X_{n-4} - \iota(X^{\downarrow a}_{n-2})$;
6. $\mathbf{c}(X^{\uparrow a})_l = X_l$, for $l < n-4$.

We need to verify the equality $(\mathbf{c}X)^{\uparrow a}_l = \mathbf{c}(X^{\uparrow a})_l$ for $l = n-1, \ldots, n-4$.

In dimension $n-1$, it is enough to show that if $x \in X_{n-1}$ and $z \in X_n$ so that $\gamma^{(n-3)}(x) \not\leq^+ a$ and $x \in \delta(z)$ then $\gamma^{n-3}(z) \not\leq^+ a$.

So assume that $x \in X_{n-1}$, $\gamma^{(n-3)}(x) \not\leq^+ a$, $z \in X_n$ such that $x \in \delta(z)$. Hence $x \lhd^+ \gamma(z)$. By Lemma 5.9.5 $\gamma^{(n-3)}(x) \leq^+ \gamma^{(n-3)}(z)$. Therefore $\gamma^{(n-3)}(z) \not\leq^+ a$ (otherwise we would have $\gamma^{(n-3)}(x) \not\leq^+ a$), as required.

In dimension $n-2$, it is enough to show that if $x \in X_{n-2}$ and $z \in X_n$ so that $x \not\leq^+ a$ and $x \in \iota(z)$ then $\gamma^{n-3}(z) \not\leq^+ a$.

So assume that $x \in X_{n-2}$, $z \in X_n$ so that $x \not\leq^+ a$ and $x \in \iota(z)$. Hence $x \leq^+ \gamma\gamma(z)$. By Lemma 5.9.5 $\gamma(x) \leq^+ \gamma^{(n-3)}(z)$. Therefore $\gamma^{(n-3)}(z) \not\leq^+ a$, as required.

The equality in dimension $n-3$ follows immediately from Lemma 5.19.4.

To show that in dimension $n-4$, the above equation also holds we shall show that

$$\iota(X_{n-2}^{\downarrow a}) \subseteq \iota(\{x \in X_{n-2} : x \notin \iota(X_n),\ \gamma(x) \leq^+ a\})$$

Note that, by Lemma 5.3.1, if $t \in X_{n-4}$ and $x \in X_{n-2}$, $y \in X_{n-1}$, $t \in \iota(x)$ and $\gamma(x) \leq^+ a$ and $x = \gamma(y)$ then there is $x' \in \delta(y)$ (i.e. $x' \lhd^+ x$ and hence $\gamma(x') \leq^+ a$) such that $t \in \iota(x')$.

Thus, as $<^+$ is well founded, from the above observation follows that, for any $t \in X_{n-4}$ and $x \in X_{n-2}$ that $t \in \iota(x)$ and $\gamma(x) \leq^+ a$, there is $x'' \leq^+ x$ such that $t \in \iota(x'')$ and $x'' \notin \gamma(X)$. Then we clearly have that $x'' \notin \iota(X)$ and $\gamma(x'') \leq^+ a$, as required. $\Box$

Lemma 8.7 *Let T, T_1, T_2 be positive face structures, $dim(T_1), dim(T_2) > k$, such that $\mathbf{c}^{(k)}(T_1) = \mathbf{d}^{(k)}(T_2)$ and $T = T_1 +_k T_2$. Then $\mathbf{c}^{(k)}(T_1)_k \cap \gamma(T_1) \neq \emptyset$. For any $a \in \mathbf{c}^{(k)}(T_1)_k \cap \gamma(T_1)$ we have $a \in Sd(T)_k$ and either $T_1 = T^{\downarrow a}$ and $T_2 = T^{\uparrow a}$ or $a \in Sd(T_1)_k$, $T^{\downarrow a} = T_1^{\downarrow a}$ and $T^{\uparrow a} = T_1^{\uparrow a} +_k T_2$.*

Proof. By assumption $(T_1)_{k+1} \neq \emptyset$ and $(T_2)_{k+1} \neq \emptyset$. So $\mathbf{c}^{(k)}(T_1) \cap \gamma(T_1) \neq \emptyset$. Fix $a \in \mathbf{c}^{(k)}(T_1) \cap \gamma(T_1) \neq \emptyset$. Then $T_{k+1}^{\downarrow a} \neq \emptyset$. As $T_{k+1}^{\downarrow a} \cap (T_2)_{k+1} = \emptyset$ we must have $a \in Sd(T)_k$.

Assume that $T_1 \neq T^{\downarrow a}$. Then $T^{\downarrow a} \subsetneq T_1$. Hence $(T_1) - (T^{\downarrow a}) \neq \emptyset$. But this means that $a \in Sd(T_1)_k$. The verification that the equalities $T^{\downarrow a} = T_1^{\downarrow a}$ and $T^{\uparrow a} = T_1^{\uparrow a} +_k T_2$ hold in this case is left as an exercise. $\Box$

9 S^* is a positive-to-one computad

Proposition 9.1 *Let S be a weak positive face structure. Then S^* is a positive-to-one computad whose k-indets correspond to faces in S_k.*

Proof. Note that in the proof of Lemma 6.1.4 we were not using pencil linearity. So if S is a weak positive face structure and $a \in S_k$ then the subhypergraph $[a]$ of S is a principal positive face structure.

The proof is by induction on dimension n of the weak positive face structure S. For $n = 0, 1$ the Proposition is obvious.

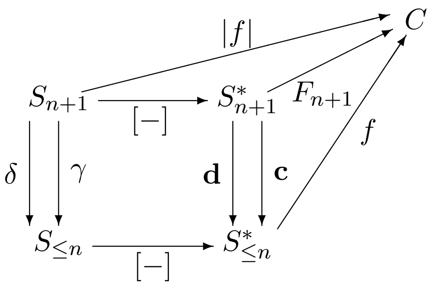
So assume that for any weak positive face structure T of dimension n, T^* is a positive-to-one computad of dimension n, generated by faces in T. Suppose that S is a weak positive face structure of dimension $n+1$. We shall show that S^* is a computad generated by faces in S. Since $S_{\leq n}$ is a weak positive face structure, by inductive assumption $S^*_{\leq n}$ is the computad generated by faces in $S_{\leq n}$. So we need to verify that for any ω-functor $f : S^*_{\leq n} \longrightarrow C$ to any ω-category C, and any function $|f| : S_{n+1} \longrightarrow C_{n+1}$ such, that for $a \in S_{n+1}$

$$d_C(|f|(a)) = f(\mathbf{d}([a])), \qquad c_C(|f|(a)) = f(\mathbf{c}([a])),$$

there is a unique ω-functor $F : S^* \longrightarrow C$, such that

$$F_{n+1}([a]) = |f|(a), \qquad F_{\leq n} = f$$

as in the diagram



We define F_{n+1} as follows. For $X \in S^*_{n+1}$

$$F_{n+1}(X) = \begin{cases} id_{f(X)} & \text{if } dim(X) \leq n, \\ |f|(a) & \text{if } dim(X) = n+1,\ X \text{ is principal and } X = [a], \\ F_{n+1}(X^{\downarrow a});_l F_{n+1}(X^{\uparrow a}) & \text{if } dim(X) = n+1,\ a \in Sd(X)_l. \end{cases}$$

$;_l$ refers to the composition in C. Clearly $F_k = f_k$ for $k \leq n$. The above morphism, if well defined, clearly preserves identities. We need to verify, for $X \in S^*_{n+1}$ and $dim(X) = n+1$, three conditions:

I F is well defined, i.e. $F_{n+1}(X) = F_{n+1}(X^{\downarrow a});_l F_{n+1}(X^{\uparrow a})$ does not depend on the choice of the saddle face $a \in Sd(X)$;

II F preserves the domains and codomains i.e., $F_n(d(X)) = d(F_{n+1}(X))$ and $F_n(c(X)) = c(F_{n+1}(X))$;

III F preserves compositions i.e., $F_{n+1}(X) = F_{n+1}(X_1);_k F_{n+1}(X_2)$ whenever $X = X_1 +_k X_2$ and $dim(X_1), dim(X_2) > k$.

We have an embedding $[-] : S_{\leq n} \longrightarrow S^*_{\leq n}$.

So let assume that for positive face substructures of S of size less than $size(X)$ the above assumption holds. If $size(X)_{n+1} = 0$ or X is principal all three conditions are obvious. So assume that X is not principal and $dim(X) = n+1$. To save on notation we write F for F_{n+1}.

Ad **I**. First we will consider two saddle faces $a, x \in Sd(X)$ of different dimension $k = dim(x) < dim(a) = m$. Using Lemma 8.4 we have

$$\begin{array}{rl} F(X^{\downarrow a});_m F(X^{\uparrow a}) = & \textit{ind. hyp. I} \\ = (F(X^{\downarrow a \downarrow x});_k F(X^{\downarrow a \uparrow x}));_m (F(X^{\uparrow a \downarrow x});_k F(X^{\uparrow a \uparrow x})) = & MEL \\ = (F(X^{\downarrow a \downarrow x});_m F(X^{\uparrow a \downarrow x}));_k (F(X^{\downarrow a \uparrow x});_m F(X^{\uparrow a \uparrow x})) = & \\ = (F(X^{\downarrow x \downarrow a});_m F(X^{\downarrow x \uparrow a}));_k (F(X^{\uparrow x \downarrow a});_m F(X^{\uparrow x \uparrow a})) = & \textit{ind. hyp. III} \\ = F(X^{\downarrow x});_m F(X^{\uparrow x}) & \end{array}$$

Now we will consider two saddle faces $a, b \in Sd(X)$ of the same dimension $dim(a) = dim(b) = m$. We shall use Lemma 8.5. Assume that $a <^-_l b$, for some $l < m$. If $\gamma^{(k)}(a) \in Sd(X)$, for some $k < m$, then this case reduces to the previous one for two pairs $a, \gamma^{(k)}(a) \in Sd(X)$ and $b, \gamma^{(k)}(a) \in Sd(X)$. Otherwise $a \in Sd(X^{\uparrow b})$ and $a \in Sd(X^{\uparrow b})$ and we have

$$\begin{array}{rl} F(X^{\downarrow a});_k F(X^{\uparrow a}) = & \textit{ind. hyp I} \\ = F(X^{\downarrow a});_k (F(X^{\uparrow a \downarrow b});_k F(X^{\uparrow a \uparrow b})) = & \\ = (F(X^{\downarrow a});_k F(X^{\uparrow a \downarrow b}));_k F(X^{\uparrow b \uparrow a}) = & \textit{ind hyp III} \\ = F(X^{\downarrow a};_k X^{\uparrow a \downarrow b});_k F(X^{\uparrow b \uparrow a}) = & \\ = F(X^{\downarrow b};_k X^{\uparrow b \downarrow a});_k F(X^{\uparrow b \uparrow a}) = & \textit{ind hyp III} \\ = (F(X^{\downarrow b});_k F(X^{\uparrow b \downarrow a}));_k F(X^{\uparrow b \uparrow a}) = & \\ = F(X^{\downarrow b});_k (F(X^{\uparrow b \downarrow a});_k F(X^{\uparrow b \uparrow a})) = & \textit{ind hyp I} \\ = F(X^{\downarrow b});_k F(X^{\uparrow b}) & \end{array}$$

Finally, we consider the case $a <^+ b$. We have

$$\begin{aligned} F(X^{\downarrow a});_k F(X^{\uparrow a}) = & \quad \textit{ind. hyp I} \\ = F(X^{\downarrow a});_k (F(X^{\uparrow a \downarrow b});_k F(X^{\uparrow a \uparrow b})) = & \\ = F(X^{\downarrow b \downarrow a});_k (F(X^{\downarrow b \uparrow a}));_k F(X^{\uparrow b}) = & \\ = (F(X^{\downarrow b \downarrow a});_k F(X^{\downarrow b \uparrow a}));_k F(X^{\uparrow b}) = & \quad \textit{ind hyp I} \\ = F(X^{\downarrow b});_k F(X^{\uparrow b}) & \end{aligned}$$

This shows that $F(X)$ is well defined.

Ad **II**. We shall show that the domains are preserved. The proof that, the codomains are preserved, is similar.

The fact that if $Sd(X) = \emptyset$ then F preserves domains and codomains follows immediately from the assumption on f and $|f|$. So assume that $Sd(X) \neq \emptyset$ and let $a \in Sd(X)$, $dim(a) = k$. We use Lemma 8.6. We have to consider two cases $k < n$, and $k = n$.

If $k < n$ then

$$\begin{aligned} F_n(d(X)) = F_n(d(X^{\downarrow a} +_k X^{\uparrow a})) = & \\ = F_n(d(X^{\downarrow a});_k d((X^{\uparrow a})) = & \\ = F_n(d(X)^{\downarrow a};_k d(X)^{\uparrow a}) = & \quad \textit{ind hyp III} \\ = F_n(d(X)^{\downarrow a});_k F_n(d(X)^{\uparrow a}) = & \\ = F_n(d(X^{\downarrow a}));_k F_n(d(X^{\uparrow a})) = & \quad \textit{ind hyp II} \\ = d(F_{n+1}(X^{\downarrow a}));_k d(F_{n+1}(X^{\uparrow a})) = & \\ = d(F_{n+1}(X^{\downarrow a});_k F_{n+1}(X^{\uparrow a})) = & \quad \textit{ind hyp I} \\ = d(F_{n+1}(X)) & \end{aligned}$$

If $k = n$ then

$$\begin{aligned} F_n(d(X)) = F_n(d(X^{\downarrow a} +_n X^{\uparrow a})) = & \\ = F_n(d(X^{\downarrow a})) = & \quad \textit{ind hyp II} \\ = d(F_{n+1}(X^{\downarrow a})) = & \\ = d(F_{n+1}(X^{\downarrow a});_n F_{n+1}(X^{\uparrow a})) = & \quad \textit{ind hyp I} \\ = d(F_{n+1}(X)) & \end{aligned}$$

Ad **III**. Suppose that $X = X_1;_k X_2$ and $dim(X) \leq n+1$. We shall show that F preserves this composition. If $dim(X_1) = k$ then $X = X_2$, $X_1 = \mathbf{d}^{(k)}(X_2)$. We have

$$F_{n+1}(X) = F_{n+1}(X_2) = 1^{(n+1)}_{F_k(\mathbf{d}^{(k)}(X_2))};_k F_{n+1}(X_2) =$$

$$= 1^{(n+1)}_{F_k(X_1)};_k F_{n+1}(X_2) = F_{n+1}(X_1);_k F_{n+1}(X_2)$$

The case $dim(X_2) = k$ is similar. So now assume that $dim(X_1), dim(X_2) > k$. We shall use Lemma 8.7. Fix $a \in \mathbf{c}^{(k)}(X_1)_k \cap \gamma(X_1)$. So $a \in Sd(X)_k$. If $X_1 = X^{\downarrow a}$ and $X_2 = X^{\uparrow a}$ then we have

$$F(X) = F(X^{\downarrow a});_k F(X^{\uparrow a}) = F(X_1);_k F(X_2).$$

If $a \in Sd(X_1)_k$ then

$$\begin{aligned} F(X) = F(X^{\downarrow a});_k F(X^{\uparrow a}) = & \quad \textit{ind hyp II} \\ = F(X^{\downarrow a});_k (F(X_1^{\uparrow a});_k F(X_2)) = & \\ = (F(X_1^{\downarrow a});_k F(X_1^{\uparrow a}));_k F(X_2) = & \quad \textit{ind hyp II} \\ F(X_1);_k F(X_2) & \end{aligned}$$

So in any case the composition is preserved.

This ends the proof of the Lemma. $\Box$

For $n \in \omega$, we have a functor

$$(-)^{\sharp,n} : \mathbf{wFs}_n^{+/1} \longrightarrow \mathbf{Comma}_n^{+/1}$$

such that, for S in $\mathbf{wFs}_n^{+/1}$

$$S^{\sharp,n} = (S_n, S^*_{<n}, [\delta], [\gamma])$$

and for $f : S \to T$ in $\mathbf{wFs}_n^{+/1}$ we have

$$f^{\natural,n} = (f_n, (f_{<n})^*).$$

Corollary 9.2 *For every $n \in \omega$, the functor $(-)^{\sharp,n}$ is well defined, full, faithful, and it preserves existing pushouts. Moreover, for S in $\mathbf{wFs}_n^{+/1}$ we have $S^* = \overline{S^{\sharp,n}}^n$.*

Proof. The functor $\overline{(-)}^n : \mathbf{Comma}_n^{+/1} \longrightarrow \mathbf{Comp}_n^{+/1}$ which is an equivalence of categories is described in the Appendix.

Fullness and faithfulness of $(-)^{\sharp,n}$ is left for the reader. We shall show simultaneously that for every $n \in \omega$, both functors

$$(-)^{\sharp,n} : \mathbf{wFs}_n^{+/1} \longrightarrow \mathbf{Comma}_n^{+/1}, \qquad (-)^{*,n} : \mathbf{wFs}_n^{+/1} \longrightarrow \mathbf{Comp}_n^{+/1}$$

preserve existing pushouts. For $n = 0$ there is nothing to prove. For $n = 1$ this is obvious. So assume that $n > 1$ and that $(-)^{\sharp,n}$ preserves existing pushouts. Let

$$\begin{array}{ccc} S & \longrightarrow & S +_R T \\ \uparrow & & \uparrow \\ R & \longrightarrow & T \end{array}$$

be a pushout in $\mathbf{wFs}_{n+1}^{+/1}$. Clearly its n-truncation is a pushout in $\mathbf{wFs}_n^{+/1}$. Hence by inductive hypothesis it is preserved by $(-)^{*,n}$. In dimension $n + 1$, the functor $(-)^{\sharp,n+1}$ is an inclusion. Hence, in dimension $n + 1$, this square is a pushout (of monos) in Set. So the whole square

$$\begin{array}{ccc} S^{\sharp,n+1} & \longrightarrow & (S +_R T)^{\sharp,n+1} \\ \uparrow & & \uparrow \\ R^{\sharp,n+1} & \longrightarrow & T^{\sharp,n+1} \end{array}$$

is a pushout in $\mathbf{Comma}_{n+1}$, i.e. $(-)^{\sharp,n+1}$ preserves pushouts. As $(-)^{*,n+1}$ is a composition of $(-)^{\sharp,n+1}$ with an equivalence of categories it preserves the pushouts, as well. $\Box$

Corollary 9.3 *The functor*

$$(-)^* : \mathbf{wFs}^{+/1} \longrightarrow \mathbf{Comp}^{+/1}$$

is full and faithful and preserves special pushouts.

Proof. This follows from the previous Corollary and the fact that the functor $\overline{(-)}^n$: $\mathbf{Comma}_n^{+/1} \longrightarrow \mathbf{Comp}_n^{+/1}$ (see Appendix) is an equivalence of categories. $\Box$

Let P be a positive-to-one computad, a a k-cell in P. A *description of the cell a* is a pair

$$< T_a, \tau_a : T_a^* \longrightarrow P >$$

where T_a is a positive face structure and τ_a is a computad map such that

$$\tau_a(T_a) = a.$$

In the remainder of this section we shall define some specific positive face structures that will be used later. First we define α^n, for $n \in \omega$. We put

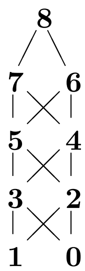
$$\alpha^n_l \;=\; \begin{cases} \emptyset & \text{if } l > n \\ \{2n\} & \text{if } l = n \\ \{2l+1,\, 2l\} & \text{if } 0 \leq l < n \end{cases}$$

$$d,\, c : \alpha^n_l \longrightarrow \alpha^n_{l-1}$$

$$d(x) = \{2l-1\} \qquad c(x) = 2l-2$$

for $x \in \alpha^n_l$, and $1 \leq l \leq n$.

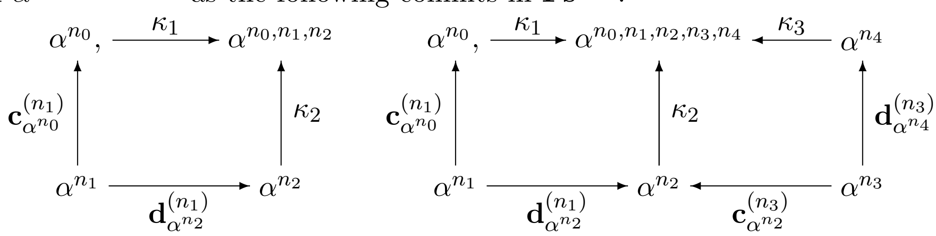
For example α^4 can be pictured as follows:



i.e. 8 is the unique cell of dimension 4 in α^4 that has 7 as its domain and 6 as its codomain, 7 and 6 have 5 as its domain and 4 as its codomain, and so on. Note that, for any $k \leq n$, we have

$$\mathbf{d}^{(k)}\alpha^n = \alpha^k = \mathbf{c}^{(k)}\alpha^n.$$

Let $n_1 < n_0, n_2$ and $n_3 < n_2, n_4$. We define the positive face structures α^{n_0,n_1,n_2} and $\alpha^{n_0,n_1,n_2,n_3,n_4}$ as the following colimits in $\mathbf{Fs}^{+/1}$:



We have

Proposition 9.4 *The above colimits are preserved by the functor*

$$(-)^* : \mathbf{Fs}^{+/1} \longrightarrow \mathbf{Comp}^{+/1}.$$

Moreover for any ω-category C we have bijective correspondences

$$\omega Cat((\alpha^n)^*, C) = C_n$$

$$\omega Cat((\alpha^{n_0,n_1,n_2})^*, C) = \{(x,y) \in C_{n_0} \times C_{n_2} : c^{(n_1)}(x) = d^{(n_1)}(y)\}$$

$$\omega Cat((\alpha^{n_0,n_1,n_2,n_3,n_4})^*, C) =$$

$$= \{(x,y,z) \in C_{n_0} \times C_{n_2} \times C_{n_4} : c^{(n_1)}(x) = d^{(n_1)}(y) \;\text{ and }\; c^{(n_3)}(y) = d^{(n_3)}(z)\}$$

which are natural in C.

Proof. As both positive face structures α^{n_0,n_1,n_2} and $\alpha^{n_0,n_1,n_2,n_3,n_4}$ are obtained vis special pushout (in the second case applied twice) these colimits are preserved by $(-)^*$. □

The essential image of the full and faithful functor $(-)^* : \mathbf{Fs}^{+/1} \longrightarrow \mathbf{Comp}^{+/1}$ is the category of the *positive computypes* and it will be denoted by $\mathcal{C}types^{+/1}$. Thus the categories $\mathbf{Fs}^{+/1}$ and $\mathcal{C}types^{+/1}$ are equivalent. The full image of the functor $(-)^* : \mathbf{Fs}^{+/1} \longrightarrow \omega Cat$ will be denoted by $\mathcal{C}types^{+/1}_\omega$. The objects of $\mathcal{C}types^{+/1}_\omega$ are ω-categories isomorphic to those of form S^* for S being positive face structure and the morphism in $\mathcal{C}types^{+/1}_\omega$ are all ω-functors.

10 The inner-outer factorization in $\mathcal{C}types^{+/1}_\omega$

Let $f : S^* \longrightarrow T^*$ be a morphism in $\mathcal{C}types^{+/1}_\omega$. We say that f is *outer*[1] if there is a map of face structures $g : S \longrightarrow T$ such that $g^* = f$. We say that f is *inner* iff $f_{dim(S)}(S) = T$. From Corollary 9.3 we have

Lemma 10.1 *An ω-functor $f : S^* \longrightarrow T^*$ is outer iff it is a computad map.* □

Proposition 10.2 *Let $f : S^* \longrightarrow T^*$ be an inner map, $dim(S) = dim(T) > 0$. The maps $\mathbf{d}f : \mathbf{d}S \longrightarrow \mathbf{d}T$ and $\mathbf{c}f : \mathbf{c}S \longrightarrow \mathbf{c}T$, being the restrictions of f, are well defined, inner and the squares*

$$\begin{array}{ccccc}
(\mathbf{d}T)^* & \xrightarrow{\mathbf{d}^*_T} & T^* & \xleftarrow{\mathbf{c}^*_T} & (\mathbf{c}T)^* \\
\uparrow{\scriptstyle \mathbf{d}f} & & \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle \mathbf{c}f} \\
(\mathbf{d}S)^* & \xrightarrow[\mathbf{d}^*_S]{} & S^* & \xleftarrow[\mathbf{c}^*_S]{} & (\mathbf{c}S)^*
\end{array}$$

commute.

Proof. So suppose that $f : S^* \to T^*$ is an inner map. So $f(S) = T$. Since f is an ω-functor we have

$$f(\mathbf{d}S) = \mathbf{d}f(S) = \mathbf{d}T \quad \text{and} \quad f(\mathbf{c}S) = \mathbf{c}f(S) = \mathbf{c}T.$$

This shows the proposition. □

We have

Proposition 10.3 *The inner and outer morphisms form a factorization system in $\mathcal{C}types^{+/1}_\omega$. So any ω-functor $f : S^* \longrightarrow T^*$ can be factored essentially uniquely by inner map $\overset{\bullet}{f}$ followed by outer map $\overset{\circ}{f}$:*

$$\begin{array}{ccc}
S^* & \xrightarrow{f} & T^* \\
& {\scriptstyle \overset{\bullet}{f}}\searrow \quad \nearrow{\scriptstyle \overset{\circ}{f}} & \\
& f(S)^* &
\end{array}$$

Proof. This is almost tautological. □

[1] The names 'inner' and 'outer' are introduced in analogy with the morphism with the same name and role in the category of disks in [J].

11 The terminal positive-to-one computad

In this section we shall describe the terminal positive-to-one computad $\mathcal{T}$.

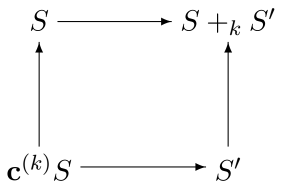
The set of n-cell $\mathcal{T}_n$ consists of (isomorphisms classes of) positive face structures of dimension less than or equal to n. For $n > 0$, the operations of domain and codomain $d^{\mathcal{T}}, c^{\mathcal{T}} : \mathcal{T}_n \to \mathcal{T}_{n-1}$ are given, for $S \in \mathcal{T}_n$ by

$$d(S) = \begin{cases} S & \text{if } dim(S) < n, \\ \mathbf{d}S & \text{if } dim(S) = n, \end{cases}$$

and

$$c(S) = \begin{cases} S & \text{if } dim(S) < n, \\ \mathbf{c}S & \text{if } dim(S) = n. \end{cases}$$

and, for $S, S' \in \mathcal{T}_n$ such that $c^{(k)}(S) = d^{(k)}(S')$ the composition in $\mathcal{T}$ is just the pushout $S;_k S' = S +_k S'$ i.e.



The identity $id_{\mathcal{T}} : T_{n-1} \to \mathcal{T}_n$ is the inclusion map.

The n-indets in $\mathcal{T}$ are the principal positive n-face structures.

Proposition 11.1 *$\mathcal{T}$ just described is the terminal positive-to-one computad.*

Proof. The fact that $\mathcal{T}$ is an ω-category is easy. The fact that $\mathcal{T}$ is free with free n-generators being principal n-face structures can be shown much like the freeness of S^* before. The fact that $\mathcal{T}$ is terminal follows from the following observation:

Observation. For every pair of parallel n-face structures N and B (i.e. $\mathbf{d}N = \mathbf{d}B$ and $\mathbf{c}N = \mathbf{c}B$) such that N is normal and B is principal there is a unique (up to an iso) principal $(n+1)$-face structure $N^\bullet$ such that $\mathbf{d}N^\bullet = N$ and $\mathbf{c}N^\bullet = B$. □

Lemma 11.2 *Let S be a positive face structure and $! : S^* \longrightarrow \mathcal{T}$ the unique map from S^* to $\mathcal{T}$. Then, for $T \in S^*_k$ we have*

$$!_k(T) = T.$$

Proof. The proof is by induction on $k \in \omega$ and the size of T in S^*_k. For $k = 0, 1$ the lemma is obvious. Let $k > 1$ and assume that lemma holds for $i < k$.

If $dim(T) = l < k$ then, using the inductive hypothesis and the fact that ! is an ω-functor, we have

$$!_k(T) = !_k(1^{(k)}_T) = 1^{(k)}_{!_l(T)} = 1^{(k)}_T = T$$

Suppose that $dim(T) = k$ and T is principal. As ! is a computad map $!_k(T)$ is an indet, i.e. it is principal, as well. We have, using again the inductive hypothesis and the fact that ! is an ω-functor,

$$d(!_k(T)) = !_{k-1}(\mathbf{d}T) = \mathbf{d}T$$

$$c(!_k(T)) = !_{k-1}(\mathbf{c}T) = \mathbf{c}T$$

As T is the only (up to a unique iso) positive face structure with the domain $\mathbf{d}T$ and the codomain $\mathbf{c}T$, it follows that $!_k(T) = T$, as required.

Finally, suppose that $dim(T) = k$, T is not principal, and for the positive face structures of size smaller than the size of T the lemma holds. Thus there are $l \in \omega$ and $a \in Sd(T)_l$ so that

$$!_k(T) = !_k(T^{\downarrow a};_l T^{\uparrow a}) = !_k(T^{\downarrow a});_l !_k(T^{\uparrow a}) = T^{\downarrow a};_l T^{\uparrow a} = T,$$

as required □

12 A description of the positive-to-one computads

In this section we shall describe all the cells in positive-to-one computads using positive face structures, in other words we shall describe in concrete terms the functor:

$$\overline{(-)} : \mathbf{Comma}^{+/1}_n \longrightarrow \mathbf{Comp}^{+/1}_n$$

More precisely, the positive-to-one computads of dimension 1 (and all computads as well) are free computads over graphs and are well understood. So suppose that $n > 1$, and we are given an object of $\mathbf{Comma}^{+/1}_n$, i.e. a quadruple $(|P|_n, P, d, c)$ such that

1. a positive-to-one $(n-1)$-computad P;
2. a set $|P|_n$ with two functions $c : |P|_n \longrightarrow |P|_{n-1}$ and $d : |P|_n \longrightarrow P_{n-1}$ such that for $x \in |P|_n$, $cc(x) = cd(x)$ and $dc(x) = dd(x)$; we assume that $d(x)$ is not an identity for any $x \in |P|_n$.

If the maps d and c in the object $(|P|_n, P, d, c)$ are understood from the context we can abbreviate notation to $(|P|_n, P)$.

For a positive face structure S, with $dim(S) \leq n$, we denote by $S^{\sharp,n}$ the object $(S_n, (S_{<n})^*, [\delta], [\gamma])$ in $\mathbf{Comma}^{+/1}_n$. In fact, we have an obvious functor

$$(-)^{\sharp,n} : \mathbf{Fs}^{+/1} \longrightarrow \mathbf{Comma}^{+/1}_n$$

such that

$$S \mapsto (S_n, (S_{<n})^*, [\delta], [\gamma])$$

Any positive-to-one computad P can be restricted to its part in $\mathbf{Comma}^{+/1}_n$. So we have an obvious forgetful functor

$$(-)^{\natural,n} : \mathbf{Comp}^{+/1} \longrightarrow \mathbf{Comma}^{+/1}_n$$

such that

$$P \mapsto (|P|_n, P_{<n}, d, c)$$

We shall describe the positive-to-one n-computad $\overline{P}$ whose $(n-1)$-truncation is P and whose n-indets are $|P|_n$ with the domains and codomains given by c and d.

n-cells of $\overline{P}$. An n-cell in $\overline{P}_n$ is a(n equivalence class of) pair(s) (S, f) where

1. S is a positive face structure, $dim(S) \leq n$;
2. $f : (S_n, (S_{<n})^*, [\delta], [\gamma]) \longrightarrow (|P|_n, P, d, c)$ is a morphism in $\mathbf{Comma}^{+/1}_n$, i.e.

$$\begin{array}{ccc} S_n & \xrightarrow{\ |f|_n\ } & |P|_n \\ {\scriptstyle [\delta]}\downarrow\downarrow{\scriptstyle [\gamma]} & & {\scriptstyle d}\downarrow\downarrow{\scriptstyle c} \\ S^*_{n-1} & \xrightarrow[\ f_{n-1}\ ]{} & P_{n-1} \end{array}$$

commutes.

We identify two pairs (S, f), (S', f') if there is an isomorphism $h : S \longrightarrow S'$ such that the triangles of sets and of $(n-1)$-computads

$$\begin{array}{ccc} S_n & \xrightarrow{\ h_n\ } & S'_n \\ & {\scriptstyle f_n}\searrow \quad \swarrow{\scriptstyle f'_n} & \\ & |P|_n & \end{array} \qquad \begin{array}{ccc} (S_{<n})^* & \xrightarrow{\ (h_{<n})^*\ } & (S'_{<n})^* \\ & {\scriptstyle f_{<n}}\searrow \quad \swarrow{\scriptstyle f'_{<n}} & \\ & P & \end{array}$$

commute. Clearly, such an h, if exists, is unique. Even if formally cells in P_n are equivalence classes of triples we will work on triples themselves as if they were cells understanding that equality between such cells is an isomorphism in the sense defined above.

Domains and codomains. The domain and codomain functions

$$d^{(k)}, c^{(k)} : \overline{P}_n \longrightarrow \overline{P}_k$$

are defined for an n-cell (S, f) as follows:

$$d^{(k)}(S, f) = (\mathbf{d}^{(k)}S, \mathbf{d}^{(k)}f)$$

where, for $x \in (\mathbf{d}^{(k)}S)_k$

$$(\mathbf{d}^{(k)}f)_k(x) = f_k([x])(x)$$

(i.e. we take the sub-face structure $[x]$ of S, then value of f on it, and then we evaluate the map in $\mathbf{Comma}_n^{+/1}$ on x the only element of $[x]_k$),

$$(\mathbf{d}^{(k)}f)_l = f_l$$

for $l < k$;

$$c^{(k)}(S, f) = (\mathbf{c}^{(k)}S, \mathbf{c}^{(k)}f)$$

where, for $x \in (\mathbf{c}^{(k)}S)_k$

$$(\mathbf{c}^{(k)}f)_k(x) = f_k([x])(x)$$

and

$$(\mathbf{d}^{(k)}f)_l = f_l$$

for $l < k$, i.e. we calculate the k-th domain and k-th codomain of an n-cell (S, f) by taking $\mathbf{d}^{(k)}$ and $\mathbf{c}^{(k)}$ of the domain S of the cell f, respectively, and by restricting the maps f accordingly.

Identities. The identity function

$$\mathbf{i} : \overline{P}_{n-1} \longrightarrow \overline{P}_n$$

is defined for an $(n-1)$-cell $((S, f)$ in P_{n-1}, as follows:

$$\mathbf{i}(S, f) = \begin{cases} (S, f) & \text{if } dim(S) < n-1, \\ (S, \overline{f}) & \text{if dim(S)=n-1} \end{cases}$$

Note that $\overline{f}$ is the map $\mathbf{Comp}_{n-1}^{+/1}$ which is the value of the functor $\overline{(-)}$ on a map f from $\mathbf{Comma}_{n-1}^{+/1}$. So it is in fact defined as 'the same $(n-1)$-cell' but considered as an n-cell.

Compositions. Suppose that (S^i, f^i) are n-cells for $i = 0, 1$, such that

$$c^{(k)}(S^0, f^0) = d^{(k)}(S^1, f^1).$$

Then their composition is defined, via pushout in $\mathbf{Comma}_n^{+/1}$, as

$$(S^0, f^0);_k (S^1, f^1) = (S^0 +_k S^1, [f^0, f^1])$$

i.e.

$$\begin{array}{ccc} S_n^0 \sqcup S_n^1 & \xrightarrow{[f_n^0, f_n^1]} & |P|_n \\ {\scriptstyle [\delta]}\downarrow\downarrow{\scriptstyle [\gamma]} & & {\scriptstyle d}\downarrow\downarrow{\scriptstyle c} \\ ((S^0 +_k S^1)_{\leq n-1})^*_{n-1} & \xrightarrow[{[f_{n-1}^0, f_{n-1}^1]}]{} & P_{n-1} \end{array}$$

This ends the description of the computad $\overline{P}$.

Now let $h : P \to Q$ be a morphism in $\mathbf{Comma}_n^{+/1}$, i.e. a function $h_n : |P|_n \longrightarrow |Q|_n$ and a $(n-1)$-computad morphism $h_{<n} : P_{<n} \longrightarrow Q_{<n}$ such that the square

$$\begin{array}{ccc} |P|_n & \xrightarrow{h_n} & |Q|_n \\ {\scriptstyle d}\downarrow\downarrow{\scriptstyle c} & & {\scriptstyle d}\downarrow\downarrow{\scriptstyle c} \\ P_{n-1} & \xrightarrow[h_{n-1}]{} & Q_{n-1} \end{array}$$

commutes serially. We define

$$\bar{h} : \bar{P} \longrightarrow \bar{Q}$$

by putting $\bar{h}_k = h_k$ for $k < n$, and for $(S, f) \in \bar{P}_n$, we put

$$\bar{h}(S, f) = (S, h \circ f).$$

Notation. Let $x = (S, f)$ be a cell in $\bar{P}_n$ as above, and $a \in Sd(S)$. Then by $x^{\downarrow a} = (S^{\downarrow a}, f^{\downarrow a})$ and $x^{\uparrow a} = (S^{\uparrow a}, f^{\uparrow a})$ we denote the cells in $\bar{P}_n$ that are the obvious restriction x. Clearly, we have $c^{(k)}(x^{\downarrow a}) = d^{(k)}((x^{\uparrow a})$ and that $x = x^{\downarrow a};_k x^{\uparrow a}$, where $k = dim(a)$.

The following Proposition states several statements concerning the above construction. This includes that the above construction is correct. We have put all these statement together as we need to prove them together, that is by simultaneous induction.

Proposition 12.1 *Let $n \in \omega$. We have*

1. *Let P be an object of $\mathbf{Comma}_n^{+/1}$. We define the function*

$$\eta_P : |P|_n \longrightarrow \overline{P}_n$$

as follows. Let $x \in |P|_n$. As $c(x)$ is an indet $d(x)$ is a normal cell of dimension $n-1$. Thus there is a unique description of the cell $d(x)$

$$< T_{d(x)}, \tau_{d(x)} : T^*_{d(x)} \longrightarrow P_{<n} >$$

with $T_{d(x)}$ being normal positive face structure. Then we have a unique n-cell in $\bar{P}$:

$$\bar{x} = < T^\bullet_{d(x)},\ \ |\overline{\tau}_x|_n : \{T^\bullet_{d(x)}\} \to |P|_n,\ \ (\overline{\tau}_x)_{<n} : (T^\bullet_{d(x)})^*_{<n} \to P_{<n} >$$

(note: $|T^\bullet_{d(x)}|_n = \{T^\bullet_{d(x)}\}$) such that

$$|\overline{\tau}_x|_n(T^\bullet_{d(x)}) = x$$

and

$$(\overline{\tau}_x)_{n-1}(S) = \begin{cases} c(x) & \text{if } S = \mathbf{c}(T^\bullet_{d(x)}) \\ (\tau_{dx})_{n-1}(S) & \text{if } S \subseteq T_{dx} \end{cases}$$

and $(\overline{\tau}_x)_{<(n-1)} = (\tau_{dx})_{<(n-1)}$. We put $\eta_P(x) = \bar{x}$.

Then $\bar{P}$ is a positive-to-one computad with η_P the inclusion of n-indeterminates. Moreover any positive-to-one n-computad Q is equivalent to a computad $\bar{P}$, for some P in $\mathbf{Comma}_n^{+/1}$.

2. *Let P be an object of $\mathbf{Comma}_n^{+/1}$, $! : \overline{P} \longrightarrow \mathcal{T}$ the unique morphism into the terminal object $\mathcal{T}$ and $f : S^{\sharp,n} \to P$ a cell in $\overline{P}_n$. Then*

$$!_n(f : S^{\sharp,n} \to P) = S.$$

3. *Let $h : P \to Q$ be an object of $\mathbf{Comma}_n^{+/1}$. Then $\bar{h} : \bar{P} \longrightarrow \bar{Q}$ is a computad morphism,*

4. *Let S be a positive face structure of dimension at most n. Moreover, for a morphism $f : S^{\sharp,n} \longrightarrow P$ in $\mathbf{Comma}_n^{+/1}$ we have that*

$$\overline{f}_k(T) = f \circ (i_T)^{\sharp,n}$$

where $k \leq n$, $T \in S_k^*$ and $i_T : T \longrightarrow S$ is the inclusion.

5. *Let S be a positive face structure of dimension n, P positive-to-one computad, $g, h : S^* \longrightarrow P$ computad maps. Then*

$$g = h \qquad \text{iff} \qquad g_n(S) = h_n(S).$$

6. *Let S be a positive face structure of dimension at most n, P be an object in $\mathbf{Comma}_n^{+/1}$. Them we have a bijective correspondence*

$$\frac{f : S^{\sharp,n} \longrightarrow P \quad \in \mathbf{Comma}_n^{+/1}}{\overline{f} : S^* \longrightarrow \overline{P} \quad \in \mathbf{Comp}_n^{+/1}}$$

such that, $\overline{f}_n(S) = f$, and for $g : S^* \longrightarrow \overline{P}$ we have $g = \overline{g_n(S)}$.

7. *The map*

$$\kappa_n^P : \coprod_S \mathbf{Comp}(S^*, \overline{P}) \longrightarrow \overline{P}_n$$

$$g : S^* \to \overline{P} \quad \mapsto \quad g_n(S)$$

where coproduct is taken over all (up to iso) positive face structures S of dimension at most n, is a bijection. In other words, any cell in $\bar{P}$ has a unique description.

Proof. Ad 1. We have to verify that $\overline{P}$ satisfy the laws of ω-categories and that it is free in the appropriate sense.

Laws ω-categories are left for the reader. We shall show that $\overline{P}$ is free in the appropriate sense.

Let C be an ω-category, $g_{<n} : P_{<n} \to C_{<n}$ and $(n-1)$-functor and $g_n : |P|_n \to C_n$ a function so that the diagram

$$\begin{array}{ccc} |P|_n & \xrightarrow{\quad g_n \quad} & C_n \\ d \downarrow \downarrow c & & d \downarrow \downarrow c \\ P_{n-1} & \xrightarrow[\quad g_{n-1} \quad]{} & C_{n-1} \end{array}$$

commutes serially. We shall define an n-functor $\overline{g} : \overline{P} \to C$ extending g and g_n. For $x = (S, f) \in \overline{P}_n$ we put

$$\overline{g}_n(x) = \begin{cases} 1_{g_{n-1} \circ f_{n-1}(S)} & \text{if } dim(S) < n, \\ g_n \circ f_n(m_S) & \text{if } dim(S) = n, \ S \text{ is principal, } S_n = \{m_S\} \\ \overline{g}_n(x^{\downarrow a});_k \overline{g}_n(x^{\uparrow a}) & \text{if } dim(S) = n, \ a \in Sd(S)_k \end{cases}$$

We need to check that $\overline{g}$ is well defined, unique one that extends g, preserves domains, codomains, compositions and identities.

All these calculations are similar, and they are very much like those in the proof of Proposition 9.1. We shall check, assuming that we already know that $\overline{g}$ is well defined, and preserves identities that compositions are preserved. So let T, T_1, T_2 be positive face structures such, that $T = T_1 +_k T_2$. Since $\overline{g}$ preserves identities, we can restrict to the case $dim(T_1), dim(T_2) > k$.

Fix $a \in \mathbf{c}^{(k)}(T_1)_k \cap \gamma(T_1)$. So $a \in Sd(T)_k$. If $T_1 = T^{\downarrow a}$ and $T_2 = T^{\uparrow a}$ then we have

$$\overline{g}(T) = \overline{g}(T^{\downarrow a});_k \overline{g}(T^{\uparrow a}) = \overline{g}(T_1);_k \overline{g}(T_2).$$

If $a \in Sd(T_1)_k$ then

$$\begin{array}{r} \overline{g}(T) = \overline{g}(T^{\downarrow a});_k \overline{g}(T^{\uparrow a}) = \\ = \overline{g}(T^{\downarrow a});_k (\overline{g}(T_1^{\uparrow a});_k \overline{g}(T_2)) = \\ = (\overline{g}(T_1^{\downarrow a});_k \overline{g}(T_1^{\uparrow a}));_k \overline{g}(T_2) = \\ \overline{g}(T_1);_k \overline{g}(T_2) \end{array}$$

The remaining verifications are similar.

Ad 2. Let $! : \overline{P} \longrightarrow \mathcal{T}$ be the unique computad map into the terminal object, S a positive face structure such that $dim(S) = l \leq n$, $f : S^{\sharp,n} \longrightarrow P$ a cell in $\overline{P}_n$.

If $l < n$ then by induction we have $!_n(f) = S$. If $l = n$ and S is principal then we have, by induction

$$!_n(d(f) : (\mathbf{d}S)^{\sharp,n} \to P) = \mathbf{d}S, \qquad !_n(c(f) : (\mathbf{c}S)^{\sharp,n} \to P) = \mathbf{c}S.$$

As f is an indet in $\overline{P}$, $!_n(f)$ is a principal positive face structure. But the only (up to an iso) principal positive face structure B such that

$$\mathbf{d}B = \mathbf{d}S, \qquad \mathbf{d}B = \mathbf{d}S$$

is S itself. Thus, in this case, $!_n(f) = S$.

Now assume that $l = n$, and S is not principal, and that for positive face structures T of smaller size than S the statement holds. Let $a \in Sd(S)_k$. We have

$$!_n(f) = !_n(f^{\downarrow a};_k f^{\uparrow a}) = !_n(f^{\downarrow a});_k !_n(f^{\uparrow a}) = S^{\downarrow a};_k S^{\uparrow a} = S$$

where $f^{\downarrow a} = f \circ (\kappa^{\downarrow a})^{\sharp,n}$ and $f^{\uparrow a} = f \circ (\kappa^{\uparrow a})^{\sharp,n}$ and $\kappa^{\downarrow a}$ and $\kappa^{\uparrow a}$ are the maps as in the following pushout

$$\begin{array}{ccc} S^{\downarrow a} & \xrightarrow{\kappa^{\downarrow a}} & S \\ \uparrow & & \uparrow \kappa^{\uparrow a} \\ \mathbf{c}^{(k)}S & \longrightarrow & S^{\uparrow a} \end{array}$$

Ad 3. The main thing is to show that $\overline{h}$ preserves compositions. This follows from the fact that the functor

$$(-)^{\sharp,n} : \mathbf{Fs}_n^{+/1} \longrightarrow \mathbf{Comma}_n^{+/1}$$

preserves special pullbacks.

Ad 4. This is an immediate consequence of 3.

Ad 5. Let S be a positive face structure S of dimension at most n. To prove 5., we are going to use the description of the n-cells in positive-to-one computads given in 1. Moreover, note that by 3. and Lemma 11.2 we have that for $T \in S_k^*$, the value of g at T is a map in $\mathbf{Comma}_k^{+/1}$, such that $g_k(T) : T^{\sharp,k} \longrightarrow P^{\natural,k}$, i.e. the domain of $g_k(T)$ is necessarily $T^{\sharp,k}$.

The implication $\Rightarrow$ is obvious. So assume that $g, h : S^* \longrightarrow P$ are different computad maps. Then there is $k \leq n$ and $x \in S_k$ such that $g_k([x]) \neq h_k([x])$. We shall show, by induction on size of T, that for any $T \in S^*_l$, such that $x \in T$, we have

$$g_k(T) \neq h_k(T) \tag{6}$$

$T = [x]$ has the least size among those positive face structures that contain x. Clearly, (6) holds in this case by assumption.

Suppose that (6) holds for all $U \in S^*_{l'}$ whenever for $l' < l$ and $x \in U$. Suppose that $T = [y]$ for some $y \in S_l$, and $x \in [y]$. Then either $x \in \mathbf{d}[y]$ or $x \in \mathbf{c}[y]$. In the former case we have, by inductive hypothesis, that $g_k(\mathbf{d}T) \neq h_k(\mathbf{d}T)$.Thus

$$d(g_k(T)) = g_k(\mathbf{d}T) \neq h_k(\mathbf{d}T) = d(g_k(T))$$

But then (6) holds as well. The later case ($x \in \mathbf{c}[y]$) is similar.

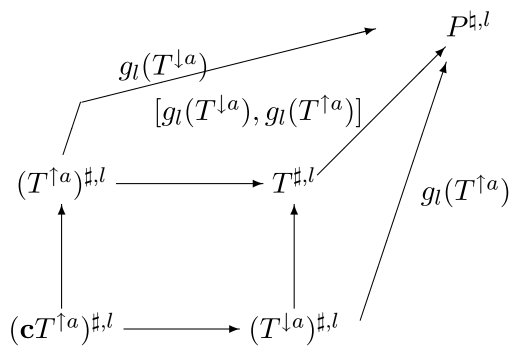
Now suppose that T is not principal $x \in T$ and that for U of a smaller size with $x \in U$ the condition (6) holds. Let $a \in Sd(T)_r$. Then either $x \in T^{\downarrow a}$ or $x \in T^{\uparrow a}$. Both cases are similar, so we will consider the first one only. Thus, as $T^{\downarrow a}$ has a smaller size than T, by inductive hypothesis we have

$$g_k(T^{\downarrow a}) \neq h_k(T^{\downarrow a}) \tag{7}$$

As the compositions in P are calculated via pushouts we have that

$$g_l(T^{\downarrow a});_r g_l(T^{\uparrow a}) = [g_l(T^{\downarrow a}), g_l(T^{\uparrow a})]$$

where $[g_l(T^{\downarrow a}), g_l(T^{\uparrow a})]$ is the unique morphism from the pushout as in the following diagram:



Similarly

$$h_l(T^{\downarrow a});_r h_l(T^{\uparrow a}) = [h_l(T^{\downarrow a}), h_l(T^{\uparrow a})]$$

As morphism form the pushout are equal if and only if their both components are equal we have

$$g_l(T) = g_l(T^{\downarrow a};_r T^{\uparrow a}) = g_l(T^{\downarrow a});_r g_l(T^{\uparrow a}) =$$
$$[g_l(T^{\downarrow a}), g_l(T^{\uparrow a})] \neq [h_l(T^{\downarrow a}), h_l(T^{\uparrow a})] =$$
$$= h_l(T^{\downarrow a});_r h_l(T^{\uparrow a}) = h_l(T^{\downarrow a};_r T^{\uparrow a}) = h_l(T)$$

Thus (6) holds for all $T \in S^*$ such that $x \in T$. As $x \in S$, we get that

$$g_n(S) \neq h_n(S)$$

as required.

Ad 6. Fix a positive face structure S of dimension at most n.

Let $f : S^{\sharp,n} \longrightarrow P$ be a cell in $\overline{P}_n$. By 4, we have

$$\overline{f}_n(S) = f \circ (i_S)^{\sharp,n} = f \circ (1_S)^{\sharp,n} = f \circ (1_S^{\sharp,n}) = f.$$

Let $g : S^* \longrightarrow \overline{P}$ be a computad map. To show that $g = \overline{g_n(S)}$, by 5, it is enough to show that

$$(\overline{g_n(S)})_n(S) = g_n(S).$$

Using 4 again, we have,

$$(\overline{g_n(S)})_n(S) = g_n(S) \circ (i_S)^{\sharp,n} =$$

$$= g_n(S) \circ i_{S^{\sharp,n}} = g_n(S) \circ 1_{S^{\sharp,n}} = g_n(S).$$

Thus, by 5, $(\overline{g_n(S)}) = g$.

Ad 7. It follows immediately from 6. □

From the Proposition 12.1.7 we know that each cell in a positive-to-one computad has (up to an isomorphism) a unique description. The following Proposition is a bit more specific.

Proposition 12.2 *Let P be a positive-to-one computad, $n \in \omega$, and $a \in P_n$. Let T_a be $!^P_n(a)$ (where $!^P : P \longrightarrow \mathcal{T}$ is the unique morphism into the terminal computad). Then there is a unique computad map $\tau_a : T_a^* \longrightarrow P$ such that $(\tau_a)_n(T_a) = a$, i.e. each cell has an essentially unique description. Moreover, we have:*

1. *for any $a \in P$ we have*

$$\tau_{da} = d(\tau_a) = \tau_{da} = \tau_a \circ (\mathbf{d}_{T_a})^*, \qquad \tau_{c(a)} = c(\tau_a) = \tau_{c(a)} = \tau_a \circ (\mathbf{c}_{T_a})^*,$$

$$\tau_{1_a} = \tau_a$$

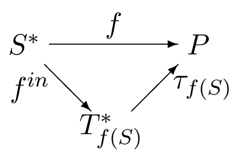
2. *for any $a, b \in P$ such that $c^{(k)}(a) = d^{(k)}(b)$ we have*

$$\tau_{a;_k b} = [\tau_a, \tau_b] : T_a^* +_{\mathbf{c}^{(k)} T_a^*} T_b^* \longrightarrow P,$$

3. *for any positive face structure S, for any computad map $f : S^* \longrightarrow P$,*

$$\overline{\tau_{f_n(S)}} = f.$$

4. *for any positive face structure S, any ω-functor $f : S^* \longrightarrow P$ can be essentially uniquely factorized as*



where f^{in} is an inner map and $(\tau_{f(S)}, T_{f(S)})$ is the description of the cell $f(S)$.

Proof. Using the above description of the positive-to-one computad P we have that $a : (T_a)^{\sharp,n} \longrightarrow P^{\natural,n}$. We put $\tau_a = \overline{a}$. By Proposition 12.1 point 6, we have that $(\tau_a)_n(T_a) = \overline{a}_n(T_a) = a$, as required.

The uniqueness of (T_a, τ_a) follows from Proposition 12.1 point 5.

The remaining part is left for the reader. □

13 Positive-to-one computads form a presheaf category

In this section we want to prove that the category $\mathbf{Comp}^{+/1}$ is equivalent to the presheaf category $Set^{(\mathbf{pFs}^{+/1})^{op}}$. In fact, we will show that both categories are equivalent to the category $sPb((\mathbf{Fs}^{+/1})^{op}, Set)$ of special pullbacks preserving functors from $(\mathbf{Fs}^{+/1})^{op}$ to Set.

First note that the inclusion functor $\mathbf{i} : \mathbf{pFs}^{+/1} \longrightarrow \mathbf{Fs}^{+/1}$ induces the adjunction

$$Set^{(\mathbf{pFs}^{+/1})^{op}} \underset{\mathbf{i}^*}{\overset{Ran_{\mathbf{i}}}{\rightleftarrows}} Set^{(\mathbf{Fs}^{+/1})^{op}}$$

with $\mathbf{i}^*$ is the functor of composing with $\mathbf{i}$ and $Ran_{\mathbf{i}}$ is a the right Kan extension along $\mathbf{i}$. Recall that for F in $Set^{(\mathbf{pFs}^{+/1})^{op}}$, S in $\mathbf{Fs}^{+/1}$, it is defined as the following limit

$$(Ran_{\mathbf{i}}F)(S) = LimF \circ \Sigma^{S,op}$$

where $\Sigma^{S,op}$ is the dual of the functor Σ^S defined before Lemma 6.4. Note that as $(\mathbf{pFs}^{+/1} \downarrow S)^{op} = S \downarrow (\mathbf{pFs}^{+/1})^{op}$ we have

$$\Sigma^{S,op} : S \downarrow (\mathbf{pFs}^{+/1})^{op} \longrightarrow (\mathbf{pFs}^{+/1})^{op}.$$

As $\mathbf{i}$ is full and faithful the right Kan extension $Ran_{\mathbf{i}}(F)$ is an extension. Therefore the counit of this adjunction

$$\varepsilon_F : (Ran_{\mathbf{i}}\, F) \circ \mathbf{i} \longrightarrow F$$

is an isomorphism. The functor $Ran_{\mathbf{i}}F$ is so defined that it preserves special limits. Hence by Lemma 6.4 it preserves special pullbacks. It is easy to see, that for G in $Set^{(\mathbf{Fs}^{+/1})^{op}}$ the unit of adjunction

$$\eta_G : G \longrightarrow Ran_{\mathbf{i}}(G \circ \mathbf{i})$$

is an isomorphism iff G preserves special pullbacks. Thus we have

Proposition 13.1 *The above adjunction restricts to the following equivalence of categories*

$$Set^{(\mathbf{pFs}^{+/1})^{op}} \underset{\mathbf{i}^*}{\overset{Ran_{\mathbf{i}}}{\rightleftarrows}} sPb((\mathbf{Fs}^{+/1})^{op}, Set)$$

□

Now we will set up the adjunction

$$sPb((\mathbf{Fs}^{+/1})^{op}, Set) \underset{\widehat{(-)} = \mathbf{Comp}^{+/1}((\simeq)^*, -)}{\overset{\widetilde{(-)}}{\rightleftarrows}} \mathbf{Comp}^{+/1}$$

which will turn out to be an equivalence of categories. The functor $\widehat{(-)}$ sends a positive-to-one computad P to a functor

$$\widehat{P} = \mathbf{Comp}^{+/1}((-)^*, P) : (\mathbf{Fs}^{+/1})^{op} \longrightarrow Set$$

$\widehat{(-)}$ is defined on morphism in the obvious way, by composition. We have

Lemma 13.2 *Let P be a positive-to-one computad. Then $\widehat{P}$ defined above is a special pullbacks preserving functor.*

Proof. This is an immediate consequence of the fact that the functor $(-)^*$ preserves special pushouts. $\Box$

Now suppose we have a special pullbacks preserving functor $F : (\mathbf{Fs}^{+/1})^{op} \longrightarrow Set$. We shall define a positive-to-one computad $\widetilde{F}$.

As n-cells of $\widetilde{F}$ we put

$$\widetilde{F}_n = \coprod_S F(S)$$

where the coproduct is taken over all[2] (up to iso) positive face structures S of dimension at most n.

If $k \leq n$, the identity map

$$1^{(n)} : \widetilde{F}_k \longrightarrow \widetilde{F}_n$$

is the obvious embedding induced by identity maps on the components of the coproducts.

Now we shall describe the domains and codomains in $\widetilde{F}$. Let S be a positive face structure of dimension at most n, $a \in F(S) \hookrightarrow \widetilde{F}_n$. We have in $\mathbf{Fs}^{+/1}$ the k-th domain and the k-th codomain morphisms:

$$\begin{array}{ccccc} & & S & & \\ & {\scriptstyle \mathbf{d}^{(k)}_S} \nearrow & & \nwarrow {\scriptstyle \mathbf{c}^{(k)}_S} & \\ \mathbf{d}^{(k)}S & & & & \mathbf{c}^{(k)}S \end{array}$$

We put

$$d^{(k)}(a) = F(\mathbf{d}^{(k)}_S)(a) \in F(\mathbf{d}^{(k)}S) \hookrightarrow \widetilde{F}_k,$$

$$c^{(k)}(a) = F(\mathbf{c}^{(k)}_S)(a) \in F(\mathbf{c}^{(k)}S) \hookrightarrow \widetilde{F}_k.$$

Finally, we define the compositions in $\widetilde{F}$. Let $n_1, n_2 \in \omega$, $n = max(n_1, n_2)$, $k < min(n_1, n_2)$, and

$$a \in F(S) \hookrightarrow \widetilde{F}_{n_1} \qquad b \in F(T) \hookrightarrow \widetilde{F}_{n_2},$$

such that

$$c^{(k)}(a) = F(\mathbf{c}^{(k)}_S)(a) = F(\mathbf{d}^{(k)}_T)(b) = d^{(k)}(b).$$

We shall define the cell $a;_k b \in \widetilde{F}_n$. We take a special pushout in $\mathbf{Fs}^{+/1}$:

$$\begin{array}{ccc} S & \xrightarrow{\kappa_1} & S +_k T \\ {\scriptstyle \mathbf{c}^{(k)}_S} \uparrow & & \uparrow {\scriptstyle \kappa_2} \\ \mathbf{c}^{(k)}S & \xrightarrow[\mathbf{d}^{(k)}_T]{} & T \end{array}$$

As F preserves special pullbacks (from $(\mathbf{Fs}^{+/1})^{op}$) it follows that the square

[2] In fact, we think about such a coproduct $\coprod_S F(S)$ as if it were to be taken over sufficiently large (so that each isomorphism type of positive face structures is represented) set of positive face structures S of dimension at most n. Then, if positive face structures S and S' are isomorphic via (necessarily unique) isomorphism h, then the cells $x \in F(S)$ and $x' \in F(S')$ are considered equal iff $F(h)(x) = x'$.

$$\begin{array}{ccc} F(S) & \xleftarrow{F(\kappa_1)} & F(S +_k T) \\ {\scriptstyle F(\mathbf{c}_S^{(k)})}\downarrow & & \downarrow{\scriptstyle F(\kappa_2)} \\ F(\mathbf{c}^{(k)}S) & \xleftarrow[F(\mathbf{d}_T^{(k)})]{} & F(T) \end{array}$$

is a pullback in Set. Thus there is a unique element

$$x \in F(S +_k T) \hookrightarrow \widetilde{F}_n$$

such that

$$F(\kappa_1)(x) = a, \qquad F(\kappa_1)(x) = b.$$

We put

$$a;_k b = x.$$

This ends the definition of $\widetilde{F}$.

For a morphism $\alpha : F \longrightarrow G$ in $sPb((\mathbf{Fs}^{+/1})^{op}, Set)$ we put

$$\widetilde{\alpha} = \{\widetilde{\alpha}_n : \widetilde{F}_n \longrightarrow \widetilde{G}_n\}_{n\in\omega}$$

such that

$$\widetilde{\alpha}_n = \coprod_S \alpha_S : \widetilde{F}_n \longrightarrow \widetilde{G}_n$$

where the coproduct is taken over all (up to iso) positive face structures S of dimension at most n. This ends the definition of the functor $\widetilde{(-)}$.

We have

Proposition 13.3 *The functor*

$$\widetilde{(-)} : sPb((\mathbf{Fs}^{+/1})^{op}, Set) \longrightarrow \mathbf{Comp}^{+/1}$$

is well defined.

Proof. The verification that $\widetilde{(-)}$ is a functor into ωCat is left for the reader. We shall verify that for any special pullbacks preserving functor $F : {\mathbf{Fs}^{+/1}}^{op} \longrightarrow Set$, $\widetilde{F}$ is a positive-to-one computad, whose n-indets are

$$|\widetilde{F}|_n = \coprod_{B\in \mathbf{pFs}^{+/1}, dim(B)=n} F(B) \hookrightarrow \coprod_{S\in \mathbf{Fs}^{+/1}, dim(S)\leq n} F(S) = \widetilde{F}_n.$$

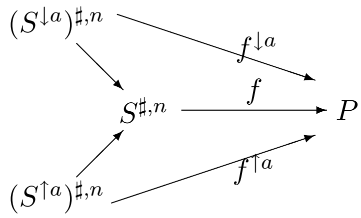
Let P be the truncation of $\widetilde{F}$ in $\mathbf{Comma}_n^{+/1}$, i.e. $P = \widetilde{F}^{\natural,n}$. We shall show that $\widetilde{F}_n$ is in a bijective correspondence with $\overline{P}_n$ described in the previous section. We define a function

$$\varphi : \overline{P}_n \longrightarrow \widetilde{F}_n$$

so that for a cell $f : S^{\sharp,n} \longrightarrow P$ in $\overline{P}_n$ we put

$$\varphi(f) = \begin{cases} 1_{f_{n-1}(S)} & \text{if } dim(S) < n, \\ f_n(m_S) & \text{if } dim(S) = n,\ S \text{ principal},\ S_n = \{m_S\} \\ \varphi(f^{\downarrow a});_k \varphi(f^{\uparrow a}) & \text{if } dim(S) = n,\ a \in Sd(S)_k. \end{cases}$$

and the morphisms in $\varphi(f^{\downarrow a})$ and $\varphi(f^{\uparrow a})$ in $\mathbf{Comma}_n^{+/1}$ are obtained by compositions so that the diagram



commutes. We need to verify, by induction on n, that φ is well defined, bijective and that it preserves compositions, domains, and codomains.

We shall only verify (partially) that φ is well defined, i.e. the definition φ for any non-principal positive face structure S of dimension n does not depend on the choice of the saddle point of S. Let $a, x \in Sd(S)$ so that $k = dim(x) < dim(a) = m$. Using Lemma 8.4 and the fact that $(-)^{\sharp,n}$ preserves special pushouts, we have

$$\begin{aligned}
&\varphi(f^{\downarrow a});_m \varphi(f^{\uparrow a}) = \\
&= (\varphi(f^{\downarrow a\downarrow x});_k \varphi(f^{\downarrow a\uparrow x}));_m (\varphi(f^{\uparrow a\downarrow x});_k \varphi(f^{\uparrow a\uparrow x})) = \\
&= (\varphi(f^{\downarrow a\downarrow x});_m \varphi(f^{\uparrow a\downarrow x}));_k (\varphi(f^{\downarrow a\uparrow x});_m \varphi(f^{\uparrow a\uparrow x})) = \\
&= (\varphi(f^{\downarrow x\downarrow a});_m \varphi(f^{\downarrow x\uparrow a}));_k (\varphi(f^{\uparrow x\downarrow a});_m \varphi(f^{\uparrow x\uparrow a})) = \\
&= \varphi(f^{\downarrow x});_m \varphi(f^{\uparrow x})
\end{aligned}$$

as required in this case. The reader can compare these calculations with the those, in the same case, of Proposition 9.1 (F is replaces by φ and T is replaces by f). So there is no point to repeat the other calculations. $\Box$

For P in $\mathbf{Comp}^{+/1}$ we define a computad map

$$\eta_P : P \longrightarrow \widehat{\widetilde{P}}$$

so that for $x \in P_n$ we put

$$\eta_{P,n}(x) = \tau_x : T_x^* \to P.$$

For F in $sPb((\mathbf{Fs}^{+/1})^{op}, Set)$ we define a natural transformation

$$\varepsilon_F : \widehat{\widetilde{F}} \longrightarrow F,$$

such that, for a positive face structure S of dimension n,

$$(\varepsilon_F)_S : \widehat{\widetilde{F}}(S) \longrightarrow F(S)$$

and $g : S^* \to \widetilde{F} \in \widehat{\widetilde{F}}(S)$ we put

$$(\varepsilon_F)_S(g) = g_n(S).$$

Proposition 13.4 *The functors*

$$sPb((\mathbf{Fs}^{+/1})^{op}, Set) \underset{\widehat{(-)} = \mathbf{Comp}^{+/1}((\simeq)^*, -)}{\overset{\widetilde{(-)}}{\rightleftarrows}} \mathbf{Comp}^{+/1}$$

together with the natural transformations η and ε defined above form an adjunction ($\widehat{(-)} \dashv \widetilde{(-)}$). It establishes the equivalence of categories $sPb((\mathbf{Fs}^{+/1})^{op}, Set)$ and $\mathbf{Comp}^{+/1}$.

Proof. The fact that both η and ε are bijective on each component follows immediately from Proposition 12.1 point 6. So we shall verify the triangular equalities only.

Let P be a computad, and F be a functor in $sPb((\mathbf{Fs}^{+/1})^{op}, Set)$. We need to show that the triangles

$$
\begin{array}{ccccc}
 & & \widehat{\widehat{\widehat{P}}} & & \\
 & \nearrow^{\widehat{\eta_P}} & & \searrow^{\varepsilon_{\widehat{P}}} & \\
\widehat{P} & & \xrightarrow[1_{\widehat{P}}]{} & & \widehat{P}
\end{array}
\qquad
\begin{array}{ccccc}
 & & \widetilde{\widetilde{\widetilde{F}}} & & \\
 & \nearrow^{\eta_{\widetilde{F}}} & & \searrow^{\widetilde{\varepsilon_F}} & \\
\widetilde{F} & & \xrightarrow[1_{\widetilde{F}}]{} & & \widetilde{F}
\end{array}
$$

commute. So let $f : S^* \to P \in \widehat{P}(S)$. Then, we have

$$\varepsilon_{\widehat{P}} \circ \widehat{\eta_P}(f) = \varepsilon_{\widehat{P}}(\eta_P \circ f) = (\eta_P \circ f)_n(S) =$$

$$= (\eta_P)_n(f_n(S)) = \tau_{f_n(S)} = f$$

So let $x \in F(S) \longrightarrow \widetilde{F}_n$. Then we have

$$\widetilde{\varepsilon_F} \circ \eta_{\widetilde{F}}(x) = \widetilde{\varepsilon_F}(\tau_x) = (\tau_x)_n(1_{T_x}) = x$$

So both triangles commutes, as required. □

From Propositions 13.1 and 13.4 we get immediately

Corollary 13.5 *The functor*

$$\widehat{(-)} : \mathbf{Comp}^{+/1} \longrightarrow Set^{(\mathbf{pFs}^{+/1})^{op}}$$

such that for a positive-to-one computad X,

$$\widehat{X} = \mathbf{Comp}^{+/1}((-)^*, X) : (\mathbf{pFs}^{+/1})^{op} \longrightarrow Set$$

is an equivalence of categories.

14 The principal pushouts

Recall the positive face structure α^n from section 9. A *total composition map* is an inner ω-functor whose domain is of form $(\alpha^n)^*$, for some $n \in \omega$. If S is a positive face structure of dimension n, then the total composition of S (in fact S^*) is denoted by

$$\mu^{S^*} : \alpha^{n,*} \longrightarrow S^*.$$

It is uniquely determined by the condition $\mu_n^{S^*}(\alpha^n) = S$. We have

Proposition 14.1 *Let N be a normal positive face structure. With the notation as above, the square*

$$
\begin{array}{ccc}
N^* & \xrightarrow{\mathbf{d}^*_{N^\bullet}} & N^{\bullet,*} \\
\uparrow{\scriptstyle \boldsymbol{\mu}^{N^*}} & & \uparrow{\scriptstyle \boldsymbol{\mu}^{N^{\bullet,*}}} \\
\alpha^{n,*} & \xrightarrow[\mathbf{d}^*_{\alpha^{n+1}}]{} & \alpha^{n+1,*}
\end{array}
$$

is a pushout in $\mathcal{C}types_\omega^{+/1}$. Such pushouts are called principal pushouts.

Proof. This is an easy consequence of Proposition 7.3, particularly point 4. □

From the above proposition we immediately get

Corollary 14.2 *If $n > 0$ and P is a principal positive face structure of dimension n then the square*

$$
\begin{array}{ccc}
\mathbf{d}P^* & \xrightarrow{\ \mathbf{d}^*_P\ } & P^* \\
\Big\uparrow{\scriptstyle \boldsymbol{\mu}^{\mathbf{d}P}} & & \Big\uparrow{\scriptstyle \boldsymbol{\mu}^{P}} \\
\alpha^{n-1,*} & \xrightarrow[\ \mathbf{d}_{\alpha^{n,*}}\ ]{} & \alpha^{n,*}
\end{array}
$$

is a (principal) pushout in $\mathcal{C}types^{+/1}_\omega$. $\Box$

Theorem 14.3 (V.Harnik) [3] *Let* $F : (\mathcal{C}types^{+/1}_\omega)^{op} \longrightarrow Set$ *be a special pullback preserving functor. Then* F *preserves the principal pullbacks as well.*

Theorem 14.3 is a special case of Lemma 14.6, for $k = n-1$.

The proof of the above theorem will be divided into three Lemmas. Theorem 14.3 is a special case of Lemma 14.6, for $k = n-1$.

Before we even formulate these Lemmas we need to introduce some constructions on positive face structures and define some ω-functors between computypes. Introducing these constructs and notation for them we shall make some comments how they are going to be interpreted by special pullback preserving morphisms from $(\mathcal{C}types^{+/1}_\omega)^{op}$ to Set.

Fix $k \leq n$, and a P principal positive face structure of dimension n. We say that P is k-*globular* iff $\mathbf{d}^{(l)}P$ is principal, for $k \leq l \leq n$, i.e. $\delta^{(l)}(\mathbf{p}_n)$ is a singleton, for $k \leq l \leq n$, where $P_n = \{\mathbf{p}_n\}$. The k-*globularization* $\boxed{k}P$ of P is the k-globular positive face structure of dimension n defined as follows. Note that by Lemma 7.1 $P_l = \delta(\mathbf{p}_{l+1}) \cup \{\mathbf{p}_l\}$ for $0 \leq l < n$. We put

$$\boxed{k}P_l = \begin{cases} \{\mathbf{q}_l, \mathbf{p}_l\} & \text{for } k \leq l < n, \\ P_l & \text{otherwise.} \end{cases}$$

For $x \in \boxed{k}P$,

$$\gamma^{\boxed{k}P}(x) = \begin{cases} \mathbf{p}_{l-1} & \text{if } x = \mathbf{q}_l \text{ for some } k \leq l < n, \\ \gamma^P(x) & \text{otherwise.} \end{cases}$$

and

$$\delta^{\boxed{k}P}(x) = \begin{cases} \mathbf{q}_l & \text{if } x \in \boxed{k}P_{l+1} \text{ for some } k < l < n, \\ \delta^P(\mathbf{p}_k) & \text{if } x \in \boxed{k}P_k, \\ \delta^P(x) & \text{otherwise.} \end{cases}$$

Note that $\boxed{n}P$ is P itself. Thus the elements of the shape $\boxed{k}P^*$ are k-globularized versions of the elements of the shape P^*. As the following positive face structures

$$\mathbf{c}^{(k)}P \;\cong\; \mathbf{c}^{(k)}\boxed{k}P \;\cong\; \mathbf{c}\mathbf{c}^{(k+1)}\boxed{k}P \;\cong\; \mathbf{d}\mathbf{c}^{(k+1)}\boxed{k}P \;\cong\; \mathbf{d}^{(k+1)}\boxed{k}P$$

are isomorphic, we can form the following special pushouts

$$
\begin{array}{ccc}
\mathbf{c}^{(k+1)}P^* & \xrightarrow{\ \kappa_1\ } & \mathbf{c}^{(k+1)}P +_{\mathbf{c}^{(k)}P} \boxed{k}P^* \\
\Big\uparrow{\scriptstyle \mathbf{c}^*_{\mathbf{c}^{(k+1)}P}} & & \Big\uparrow{\scriptstyle \kappa_2} \\
\mathbf{c}^{(k)}P^* & \xrightarrow[\ \mathbf{d}^{(k),*}_{\boxed{k}P}\ ]{} & \boxed{k}P^*
\end{array}
$$

[3] The original statement of V.Harnik is saying that the nerve functor from ω-categories to (all) computads is monadic. However, in the present context the argument given by V.Harnik is directly proving the present statement, i.e. that the principal pullbacks are preserved whenever the special one's are. This, statement is used to show that the category of ω-categories is equivalent to the category of special pullback preserving functors from $(\mathcal{C}types^{+/1})^{op}$ to Set, c.f. Corollary 15.2. From that statement, the monadicity of the nerve functor is an easy corollary, c.f. Theorem 16.7. In the remainder of this section the Harnik's argument, adopted to the present context, is presented.

and

$$\begin{array}{ccc} \mathbf{c}^{(k+1)}\boxed{k}P^* & \xrightarrow{\kappa'_1} & \mathbf{c}^{(k+1)}\boxed{k}P +_{\mathbf{c}^{(k)}P} \boxed{k}P^* \\ \uparrow{\scriptstyle \mathbf{c}^*_{\mathbf{c}^{(k+1)}\boxed{k}P}} & & \uparrow{\scriptstyle \kappa'_2} \\ \mathbf{c}^{(k)}P^* & \xrightarrow[\mathbf{d}^{(k),*}_{\boxed{k}P}]{} & \boxed{k}P^* \end{array}$$

We describe in details the positive face structures we have just defined:

$$\boxed{k}P, \quad P' = \mathbf{c}^{(k+1)}P +_{\mathbf{c}^{(k)}P} \boxed{k}P, \quad \text{and} \quad \mathrm{P}'' = \mathbf{c}^{(\mathrm{k}+1)}\boxed{k}\mathrm{P} +_{\mathbf{c}^{(\mathrm{k})}\mathrm{P}} \boxed{k}\mathrm{P},$$

for a $(k+1)$-globular positive face structures P, i.e. in case P is equal to $\boxed{k+1}P$.

dim	P	$\boxed{k}P$	P'	P''
n	$\{\mathbf{p}_n\}$	$\{\mathbf{p}_n\}$	$\{\mathbf{p}_n\}$	$\{\mathbf{p}_n\}$
n–1	$\{\mathbf{q}_{n-1}, \mathbf{p}_{n-1}\}$	$\{\mathbf{q}_{n-1}, \mathbf{p}_{n-1}\}$	$\{\mathbf{q}_{n-1}, \mathbf{p}_{n-1}\}$	$\{\mathbf{q}_{n-1}, \mathbf{p}_{n-1}\}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
k+1	$\{\mathbf{q}_{k+1}, \mathbf{p}_{k+1}\}$	$\{\mathbf{q}_{k+1}, \mathbf{p}_{k+1}\}$	$\{\mathbf{r}_{k+1}, \mathbf{q}_{k+1}, \mathbf{p}_{k+1}\}$	$\{\mathbf{r}_{k+1}, \mathbf{q}_{k+1}, \mathbf{p}_{k+1}\}$
k	$\delta(\mathbf{p}_{k+1}) \cup \{\mathbf{p}_k\}$	$\{\mathbf{q}_k, \mathbf{p}_k\}$	$\delta(\mathbf{p}_{k+1}) \cup \{\mathbf{q}_k, \mathbf{p}_k\}$	$\{\mathbf{r}_k, \mathbf{q}_k, \mathbf{p}_k\}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
l	P_l	P_l	P_l	P_l

$0 \leq l < k$. The functions γ and δ in both P' and P'' are easy to figure out. We give few less obvious values below. In P'

$$\delta^{P'}(\mathbf{r}_{k+1}) = \delta^P(\mathbf{p}_{k+1}), \;\; \delta^{P'}(\mathbf{q}_{k+1}) = \delta^{P'}(\mathbf{p}_{k+1}) = \mathbf{q}_k$$

$$\gamma^{P'}(\mathbf{r}_{k+1}) = \mathbf{q}_k, \;\; \gamma^{P'}(\mathbf{q}_{k+1}) = \gamma^{P'}(\mathbf{p}_{k+1}) = \mathbf{p}_k,$$

and in P''

$$\delta^{P''}(\mathbf{r}_{k+1}) = \mathbf{r}_k.$$

Now we shall define some ω-functors between computypes. To describe their meaning let us fix a special pullback preserving functors from $F : (\mathcal{C}types_\omega^{+/1})^{op} \longrightarrow Set$.

The ω-functors denoted by letter $\boldsymbol{\mu}$ are interpreted as operation that 'globularize' cells. We have two of them. The first one $\boldsymbol{\mu}^{S^*} : \alpha^{n,*} \longrightarrow S^*$ was already introduced at the beginning of this section for any positive face structure S. The second is the ω-functor

$$\boldsymbol{\mu}_{\boxed{k}} : \boxed{k}P^* \longrightarrow \boxed{k+1}P^*$$

such that

$$\boldsymbol{\mu}_{\boxed{k}}(X) = \begin{cases} (X - \{\mathbf{q}_k\}) \cup \delta(\mathbf{p}_{k+1}) & \text{if } \mathbf{q}_k \in X \\ X & \text{otherwise.} \end{cases}$$

for $X \in \boxed{k}P^*$, and

The fact that these operations are interpreted as globularization of cells can be explained as follows. The function

$$F(\boldsymbol{\mu}_{\boxed{k}}) : F(\boxed{k+1}P^*) \longrightarrow F(\boxed{k}P^*),$$

takes a $(k+1)$-globular n-cell $a \in F(\boxed{k+1}P^*)$ and returns a k-globular n-cell $F(\boldsymbol{\mu}_{\boxed{k}})(a) \in F(\boxed{k}P^*)$. Intuitively, $F(\boldsymbol{\mu}_{\boxed{k}})$ is composing the k-domain of a leaving the rest 'untouched'. So it is a 'one-step globularization'. On the other hand, the function

$$F(\boldsymbol{\mu}^{S^*}) : F(S^*) \longrightarrow F(\alpha^{n,*})$$

is taking an n-cell b of an arbitrary shape S^* and is returning a globular n-cell. This time $F(\boldsymbol{\mu}^{S^*})$ is composing all the domains and codomains in the cell b as much as possible, so that there is nothing left to be compose. This is the 'full globularization'. Instead of writing $F(\boldsymbol{\mu}_{\boxed{k}})(a)$ and $F(\boldsymbol{\mu}^{S^*})(b)$ we write $\boldsymbol{\mu}_{\boxed{k}}(a)$ and $\boldsymbol{\mu}^{S^*}(b)$ or even $\boldsymbol{\mu}(a)$ and $\boldsymbol{\mu}(b)$ if it does not lead to confusions. The same conventions will apply to the other operations that we introduce below.

We need a separate notation for the ω-functor $\boldsymbol{\mu}_k : \mathbf{c}^{(k+1)}\boxed{k}P^* \longrightarrow \mathbf{c}^{(k+1)}P^*$ such that

$$\boldsymbol{\mu}_k(X) = \begin{cases} \mathbf{c}^{(k+1)}P & \text{if } X = \mathbf{c}^{(k+1)}\boxed{k}P \\ \mathbf{c}^{(k)}P & \text{if } X = \mathbf{c}^{(k)}\boxed{k}P \\ X & \text{otherwise.} \end{cases}$$

for $X \in \mathbf{c}^{(k+1)}\boxed{k}P^*$. It is a version of $\boldsymbol{\mu}_{\boxed{k}}$.

The ω-functor

$$\boldsymbol{\iota}_P : P^* \longrightarrow \mathbf{d}P^*$$

is given by

$$\boldsymbol{\iota}_P(X) = \begin{cases} \mathbf{d}P & \text{if } \mathbf{c}P \subseteq X \\ X & \text{otherwise.} \end{cases}$$

for $X \in P^*$. $\boldsymbol{\iota}_P$ is a kind of degeneracy map and it is interpreted as 'a kind of identity'. For a given $(n-1)$-cell t of the shape $\mathbf{d}P^*$, $\boldsymbol{\iota}(t)$ is 'identity on t' but with the codomain composed. This is why we will write $\boldsymbol{\iota}_t$ rather than $\boldsymbol{\iota}(t)$.

The ω-functor

$$\boldsymbol{\beta}_k : \mathbf{c}^{(k)}P \longrightarrow \mathbf{d}^{(k)}P$$

such that

$$\boldsymbol{\beta}_k(X) = \begin{cases} \mathbf{d}^{(k)}P & \text{if } X = \mathbf{c}^{(k)}P \\ X & \text{otherwise.} \end{cases}$$

for $X \in P^*$, is the operation of 'composition of all the cells at the top' leaving the rest untouched. Clearly we have $\boldsymbol{\beta}_{n-1} = \mathbf{d}^*_P; \boldsymbol{\iota}_P$.

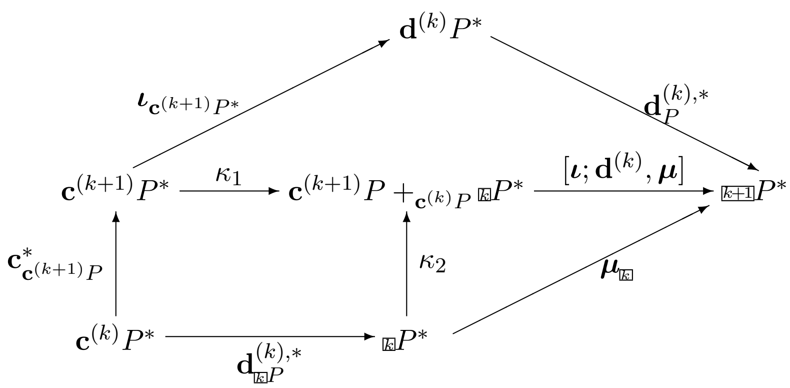
The following two ω-functors

$$[\boldsymbol{\iota}_{\mathbf{c}^{(k+1)}P}; \mathbf{d}^{(k),*}_P, \boldsymbol{\mu}_P] : \mathbf{c}^{(k+1)}P +_{\mathbf{c}^{(k)}P} \boxed{k}P^* \longrightarrow P^*$$

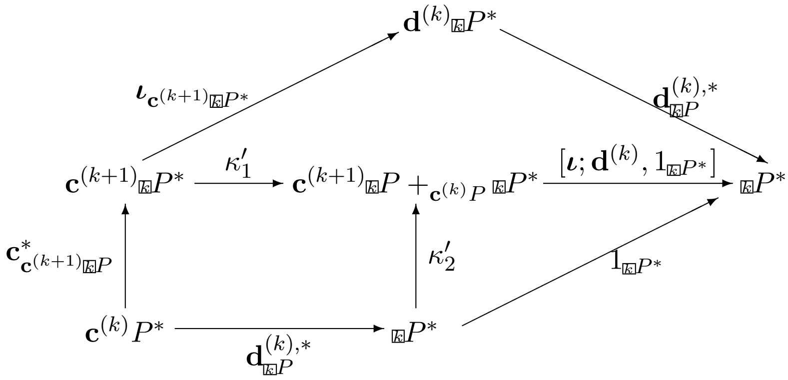
and

$$[\boldsymbol{\iota}_{\mathbf{c}^{(k+1)}\boxed{k}P}; \mathbf{d}^{(k),*}_{\boxed{k}P}, 1_{\boxed{k}P}] : \mathbf{c}^{(k+1)}\boxed{k}P +_{\mathbf{c}^{(k)}P} \boxed{k}P^* \longrightarrow \boxed{k}P^*$$

are defined as the unique ω-functors making the following diagrams



and



commute.

Finally, we introduce two maps that are a kind of a binary composition but with a wiskering. The first

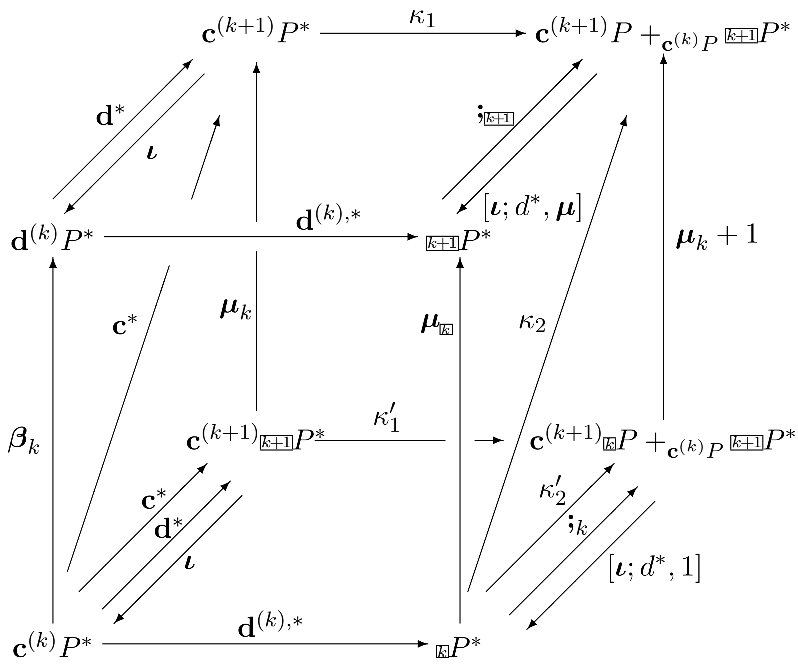
$$;_{\boxed{k+1}} : \boxed{k+1}P^* \longrightarrow \mathbf{c}^{(k+1)}P +_{\mathbf{c}^{(k)}P} \boxed{k}P$$

is given by

$$;_{\boxed{k+1}}(X) = \begin{cases} X \cup \{\mathbf{r}_{k+1}, \mathbf{q}_k\} & \text{if } X \cap \{\mathbf{q}_{k+1}, \mathbf{p}_{k+1}\} \neq \emptyset \\ X & \text{otherwise.} \end{cases}$$

for $X \in \boxed{k+1}P^*$. The other is $;_k : \boxed{k}P^* \longrightarrow \mathbf{c}^{(k+1)}\boxed{k}P +_{\mathbf{c}^{(k)}P} \boxed{k}P$ with the same defining formula, for $X \in \boxed{k}P^*$.

In the following diagram all the morphisms that we introduced above are displayed. Most of the subscripts of the morphisms are suppressed for clarity of the picture.



The above cube contains two special pushouts mentioned above. The following Lemma describe some other commutations.

Lemma 14.4 *With the notation as above we have:*

1. $\mathbf{c}^*_{\mathbf{c}^{(k+1)}P}; \iota_{\mathbf{c}^{(k+1)}P^*} = \boldsymbol{\beta}_k$,

2. $\mathbf{d}^*_{\mathbf{c}^{(k+1)}P}; \kappa_1 = \mathbf{d}^{(k),*}_P; (;_{\boxed{k+1}})$,

3. $\mathbf{d}^*_{\mathbf{c}^{(k+1)}P};\iota_{\mathbf{c}^{(k+1)}P^*} = 1_{\mathbf{d}^{(k)}P^*}$,

4. $\boldsymbol{\mu}_{\boxed{k}};(\boldsymbol{;}_{\boxed{k+1}}) = (\boldsymbol{;}_k);(\boldsymbol{\mu}_k + 1_{\boxed{k}P})$,

5. $\iota_{\mathbf{c}^{(k+1)}\boxed{k}P^*};\boldsymbol{\beta}_k = \boldsymbol{\mu}_k;\iota_{\mathbf{c}^{(k+1)}P^*}$,

6. $(\boldsymbol{;}_k);[\boldsymbol{\iota}_{\mathbf{c}^{(k+1)}\boxed{k}P^*};\mathbf{d}^{(k),*}_{\boxed{k}P}, 1_{\boxed{k}P^*}] = 1_{\boxed{k}P^*}$,

7. $(\boldsymbol{;}_{\boxed{k+1}});[\boldsymbol{\iota}_{\mathbf{c}^{(k+1)}P^*};\mathbf{d}^{(k),*}_P, \boldsymbol{\mu}_{\boxed{k}}] = 1_{P^*}$.

Proof. Routine check. □

Lemma 14.5 *Let $F : (\mathcal{C}types^{+/1}_\omega)^{op} \longrightarrow Set$ be a special pullback preserving functor, P a principal positive face structure of dimension n. Then for any $0 \le k < n$, F preserves the pullback in $(\mathcal{C}types^{+/1})^{op}$*

$$\begin{array}{ccc}
\mathbf{d}^{(k)}P^* & \xrightarrow{\mathbf{d}^{(k),*}_{\boxed{k+1}P}} & \boxed{k+1}P^* \\
\uparrow \boldsymbol{\beta}_k & & \uparrow \boldsymbol{\mu}_{\boxed{k}} \\
\mathbf{c}^{(k)}P^* & \xrightarrow[\mathbf{d}^{(k),*}_{\boxed{k}P}]{} & \boxed{k}P^*
\end{array}$$

Proof. Let $F : (\mathcal{C}types^{+/1}_\omega)^{op} \longrightarrow Set$ be a special pullback preserving functor, P a principal positive face structure of dimension n. We need to show that the square

$$\begin{array}{ccc}
F(\mathbf{d}^{(k)}P^*) & \xleftarrow{F(\mathbf{d}^{(k),*}_{\boxed{k+1}P})} & F(\boxed{k+1}P^*) \\
F(\boldsymbol{\beta}_k) \downarrow & & \downarrow F(\boldsymbol{\mu}_{\boxed{k}}) \\
F(\mathbf{c}^{(k)}P^*) & \xleftarrow[F(\mathbf{d}^{(k),*}_{\boxed{k}P})]{} & F(\boxed{k}P^*)
\end{array}$$

is a pullback in Set. Let us fix $t \in F(\mathbf{d}^{(k)}P^*)$ and $a \in F(\mathbf{c}^{(k)}P^*)$ such that $\boldsymbol{\beta}(t) = \mathbf{d}^{(k)}(a)$. We will check that it is a pullback, by showing existence and uniqueness of an element $b \in F(\boxed{k+1}P^*)$ such that $\mathbf{d}^{(k)}(b) = t$ and $\boldsymbol{\mu}(b) = a$.

Existence. By Lemma 14.4.1 we have $\mathbf{c}(\boldsymbol{\iota}_t) = \mathbf{d}^{(k)}(a)$. Since F preserves special pullbacks and $\mathbf{c}^{(k+1)}P +_{\mathbf{c}^{(k)}P} \boxed{k}P$ is a special pullback in $(\mathcal{C}types^{+/1}_\omega)^{op}$ we have an element $(\boldsymbol{\iota}_t, a) \in F(\mathbf{c}^{(k+1)}P +_{\mathbf{c}^{(k)}P} \boxed{k}P)$ such that $\kappa_1(\boldsymbol{\iota}_t, a) = \boldsymbol{\iota}_t$ and $\kappa_2(\boldsymbol{\iota}_t, a) = a$. We shall show that the element $\boldsymbol{\iota}_t\boldsymbol{;}_{\boxed{k+1}}a \in F(\boxed{k+1}P^*)$ is the element we are looking for, i.e. $\mathbf{d}^{(k)}(\boldsymbol{\iota}_t\boldsymbol{;}_{\boxed{k+1}}a) = t$ and $\boldsymbol{\mu}(\boldsymbol{\iota}_t\boldsymbol{;}_{\boxed{k+1}}a) = a$.

Using Lemma 14.4 2 and 3 we have

$$\mathbf{d}^{(k)}(\boldsymbol{\iota}_t\boldsymbol{;}_{\boxed{k+1}}a) = \mathbf{d}(\boldsymbol{\iota}_t) = t$$

and using Lemma 14.4 4, 5, 6 and assumption $\boldsymbol{\beta}(t) = \mathbf{d}^{(k)}(a)$ we have

$$\boldsymbol{\mu}(\boldsymbol{\iota}_t\boldsymbol{;}_{\boxed{k+1}}a) = \boldsymbol{\mu}(\boldsymbol{\iota}_t)\boldsymbol{;}_k a = \boldsymbol{\iota}_{\boldsymbol{\beta}(t)}\boldsymbol{;}_k a = \boldsymbol{\iota}_{\mathbf{d}^{(k)}(a)}\boldsymbol{;}_k a = a$$

This ends the proof of existence.

Uniqueness. Now suppose that we have two elements $b, b' \in F(\boxed{k+1}P^*)$ such that $\mathbf{d}^{(k)}(b) = t = \mathbf{d}^{(k)}(b')$ and $\boldsymbol{\mu}(b) = a = \boldsymbol{\mu}(b)$. Then using Lemma 14.4 7 and the assumption we have

$$b = \boldsymbol{\iota}_{\mathbf{d}^{(k)}(b)}\boldsymbol{;}_{\boxed{k+1}}\boldsymbol{\mu}(b) = \boldsymbol{\iota}_{\mathbf{d}^{(k)}(b')}\boldsymbol{;}_{\boxed{k+1}}\boldsymbol{\mu}(b') = b'.$$

So the element with this property is unique. □

Lemma 14.6 *Let $F : (\mathcal{C}types^{+/1}_\omega)^{op} \longrightarrow Set$ be a special pullback preserving functor, P a principal positive face structure of dimension n. Then for any $0 \leq k < n$, F preserves the following pullback in $(\mathcal{C}types^{+/1}_\omega)^{op}$*

$$\begin{array}{ccc} \mathbf{d}^{(k)}P^* & \xrightarrow{\mathbf{d}^{(k),*}_{\boxed{k+1}P}} & \boxed{k+1}P^* \\ \Big\uparrow{\scriptstyle \boldsymbol{\mu}^{\mathbf{d}^{(k)}P^*}} & & \Big\uparrow{\scriptstyle \boldsymbol{\mu}^{\boxed{k+1}P^*}} \\ \alpha^{k,*} & \xrightarrow[\mathbf{d}^{(k),*}_{\alpha^n}]{} & \alpha^{n,*} \end{array} \tag{8}$$

Proof. The proof is by double induction, on dimension n of the principal positive face structure P, and $k < n$.

Note that if $k = 0$ then, for any $n > 0$, the vertical arrows in (8) are isomorphisms, so any functor form $(\mathcal{C}types^{+/1}_\omega)^{op}$ sends (8) to a pullback. This shows in particular that the Lemma holds for $n = 1$. As we already mentioned, if $k = n-1$, the square (8) is an arbitrary special pushout.

Thus, we assume that $F : (\mathcal{C}types^{+/1}_\omega)^{op} \longrightarrow Set$ is a special pullback preserving functor, and that P a principal positive face structure of dimension n, $0 \leq k < n$, F preserves the pullback (8). Moreover, for $n' < n$ and the principal positive face structure Q of dimension n', F preserves the principal pullback in $(\mathcal{C}types^{+/1}_\omega)^{op}$:

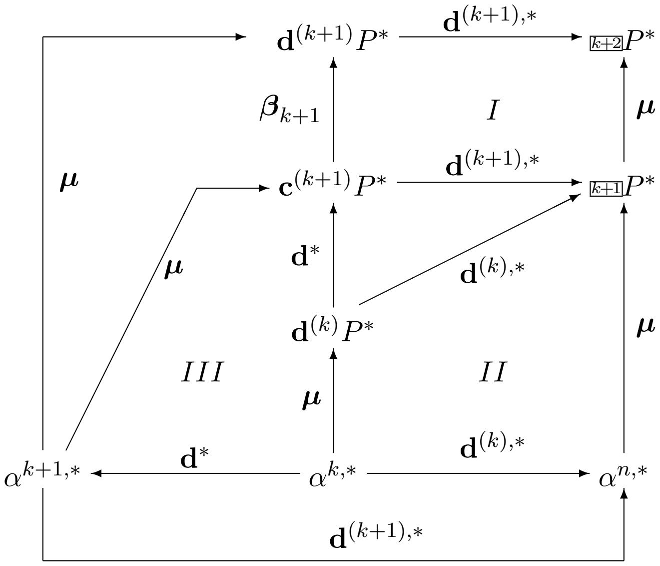
$$\begin{array}{ccc} \mathbf{d}Q^* & \xrightarrow{\mathbf{d}^*_Q} & Q^* \\ \Big\uparrow{\scriptstyle \boldsymbol{\mu}^{\mathbf{d}Q}} & & \Big\uparrow{\scriptstyle \boldsymbol{\mu}^{Q}} \\ \alpha^{n'-1,*} & \xrightarrow[\mathbf{d}^*_{\alpha^{n'}}]{} & \alpha^{n',*} \end{array} \quad = \quad \begin{array}{ccc} \mathbf{d}^{(n'-1)}Q^* & \xrightarrow{\mathbf{d}^{(n'-1),*}_{\boxed{n'}Q}} & \boxed{n'}Q^* \\ \Big\uparrow{\scriptstyle \boldsymbol{\mu}^{\mathbf{d}^{(n'-1)}Q^*}} & & \Big\uparrow{\scriptstyle \boldsymbol{\mu}^{\boxed{n'}Q^*}} \\ \alpha^{n'-1,*} & \xrightarrow[\mathbf{d}^{(n'-1),*}_{\alpha^{n'}}]{} & \alpha^{n',*} \end{array}$$

We shall show that F preserves the pullback $(\mathcal{C}types^{+/1}_\omega)^{op}$

$$\begin{array}{ccc} \mathbf{d}^{(k+1)}P^* & \xrightarrow{\mathbf{d}^{(k+1),*}_{\boxed{k+2}P}} & \boxed{k+2}P^* \\ \Big\uparrow{\scriptstyle \boldsymbol{\mu}^{\mathbf{d}^{(k+1)}P^*}} & & \Big\uparrow{\scriptstyle \boldsymbol{\mu}^{\boxed{k+2}P^*}} \\ \alpha^{k+1,*} & \xrightarrow[\mathbf{d}^{(k+1),*}_{\alpha^n}]{} & \alpha^{n,*} \end{array} \tag{9}$$

as well.

In the following diagram (most of the subscripts and some superscripts were suppressed for clarity):



all the squares and triangles commute. Moreover, F sends the squares I, II, III to pullbacks in Set: I by Lemma 14.5, II by inductive hypothesis for k, III by inductive hypothesis since $dim(\mathbf{c}^{(k+1)}P) < n$.

Let $f : X \longrightarrow F(\mathbf{d}^{(k+1)}P^*)$ and $g : X \longrightarrow F(\boxed{k+1}P^*)$ be functions such that

$$F(\boldsymbol{\mu}^{\mathbf{d}^{(k+1)}P^*}); f = F(\mathbf{d}^{(k+1),*}_{\alpha^n}); g$$

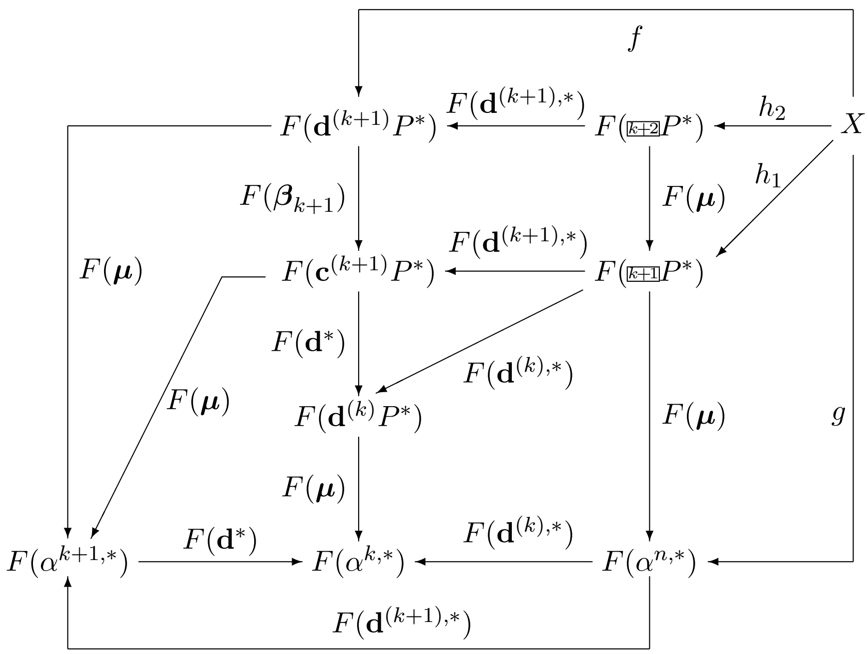
Since F applied to II is a pullback in Set, and some squares and triangles in the above diagram commute, there is a unique function $h_1 : X \longrightarrow F(\boxed{k+1}P)$ such that

$$h_1; F(d^{(k),*}_{\boxed{k+1}P}) = f; F(\boldsymbol{\beta}_{k+1}); F(\mathbf{d}^*_{\mathbf{c}^{(k+1)}P}) \quad \text{and} \quad \mathrm{h}_1; \mathrm{F}(\boldsymbol{\mu}^{\boxed{k+1}\mathrm{P}^*}) = \mathrm{g}. \qquad (10)$$

To get a unique function $h_2 : X \longrightarrow F(\boxed{k+2}P^*)$ such that

$$h_2; F(\mathbf{d}^{k+1,*}_{\boxed{k+2}P}) = f \quad \text{and} \quad \mathrm{h}_2; \mathrm{F}(\boldsymbol{\mu}_{\boxed{k+1}}) = \mathrm{h}_1 \qquad (11)$$

we use the fact that F sends III to a pullback in Set. The application of F to the diagram above will give the following diagram in Set, where we added the additional functions f, g, h_1, and h_2:



Thus in order to verify that

$$f; F(\boldsymbol{\beta}_{k+1}) = h_1; F(\mathbf{d}^{(k+1),*}_{\boxed{k+1}P})$$

and to get h_2 satisfying (11), it is enough to verify that

$$f; F(\boldsymbol{\beta}_{k+1}); F(\boldsymbol{\mu}^{\mathbf{c}^{(k+1)}P^*}) = h_1; F(\mathbf{d}^{(k+1),*}_{\boxed{k+1}P}); F(\boldsymbol{\mu}^{\mathbf{c}^{(k+1)}P^*}) \qquad (12)$$

$$f; F(\boldsymbol{\beta}_{k+1}); F(\mathbf{d}^*_{\mathbf{c}^{(k+1)}P}) = h_1; F(\mathbf{d}^{(k+1),*}_{\boxed{k+1}P}); F(\mathbf{d}^*_{\mathbf{c}^{(k+1)}P}) \qquad (13)$$

For (12), we have

$$f; F(\boldsymbol{\beta}_{k+1}); F(\boldsymbol{\mu}^{\mathbf{c}^{(k+1)}P^*}) = f; F(\boldsymbol{\mu}^{\mathbf{d}^{(k+1)}P^*}) =$$

$$= g; F(\mathbf{d}^{(k+1),*}_{\alpha^n}) = h_1; F(\boldsymbol{\mu}^{\boxed{k+1}P^*}); F(\mathbf{d}^{(k+1),*}_{\alpha^n}) =$$

$$= h_1; F(\mathbf{d}^{(k+1)^*}_{\boxed{k+1}P}); F(\boldsymbol{\mu}^{\mathbf{c}^{(k+1)}P^*})$$

and for (13), we have

$$f; F(\boldsymbol{\beta}_{k+1}); F(\mathbf{d}^*_{\mathbf{c}^{(k+1)}P}) = h_1; F(d^{(k),*}_{\boxed{k+1}P}) =$$

$$= h_1; F(\mathbf{d}^{(k+1),*}_{\boxed{k+1}P}); F(\mathbf{d}^*_{\mathbf{c}^{(k+1)}P}).$$

By uniqueness of both h_1 and h_2, h_2 is the unique function such that

$$h_2; \mathbf{d}^{(k+1),*}_{\boxed{k+2}P} = f \quad \text{and} \quad \mathrm{h}_2; \boldsymbol{\mu}^{\boxed{k+2}\mathrm{P}^*} = \mathrm{g}$$

i.e. F sends (9) to a pullback in Set, as required. $\square$

15 Yet another full nerve of the ω-categories

Let $\mathcal{S}$ denote the category of simple ω-categories introduced in [MZ]. It was proved there that any simple ω-category is isomorphic to one of form $(\alpha^{\vec{u}})^*$ for some ud-vector $\vec{u}$. In fact what we need here is that any simple ω-category can be obtained from those of form $(\alpha^n)^*$, with $n \in \omega$, via special pushouts.

As every simple ω-category is a positive computype, we have an inclusion functor

$$\mathbf{k} : \mathcal{S} \longrightarrow \mathcal{C}types^{+/1}_{\omega}.$$

In [MZ] we have shown, that $sPb(\mathcal{S}^{op}, Set)$ the category of special pullbacks preserving functors from the dual of $\mathcal{S}$ to Set is equivalent to the category ω-categories. We have in fact an adjoint equivalence

$$\omega Cat \; \underset{\widehat{(-)} = \omega Cat(\simeq, -)}{\overset{\widetilde{(-)}}{\leftrightarrows}} \; sPb(\mathcal{S}^{op}, Set)$$

where

$$\widehat{C} : \mathcal{S}^{op} \longrightarrow Set$$

is given by

$$\widehat{C}(A) = \omega Cat(A, C),$$

where A is a simple category. We shall show

Proposition 15.1 *The adjunction*

$$Set^{\mathcal{S}^{op}} \; \underset{\mathbf{k}^*}{\overset{Ran_{\mathbf{k}}}{\rightleftarrows}} \; Set^{(\mathcal{C}types^{+/1}_{\omega})^{op}}$$

restricts to an equivalence of categories.

$$sPb(\mathcal{S}^{op}, Set) \; \underset{\mathbf{k}^*}{\overset{Ran_{\mathbf{k}}}{\rightleftarrows}} \; sPb((\mathcal{C}types^{+/1}_{\omega})^{op}, Set)$$

where $sPb((\mathcal{C}types^{+/1}_{\omega})^{op}, Set)$ is the category of the special pullbacks preserving functors.

Proof. First we shall describe the adjunction in details.

The counit. Let G be a functor in $sPb(\mathcal{S}^{op}, Set)$, A a simple ω-category. We have a functor

$$(\mathbf{k} \downarrow A)^{op} \xrightarrow{\pi^A} \mathcal{S}^{op} \xrightarrow{G} Set$$

with the limit, say $(LimG \circ \pi^A, \sigma^A)$. Then the counit $(\varepsilon_G)_A$ is

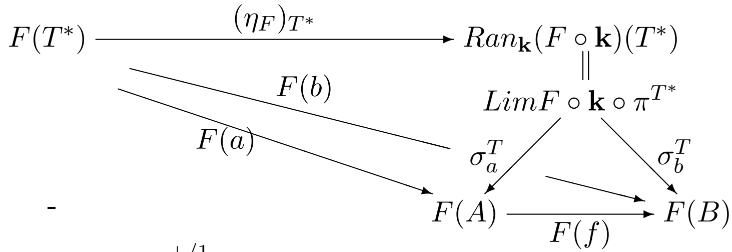
$$(\varepsilon_G)_A : (Ran_{\mathbf{k}}(G) \circ \mathbf{k})(A) = Lim(G \circ \pi^A) \xrightarrow{\sigma^A_{1_A}} G(A)$$

As $\mathbf{k}$ is full and faithful[4], for any G, ε_G is an iso. Thus ε is an iso.

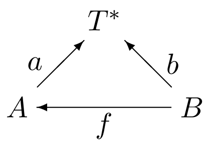
The unit. Let F be a functor in $sPb((\mathcal{C}types^{+/1}_\omega)^{op}, Set)$, T a positive face structure. We have a functor

$$(\mathbf{k} \downarrow T^*)^{op} \xrightarrow{\pi^{T^*}} \mathcal{S}^{op} \xrightarrow{\mathbf{k}} \mathcal{C}types^{+/1}_\omega \xrightarrow{F} Set$$

with the limit, say $(LimF \circ \mathbf{k} \circ \pi^{T^*}, \sigma^{T^*})$. Then the unit $(\eta_F)_{T^*}$ is the unique morphism into the limit:



where the triangle in $\mathcal{C}types^{+/1}_\omega$



commutes.

Note that, as F preserves special pullbacks, and any simple category can be obtained from those of form α^n with $n \in \omega$, we can restrict the limiting cone $(LimF \circ \mathbf{k} \circ \pi^{T^*}, \sigma^{T^*})$ to the objects of form α^n, with $n \in \omega$.

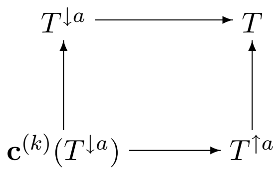
After this observation we shall prove by induction on size of a positive face structure T, that $(\eta_F)_{T^*}$ is an iso.

If $dim(T) \leq 1$ then $(\eta_F)_{T^*}$ is obviously an iso.

Suppose T is not principal, i.e. we have $a \in Sd(T)$ for some $k \in \omega$. By inductive hypothesis the morphisms

$$(\eta_F)_{(T^{\downarrow a})^*}, \quad (\eta_F)_{\mathbf{c}^{(k)}(T^{\downarrow a})^*}, \quad (\eta_F)_{(T^{\uparrow a})^*}$$

are iso, and the square



is a special pushout (see Proposition 6.2) which is sent by F to a pullback. Hence the morphism

$$(\eta_F)_{T^*} = (\eta_F)_{(T^{\downarrow a})^*} \times (\eta_F)_{(T^{\uparrow a})^*}$$

is indeed an iso in this case.

[4]This conditions translates to the fact that 1_A is the initial object in $(\mathbf{k} \downarrow A)^{op}$ and therefore that we have an iso $\sigma^A_{1_A} : LimG \circ \pi^A \cong G \circ \pi^A(1_A) = G(A)$.

If T is principal and $T = (\alpha^n)^*$ then the category $(\mathbf{k} \downarrow (\alpha^n)^*)^{op}$ has the initial object $1_{(\alpha^n)^*}$, so morphism

$$(\eta_F)_{(\alpha^n)^*} : F((\alpha^n)^*) \longrightarrow Ran_{\mathbf{k}}(F \circ \mathbf{k})((\alpha^n)^*)$$

is an iso.

Finally, let assume that $T(= P)$ is any principal positive face structure of dimension n. Thus, by Corollary 14.2, we have a principal pushout

$$\begin{array}{ccc} (\mathbf{d}P)^* & \xrightarrow{\mathbf{d}^*_P} & P^* \\ \uparrow{\scriptstyle \boldsymbol{\mu}^{\mathbf{d}P}} & & \uparrow{\scriptstyle \boldsymbol{\mu}^{P}} \\ (\alpha^{n-1})^* & \xrightarrow[\mathbf{d}^*_{\alpha^n}]{} & (\alpha^n)^* \end{array}$$

which, by Theorem 14.3, is preserved by F. By induction hypothesis the morphisms

$$(\eta_F)_{(T^{\downarrow a})^*}, \quad (\eta_F)_{\mathbf{c}^{(k)}(T^{\downarrow a})^*}, \quad (\eta_F)_{(T^{\uparrow a})^*}$$

are iso, so we have that the morphism

$$(\eta_F)_{P^*} = (\eta_F)_{(\mathbf{d}P)^*} \times (\eta_F)_{(\alpha^n)^*}$$

is iso, as well. □

Corollary 15.2 *We have a commuting triangle of adjoint equivalences*

$$\begin{array}{ccc} \omega Cat & & \\ \widehat{(-)}\downarrow\uparrow\widetilde{(-)} & \searrow\widehat{(-)} \quad \nwarrow\widetilde{(-)} & \\ sPb((\mathcal{C}types^{+/1}_\omega)^{op}, Set) & \underset{Ran_{\mathbf{k}}}{\overset{\mathbf{k}^*}{\rightleftarrows}} & sPb(\mathcal{S}^{op}, Set) \end{array}$$

In particular the categories ωCat and $sPb((\mathcal{C}types^{+/1}_\omega)^{op}, Set)$ are equivalent.

Proof. It is enough to show that in the above diagram $\mathbf{k}^* \circ \widehat{(-)} = \widehat{(-)}$. But this is obvious. □

16 A monadic adjunction

In this section we show that the inclusion functor $\mathbf{e} : \mathbf{Comp}^{+/1} \longrightarrow \omega Cat$ has a right adjoint which is monadic.

First we will give an outline of the proof. Consider the following diagram of categories and functors

$$\begin{array}{ccc} \mathbf{Comp}^{+/1} & \xrightarrow{\mathbf{e}} & \omega Cat \\ \widehat{(-)}\downarrow\uparrow\widetilde{(-)} & & \widehat{(-)}\downarrow\uparrow\widetilde{(-)} \\ sPb((\mathbf{Fs}^{+/1})^{op}, Set) & \underset{\mathbf{j}^*}{\overset{Lan_{\mathbf{j}}}{\rightleftarrows}} & sPb((\mathcal{C}types^{+/1}_\omega)^{op}, Set) \end{array}$$

where $\mathbf{e}$ is just an inclusion of positive-to-one computads into ω-categories and $\mathbf{j} = (-)^* : \mathbf{Fs}^{+/1} \longrightarrow \mathcal{C}types_\omega^{+/1}$ is an essentially surjective embedding. We have already shown (Proposition 13.4, Corollary 15.2) that the vertical morphisms constitute two adjoint equivalences. The proof that $\mathbf{e}$ has a right adjoint has two parts. First we will check that the functor $Lan_{\mathbf{j}}(F)$, the left Kan extension of a composition pullbacks preserving functor F preserves the composition pullbacks. Then we shall check that the above square commutes, i.e. $\widehat{(-)} \circ \mathbf{e} = Lan_{\mathbf{j}} \circ \widehat{(-)}$. Thus reduces the problem to check whether $\mathbf{j}^*$ the left adjoint to $Lan_{\mathbf{j}}$ is monadic.

Lemma 16.1 *The functor of the left Kan extension*

$$Lan_{\mathbf{j}} : sPb((\mathbf{Fs}^{+/1})^{op}, Set) \longrightarrow sPb((\mathcal{C}types_\omega^{+/1})^{op}, Set)$$

is well defined, i.e. whenever $F : (\mathbf{Fs}^{+/1})^{op} \longrightarrow Set$ preserves special pullbacks so does $Lan_{\mathbf{j}}(F) : (\mathcal{C}types_\omega^{+/1})^{op} \longrightarrow Set$. Moreover, the above functor $Lan_{\mathbf{j}}$ is the left adjoint to

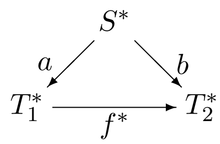
$$\mathbf{j}^* : sPb((\mathcal{C}types_\omega^{+/1})^{op}, Set) \longrightarrow sPb((\mathbf{Fs}^{+/1})^{op}, Set).$$

Proof. Note that once we will prove the first part of the statement the part following 'moreover' will follow immediately.

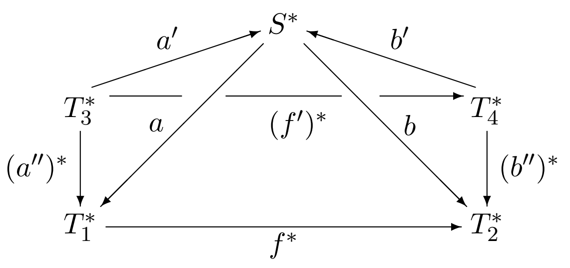
Fix F in $sPb((\mathbf{Fs}^{+/1})^{op}, Set)$ for the whole proof. First we shall describe the left Kan extension along $\mathbf{j}$ in a more convenient then the usual way, c.f. [CWM]. Fix a positive face structure S. $Lan_{\mathbf{j}}(F)(S)$ is the colimit of the following functor

$$\mathbf{j}^{op} \downarrow S \xrightarrow{\pi^S} (\mathbf{Fs}^{+/1})^{op} \xrightarrow{F} Set$$

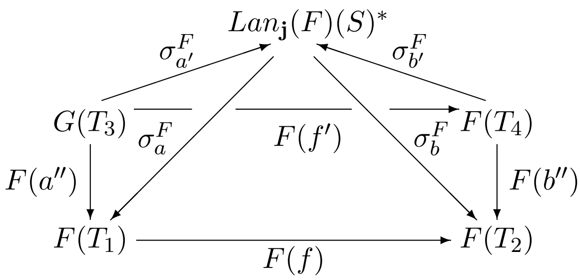
i.e. $Lan_{\mathbf{j}}(F)(S) = (F \circ \pi^S, \sigma^F)$. Note however that if we have a map $f : a \longrightarrow b$ in $\mathbf{j}^{op} \downarrow S$ i.e. there is a commuting triangle



by Lemma 10.3 we can take the inner-outer factorizations of both $a = a'; (a'')^*$ and $b = b'; (b'')^*$, with a' and b' inner. Hence, again by Lemma 10.3, there is a morphism $f' : a' \longrightarrow b'$ which must be an iso. In this way we get a commuting diagram



which correspond to the following part of the colimiting cocone



Thus if there is a morphism between two objects in $\mathbf{j}^{op} \downarrow S$ $f : a \to b$, we have a commuting diagram

$$\begin{array}{ccc} & a' & \\ {}^{a''}\swarrow & & \searrow^{f';b''} \\ a & \xrightarrow[f]{} & b \end{array}$$

with a' being the inner part of both a and b. Otherwise there are no comparison maps. But this say, that in fact

$$Lan_{\mathbf{j}}(F)(S^*) = \coprod_{a:S^*\to T^* \text{ inner}} F(T) \xleftarrow{\quad \kappa_a^{S^*} \quad} F(T)$$

where the coproduct is taken over all (up to iso) inner maps with the domain S^*, with the coprojections as shown.

If $h : S_1^* \longrightarrow S_2^*$ is an ω-functor and $a_2 : S_2^* \to T_2^*$ is inner, by Lemma 10.3, we can form a diagram

$$\begin{array}{ccc} S_1^* & \xrightarrow{h} & S_2^* \\ {\scriptstyle a_1}\downarrow & & \downarrow{\scriptstyle a_2} \\ T_1^* & \xrightarrow[(h')^*]{} & T_2^* \end{array}$$

with a_1 inner and $(h')^*$ outer. $Lan_{\mathbf{j}}(h)$ is so defined that, for any h', a_1, a_2 as above, the diagram

$$\begin{array}{ccc} Lan_{\mathbf{j}}(F)(S_2^*) = \coprod_{a_2:S_2^*\to T_2^* \text{ inner}} F(T_2) & \xrightarrow{\quad \kappa_{a_2}^{S_2^*} \quad} & F(T_2) \\ {\scriptstyle Lan_{\mathbf{j}}(F)(h)}\downarrow & & \downarrow{\scriptstyle F(h')} \\ Lan_{\mathbf{j}}(F)(S_1^*) = \coprod_{a_1:S_1^*\to T_1^* \text{ inner}} F(T_1) & \xrightarrow[\quad \kappa_{a_1}^{S_1^*} \quad]{} & F(T_1) \end{array}$$

commutes. This ends the description of the functor $Lan_{\mathbf{j}}$.

We shall use this description to show that $Lan_{\mathbf{j}}(F)$ preserves the special pushouts.

So assume that S_1 and S_2 are positive face structures such that $\mathbf{c}^{(k)}(S_1) = \mathbf{d}^{(k)}(S_2)$, i.e. we have a pushout

$$\begin{array}{ccc} S_1 & \xrightarrow{\kappa_1} & S_1;_k S_2 \\ {\scriptstyle \mathbf{c}^{(k)}_{S_1}}\uparrow & & \uparrow{\scriptstyle \kappa_2} \\ \mathbf{c}^{(k)}(S_1) & \xrightarrow[\mathbf{d}^{(k)}_{S_2}]{} & S_2 \end{array}$$

in $\mathbf{Fs}^{+/1}$. We need to show that the square

$$\begin{array}{ccc} Lan_{\mathbf{j}}(F)(S_1) & \xleftarrow{\quad Lan_{\mathbf{j}}(F)(\kappa_1) \quad} & Lan_{\mathbf{j}}(F)(S_1;_k S_2) \\ {\scriptstyle Lan_{\mathbf{j}}(F)(\mathbf{c}^{(k)}_{S_1})}\downarrow & & \downarrow{\scriptstyle Lan_{\mathbf{j}}(F)(\kappa_2)} \\ Lan_{\mathbf{j}}(F)(\mathbf{c}^{(k)}(S_1)) & \xleftarrow[\quad Lan_{\mathbf{j}}(F)(\mathbf{d}^{(k)}_{S_2}) \quad]{} & Lan_{\mathbf{j}}(F)(S_2) \end{array}$$

is a pullback in Set, i.e. that the square

$$\begin{array}{ccc}
\coprod_{a:S_1^*\to T^*\text{ inner}} F(T) & \xleftarrow{Lan_{\mathbf{j}}(F)(\kappa_1)} & \coprod_{a:(S_1;_k S_2)^*\to T^*\text{ inner}} F(T) \\
\Big\downarrow{\scriptstyle Lan_{\mathbf{j}}(F)(\mathbf{c}^{(k)}_{S_1})} & & \Big\downarrow{\scriptstyle Lan_{\mathbf{j}}(F)(\kappa_2)} \\
\coprod_{a:(\mathbf{c}^{(k)}(S_1))^*\to T^*\text{ inner}} F(T) & \xleftarrow[Lan_{\mathbf{j}}(F)(\mathbf{d}^{(k)}_{S_2})]{} & \coprod_{a:S_2^*\to T^*\text{ inner}} F(T)
\end{array}$$

is a pullback in Set. So suppose we have

$$x_1 \in F(T_1) \xrightarrow{\kappa^{s^*}_{a_1}} \coprod_{a:S_1^*\to T^*\text{ inner}} F(T)$$

$$x_2 \in F(T_1) \xrightarrow{\kappa^{s^*}_{a_2}} \coprod_{a:S_1^*\to T^*\text{ inner}} F(T)$$

such that

$$Lan_{\mathbf{j}}(F)(\mathbf{c}^{(k)}_{S_1})(x_1) = Lan_{\mathbf{j}}(F)(\mathbf{d}^{(k)}_{S_2})(x_2)$$

i.e. we have a commutative diagram in $\mathbf{Fs}^{+/1}$

$$\begin{array}{ccccc}
S_1^* & \xleftarrow{(\mathbf{c}^{(k)}_{S_1})^*} & (\mathbf{c}^{(k)}(S_1))^* = (\mathbf{d}^{(k)}(S_2))^* & \xrightarrow{(\mathbf{d}^{(k)}_{S_2})^*} & T_2^* \\
\Big\downarrow{\scriptstyle a_1} & & \Big\downarrow{\scriptstyle a_0} & & \Big\downarrow{\scriptstyle a_2} \\
T_1^* & \xleftarrow[f_1^*]{} & T^* & \xleftarrow[f_2^*]{} & T_2^*
\end{array}$$

such that

$$F(f_1)(x_1) = F(f_2)(x_2).$$

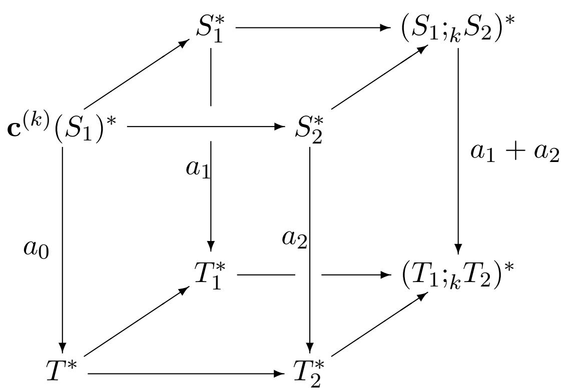
By Proposition 10.2,

$$\mathbf{c}^{(k)}a_1 = a_0 = \mathbf{d}^{(k)}a_2, \quad f_1^* = (\mathbf{c}^{(k)}_{T_1})^*, \quad f_2^* = (\mathbf{d}^{(k)}_{T_2})^*$$

and the square

$$\begin{array}{ccc}
T_1 & \xrightarrow{\kappa_1'} & T_1;_k T_2 \\
\Big\uparrow{\scriptstyle f_1} & & \Big\uparrow{\scriptstyle \kappa_2'} \\
T & \xrightarrow[f_2]{} & T_2
\end{array}$$

is a special pushout. We have a commuting diagram



where the bottom square is the above square, the top square is obvious. All the horizontal morphisms are outer. Since a_1 and a_2 are inner, $a_1(S_1) = T_1$ and $a_2(S_2) = T_2$ we have

$$(a_1 + a_2)(S_1;_k S_2) = a_1(S_1);_k a_2(S_2) = T_1;_k T_2.$$

i.e. $a_1+a_2 : (S_1;_kS_2)* \longrightarrow (T_1;_kT_2)$ is inner, as well. So in fact all vertical morphisms in the above diagram are inner.

Suppose we have another inner map u and outer maps κ''_1, κ''_2 so that the squares

$$\begin{array}{ccccc} S_1^* & \xrightarrow{\kappa_1^*} & (S_1;_kS_2)^* & \xleftarrow{\kappa_2^*} & S_2^* \\ \downarrow a_1 & & \downarrow u & & \downarrow a_2 \\ T_1^* & \xrightarrow[\kappa_1''^*]{} & U^* & \xleftarrow[\kappa_2''^*]{} & T_2^* \end{array}$$

commute. A diagram chasing shows that

$$a_0; f_1^*; \kappa_1''^* = a_0; f_2^*; \kappa_2''^*$$

As inner-outer factorization is essentially unique, it follows that

$$f_1^*; \kappa_1''^* = f_2^*; \kappa_2''^*$$

By the universal property of the pushout $(T_1;_kT_2)^*$ we have an ω-functor $v : (T_1;_kT_2)^* \longrightarrow U^*$ such that

$$\kappa''_1 = \kappa'_1; u, \ \ \kappa''_2 = \kappa'_2; u$$

Then again by a diagram chasing we get

$$\kappa_i; u = \kappa_i; (a_1 + a_2); v$$

for $i = 1, 2$. Hence by universal property of the pushout $(S_1;_kS_2)^*$ we have that $u = (a_1+a_2); v$. But both u and (a_1+a_2) are inner so by uniqueness of factorization, see Lemma 10.3, v must be an iso, as well. This means that if there is an $x \in \coprod_{a:(S_1;_kS_2)^*\to T^* \text{ inner}} F(T)$ such that

$$Lan_{\mathbf{j}}(F)(\kappa_1)(x) = x_1, \quad Lan_{\mathbf{j}}(F)(\kappa_2)(x) = x_2$$

it is necessary that x belongs to the summand of the coproduct with the index $(a_1 + a_2)$, i.e.

$$x \in F(T_1;_k T_2) \xrightarrow{\kappa^{(S_1;_kS_2)^*}_{(a_1+a_2)}} \coprod_{a:(S_1;_kS_2)^*\to T^* \text{ inner}} F(T)$$

But F sends special pushouts in $\mathbf{Fs}^{+/1}$ to pullbacks in Set so the square

$$\begin{array}{ccc} F(T_1) & \xleftarrow{F(\kappa'_1)} & F(T_1;_kT_2) \\ F(f_1) \downarrow & & \downarrow F(\kappa'_2) \\ F(T) & \xleftarrow[F(f_2)]{} & F(T_2) \end{array}$$

is a pullback in Set. Thus indeed there is a unique $x \in F(T_1;_kT_2)$‘$F(T)$ such that $F(\kappa'_i)(x) = x_i$ for $i = 1, 2$. This shows that $Lan_{\mathbf{j}}(F)$ preserves special pushouts. □

In the proof above we have described the left Kan extension $Lan_{\mathbf{j}}$ in a special way in terms coproducts. As it is a very important property the corollary below we restate this description explicitly, for the record.

Corollary 16.2 *The functor*

$$Lan_{\mathbf{j}} : sPb((\mathbf{Fs}^{+/1})^{op}, Set) \longrightarrow sPb((\mathcal{C}types_{\omega}^{+/1})^{op}, Set)$$

is defined for $F \in sPb((\mathbf{Fs}^{+/1})^{op}, Set)$ as follows. For a positive face structure S we have

$$Lan_{\mathbf{j}}(F)(S^*) = \textstyle\coprod_{a:S^*\to T^* \text{ inner}} F(T) \xleftarrow{\ \kappa_a^{S^*}\ } F(T)$$

where coproduct is taken over all up to iso inner maps in $\mathcal{C}types_{\omega}^{+/1}$ with the domain S^*, with the coprojections as shown.

If $h : S_1^* \longrightarrow S_2^*$ is an ω-functor and $a_2 : S_2^* \to T_2^*$ is inner, by Lemma 10.3, we can form a diagram

$$\begin{array}{ccc} S_1^* & \xrightarrow{\ h\ } & S_2^* \\ {\scriptstyle a_1}\downarrow & & \downarrow{\scriptstyle a_2} \\ T_1^* & \xrightarrow[(h')^*]{} & T_2^* \end{array}$$

with a_1 inner and h' a face structures map, i.e. the map $(h')^*$ is outer. $Lan_{\mathbf{j}}(h)$ is so defined that, for any h, h', a_1, a_2 as above, the diagram

$$\begin{array}{ccc} Lan_{\mathbf{j}}(F)(S_2^*) = \coprod_{a_2:S_2^*\to T_2^* \text{ inner}} F(T_2) & \xrightarrow{\ \kappa_{a_2}^{S_2^*}\ } & F(T_2) \\ {\scriptstyle Lan_{\mathbf{j}}(F)(h)}\downarrow & & \downarrow{\scriptstyle F(h')} \\ Lan_{\mathbf{j}}(F)(S_1^*) = \coprod_{a_1:S_1^*\to T_1^* \text{ inner}} F(T_1) & \xrightarrow[\kappa_{a_1}^{S_1^*}]{} & F(T_1) \end{array}$$

commutes. $\Box$

Lemma 16.3 *The following square*

$$\begin{array}{ccc} \mathbf{Comp}^{+/1} & \xrightarrow{\ \mathbf{e}\ } & \omega Cat \\ {\scriptstyle \widehat{(-)}}\downarrow & & \downarrow{\scriptstyle \widehat{(-)}} \\ sPb((\mathbf{Fs}^{+/1})^{op}, Set) & \xrightarrow{\ Lan_{\mathbf{j}}\ } & sPb((\mathcal{C}types_{\omega}^{+/1})^{op}, Set) \end{array}$$

commutes, up to an isomorphism.

Proof. We shall define two natural transformations φ and ψ which are mutually inverse, i.e. for a positive-to-one computad Q we define

$$Lan_{\mathbf{j}}\left(\mathbf{Comp}^{+/1}((-)^*, Q)\right) \underset{\psi_Q}{\overset{\varphi_Q}{\rightleftarrows}} \omega Cat((-)^*, Q)$$

Let $a : S^* \longrightarrow T^*$ be an inner map and $f : T^* \to A$ be a computad map, i.e. g is in the following coproduct

$$g \in \mathbf{Comp}^{+/1}(T^*, Q) \xrightarrow{\ \kappa_a^{S^*}\ } \textstyle\coprod_{S^*\to R^* \text{ inner}} \mathbf{Comp}^{+/1}(R^*, Q)$$

Then we put

$$\varphi_Q(g) = a; g.$$

On the other hand, for an ω-functor $f : S^* \to Q \in \omega Cat(S^*, Q)$, by Proposition 12.2.4, we have a factorization

$$\begin{array}{ccc} S^* & \xrightarrow{f} & Q \\ & {}_{f^{in}}\searrow \quad \nearrow_{\tau_{f(S)}} & \\ & T^*_{f(S)} & \end{array}$$

Then we put

$$\psi_Q(f) = \tau_{f(S)} \in \mathbf{Comp}^{+/1}(T^*_{f(S)}, Q) \xrightarrow{\kappa^{S^*}_{fin}} \coprod_{S^* \to R^* \text{ inner}} \mathbf{Comp}^{+/1}(R^*, Q)$$

The fact that these transformations are mutually inverse follows from the fact that the above factorization is essentially unique.

The verifications that these transformations and natural is left for the reader. $\Box$

Proposition 16.4 *The functor*

$$Lan_{\mathbf{j}} : sPb((\mathbf{Fs}^{+/1})^{op}, Set) \longrightarrow sPb((\mathcal{C}types_\omega^{+/1})^{op}, Set)$$

preserves connected limits.

Proof. This is an easy consequence of Lemma 16.2, where $Lan_{\mathbf{j}}$ is described in terms of pushouts. For preservation of connected limits it is sufficient to show that wide pullbacks and equalizers are preserved. We shall sketch the preservation of the binary pullbacks leaving the details and other cases to the reader.

So let

$$\begin{array}{ccc} F \times_H G & \longrightarrow & G \\ \downarrow & & \downarrow \\ F & \longrightarrow & H \end{array}$$

be a pullback in $sPb((\mathbf{Fs}^{+/1})^{op}, Set)$. Then we have, for any positive face structure S, we have

$$Lan_{\mathbf{j}}(F \times_H G)(S^*) = \coprod_{a:S^* \to T^* \text{ inner}} (F \times_H G)(T) \cong$$

$$\cong \coprod_{a:S^* \to T^* \text{ inner}} F(T) \times_{H(T)} G(T) \cong$$

$$\cong \coprod_{a:S^* \to T^* \text{ inner}} F(T) \times_{\coprod_{a:S^* \to T^* \text{ inner}} H(T)} \coprod_{a:S^* \to T^* \text{ inner}} G(T) \cong$$

$$\cong Lan_{\mathbf{j}}(F)(S^*) \times_{Lan_{\mathbf{j}}(H)(S^*)} Lan_{\mathbf{j}}(G)(S^*)$$

as required. $\Box$

From Propositions 13.4, 16.4, Lemma 16.3 and Corollary 15.2 we get immediately

Theorem 16.5 *The embedding functor*

$$\mathbf{e} : \mathbf{Comp}^{+/1} \longrightarrow \omega Cat$$

preserves connected limits. $\Box$

We have

Theorem 16.6 *The functor*

$$\mathbf{j}^* : sPb((\mathcal{C}types_\omega^{+/1})^{op}, Set) \longrightarrow sPb((\mathbf{Fs}^{+/1})^{op}, Set)$$

is monadic.

Proof. We are going to verify Beck's conditions for monadicity. As $\mathbf{j}$ is essentially surjective $\mathbf{j}^*$ is conservative. By Lemma 16.1, the adjunction $Lan_{\mathbf{j}} \dashv j^*$ restricts to the above categories. So $\mathbf{j}^*$ has a left adjoint. It remains to show that $sPb((\mathcal{C}types_\omega^{+/1})^{op}, Set)$ has coequalizers of $\mathbf{j}^*$-contractible coequalizer pairs and that $\mathbf{j}^*$ preserves them.

To this aim, let assume that we have a parallel pair of morphisms in $sPb((\mathcal{C}types_\omega^{+/1})^{op}, Set)$

$$A \underset{G}{\overset{F}{\rightrightarrows}} B$$

such that

$$A((-)^*) \underset{G_{(-)^*}}{\overset{F_{(-)^*}}{\rightrightarrows}} B((-)^*) \xrightarrow{\;q\;} Q, \qquad t : B((-)^*) \longrightarrow A((-)^*), \quad s : Q \longrightarrow B((-)^*)$$

is a split coequalizer in $sPb((\mathbf{Fs}^{+/1})^{op}, Set)$, i.e. the following equations

$$s; q = 1_Q, \qquad G_{(-)^*}; q = F_{(-)^*}; q, \qquad t; F_{(-)^*} = 1_{B((-)^*)}, \qquad t; G_{(-)^*} = q; s$$

hold.

We are going to construct a special pullbacks preserving functor $C : (\mathcal{C}types_\omega^{+/1})^{op} \longrightarrow Set$ and a natural transformation $H : B \longrightarrow C$ so that the diagram in $sPb((\mathcal{C}types_\omega^{+/1})^{op}, Set)$

$$A \underset{G}{\overset{F}{\rightrightarrows}} B \xrightarrow{\;H\;} C$$

is a coequalizer, and $H_{(-)^*} = q$.

The functor C on a morphism $f : T_1^* \longrightarrow T_2^*$ is defined as in the diagram

$$\begin{array}{ccc}
C(T_1^*) & \xrightarrow{\;C(f)\;} & C(T_2^*) \\
\| & & \| \\
Q(T_1) & & Q(T_2) \\
{\scriptstyle s_{T_1}}\downarrow & & \uparrow{\scriptstyle q_{T_2}} \\
B(T_1^*) & \xrightarrow[\;B(f)\;]{} & B(T_2^*)
\end{array}$$

i.e. $C(T_i) = Q(T_i)$, for $i = 1, 2$ and $C(f) = s_{T_1}; B(f); q_{T_2}$.

The natural transformation H is given by

$$H_{T^*} = q_T$$

for $T \in \mathbf{Fs}^{+/1}$.

It remains to verify that

1. C is a functor;

2. H is a natural transformation;

3. $C((-)^*) = Q$;

4. $H_{(-)^*} = q$;

5. C preserves the special pullbacks;

6. H is a coequalizer.

Ad 1. Let

$$T_1^* \xleftarrow{\ f\ } T_2^* \xleftarrow{\ g\ } T_3^*$$

be a pair of morphisms in $\mathcal{C}types_\omega^{+/1}$. We calculate

$$\begin{aligned}
C(f);C(g) &= s_{T_1};B(f);q_{T_2};s_{T_2};B(g);q_{T_3} = \\
&= s_{T_1};B(f);t_{T_2};G_{T_2^*};B(g);q_{T_3} = \\
&= s_{T_1};B(f);t_{T_2};A(g);G_{T_3^*};q_{T_3} = \\
&= s_{T_1};B(f);t_{T_2};A(g);F_{T_3^*};q_{T_3} = \\
&= s_{T_1};B(f);t_{T_2};F_{T_2^*};B(g);q_{T_3} = \\
&= s_{T_1};B(f);t_{T_2};F_{T_2^*};B(g);q_{T_3} = \\
&= s_{T_1};B(f);B(g);q_{T_3} = \\
&= s_{T_1};B(f;g);q_{T_3} = C(f;g)
\end{aligned}$$

i.e. C preserves compositions. If T is a positive face structure, we also have

$$C(1_{T^*}) = s_T;B(1_{T^*});q_T = s_T;q_T = 1_{Q(T)} = 1_{C(T^*)}.$$

i.e. C preserves identities, as well.

Ad 2. Let $f : T_2^* \longrightarrow T_1^*$ be a morphism in $\mathcal{C}types_\omega^{+/1}$. We have

$$\begin{aligned}
B(f);H_{T_2^*} &= B(f);q_{T_2} = \\
&= t_{T_1};F(T_1^*);B(f);q_{T_2} = \\
&= t_{T_1};A(f);F(T_2^*);q_{T_2} = \\
&= t_{T_1};A(f);G(T_2^*);q_{T_2} = \\
&= t_{T_1};G(T_1^*);B(f);q_{T_2} = \\
&= q_{T_1};s_{T_1};B(f);q_{T_2} = \\
&= q_{T_1};C(f) = H_{T_1^*};C(f)
\end{aligned}$$

i.e. H is a natural transformation.

Ad 3. Let $f : T_2 \longrightarrow T_1$ be a morphism in $\mathbf{Fs}^{+/1}$. Thus q is natural with respect to f. So we have

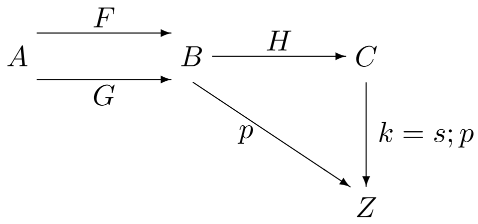
$$C(f^*) = s_{T_1};B(f^*);q_{T_2} = s_{T_1};q_{T_1};Q(f) = 1_{T_1};Q(f) = Q(f)$$

i.e. $C_{(-)^*} = Q$. Note that we still don't know that C is in $spPb((\mathcal{C}types_\omega^{+/1})^{op}, Set)$.

Ad 4. $H_{(-)^*} = q$ holds by definition.

Ad 5. Since special pullbacks involve only the outer morphisms (i.e. those that comes from $\mathbf{Fs}^{+/1}$), and Q preserves special pullbacks so does C.

Ad 6. Finally, we shall show that H is a coequalizer. Let $p : B \longrightarrow Z$ be a natural transformation in $sPb((\mathcal{C}types_\omega^{+/1})^{op}, Set)$ such that $pF = pG$. We put $k = s;p : C \longrightarrow Z$, so that we have a diagram



We need to verify that k is a natural transformation in $sPb((\mathcal{C}types_\omega^{+/1})^{op}, Set)$, such that $p = H; k$. Then, the uniqueness of k will follows from the fact that q is a split epi. Let $f : T_2^* \longrightarrow T_1^*$ be a morphism in $\mathcal{C}types_\omega^{+/1}$. Then

$$C(f); k_{T_2^*} = s_{T_1}; B(f); q_{T_2}; k_{T_2^*} =$$

$$= s_{T_1}; B(f); q_{T_2}; s_{T_2}; p_{T_2^*} =$$

$$= s_{T_1}; B(f); t_{T_2}; G_{T_2^*}; p_{T_2^*} =$$

$$= s_{T_1}; B(f); t_{T_2}; F_{T_2^*}; p_{T_2^*} =$$

$$= s_{T_1}; B(f); p_{T_2^*} =$$

$$= s_{T_1}; p_{T_1^*}; D(f) = k_{T_1^*}; D(f)$$

i.e. k a natural transformation and hence H is indeed a coequalizer of F and G in $sPb((\mathcal{C}types_\omega^{+/1})^{op}, Set)$, as required. □

Combining the above theorem with Corollaries 13.5 and 15.2 we get

Theorem 16.7 *The nerve functor*

$$\widehat{(-)} : \omega Cat \longrightarrow sPb((\mathbf{Fs}^{+/1})^{op}, Set)$$

sending the ω-category C to the presheaf

$$\omega Cat((-)^*, C) : (\mathbf{Fs}^{+/1})^{op} \longrightarrow Set$$

is monadic.

17 Appendix

A definition of the positive-to-one computads and the comma categories

The notion of a computad was introduced by Ross Street. We repeat this definition for a subcategory $\mathbf{Comp}^{+/1}$ of the category of all computads that have indeterminates of a special shape, namely their codomains are again indeterminates and their domains are not identities. We use this opportunity to introduce the notation used in the paper. In order to define $\mathbf{Comp}^{+/1}$ we define three sequences of categories $\mathbf{Comp}_n^{+/1}$, $\mathbf{Comma}_n^{+/1}$, and $\mathbf{Comma}_n$.

1. For $n = 0$, the categories $\mathbf{Comp}_n^{+/1}$, $\mathbf{Comma}_n^{+/1}$and $\mathbf{Comma}_n$ are just Set, and the functor $\overline{(-)}^n : \mathbf{Comma}_n^{+/1} \longrightarrow \mathbf{Comp}_n^{+/1}$ is the identity.

2. For $n = 1$, the categories $\mathbf{Comma}_n^{+/1}$ and $\mathbf{Comma}_n$ are the category of graphs (i.e. 1-graphs) and $\mathbf{Comp}_n^{+/1}$ is the category of free ω-categories over graphs with morphisms being the functors sending indets (=indeterminates=generators) to indets.

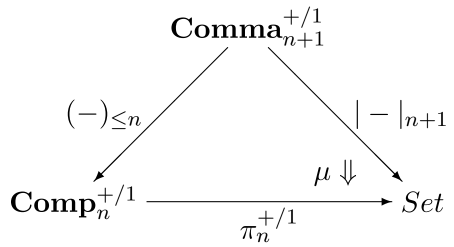
3. Let $n \geq 1$. We define the following functor

$$\pi_n^{+/1} : \mathbf{Comp}_n^{+/1} \longrightarrow Set$$

such that

$$\pi_n^{+/1}(P) = \{(a,b) : a \in (P_n - \iota(P_{n-1})),\ b \in |P|_n,\ d(a) = d(b),\ c(a) = c(b)\}$$

i.e. $\pi_n^{+/1}(P)$ consists of those parallel pairs (a,b) of n-cells of P such that a is not an identity and b is an indet. On morphisms $\pi_n^{+/1}$ is defined in the obvious way. We define $\mathbf{Comma}_{n+1}^{+/1}$ to be equal to the comma category $Set \downarrow \pi_n^{+/1}$. So we have a diagram



4. For $n \geq 1$, we can define also a functor

$$\pi_n : n\mathbf{Cat} \longrightarrow Set$$

such that

$$\pi_n(C) = \{(a,b) : a, b \in C_n,\ d(a) = d(b),\ c(a) = c(b)\}$$

i.e. $\pi_n(C)$ consists of all parallel pairs (a,b) of n-cells of the n-category C. We define $\mathbf{Comma}_{n+1}$ to be equal to the comma category $Set \downarrow \pi_n$. We often denote objects of $\mathbf{Comma}_{n+1}$ as quadruples $C = (|C|_{n+1}, C_{\leq n}, d, c)$, where $C_{\leq n}$ is an n-category, $|C|_{n+1}$ is a set and $(d,c) : |C|_{n+1} \longrightarrow \pi_n(C_{\leq n})$ is a function. Clearly, the category $\mathbf{Comma}_{n+1}^{+/1}$ is a full subcategory of $\mathbf{Comma}_{n+1}$, moreover we have a forgetful functor

$$\mathcal{U}_{n+1} : (n+1)\mathbf{Cat} \longrightarrow \mathbf{Comma}_{n+1}$$

such that for an $(n+1)$-category A

$$\mathcal{U}_{n+1}(A) = (A_{n+1}, A_{\leq n}, d, c)$$

i.e. $\mathcal{U}_{n+1}$ forgets the structure of compositions and identities at the top level. This functor has a left adjoint

$$\mathcal{F}_{n+1} : \mathbf{Comma}_{n+1} \longrightarrow (n+1)\mathbf{Cat}$$

The category $\mathcal{F}_{n+1}(|B|_{n+1}, B, d, c)$ is said to be a *free extension* of $(|B|_{n+1}, B, d, c)$ the n-category B by the indets $|B|_{n+1}$. The category of positive-to-one $(n+1)$-computads $\mathbf{Comp}_{n+1}^{+/1}$ is a subcategory of $(n+1)\mathbf{Cat}$ whose objects are free extensions of objects from $\mathbf{Comma}_{n+1}^{+/1}$. The morphisms in $\mathbf{Comp}_{n+1}^{+/1}$ are $(n+1)$-functors that sends indets to indets. Thus the functor $\mathcal{F}_{n+1}$ restricts to an equivalence of categories

$$\mathcal{F}_{n+1}^{+/1} : \mathbf{Comma}_{n+1}^{+/1} \longrightarrow \mathbf{Comp}_{n+1}^{+/1},$$

its essential inverse will be denoted by

$$\| - \|_{n+1} : \mathbf{Comp}_{n+1}^{+/1} \longrightarrow \mathbf{Comma}_{n+1}^{+/1}.$$

Thus for an $(n+1)$-computad P we have $\|P\|_{n+1} = (|P|_{n+1}, P_{\leq n}, d, c)$.

5. The category $\mathbf{Comp}^{+/1}$ is the category of such ω-categories P, that for every $n \in \omega$, $P_{\leq n}$ is a positive-to-one n-computad, and whose morphisms are ω-functors sending indets to indets.

   For $n \in \omega$, we have functors

$$| - |_n : \mathbf{Comp}^{+/1} \longrightarrow Set$$

   associating to computads their n-indets, i.e.

$$f : A \longrightarrow B \mapsto |f|_n : |A|_n \longrightarrow |B|_n,$$

   they all preserve colimits. Moreover we have a functor

$$| - | : \mathbf{Comp}^{+/1} \longrightarrow Set$$

   associating to computads all their indets, i.e.

$$f : A \longrightarrow B \mapsto |f| : |A| \longrightarrow |B|,$$

   where

$$|A| = \coprod_{n \in \omega} |A|_n.$$

   It also preserves colimits and moreover it is faithful.

6. We have a truncation functor

$$(-)_{\leq n} : \omega Cat \longrightarrow n\mathbf{Cat}$$

   such that

$$f : A \longrightarrow B \mapsto f_{\leq k} : A_{\leq k} \longrightarrow B_{\leq k}$$

   with $k \in \omega$, it preserves limits and colimits.

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