Multitopes are the same as principal ordered face structures

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Dedicated to Professor F.W.Lawvere on the occasion of his 70th birthday.

Abstract

We show that the category of principal ordered face structures **pFs** is equivalent to the category of multitopes **Mlt**. On the way we introduce the notion of a graded tensor theory to state the abstract properties of the category of ordered face structures **oFs** and show how **oFs** fits into the recent work of T. Leinster and M. Weber concerning the nerve construction.

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1 Introduction

In [Z1] the notion of a positive face structure is introduced and it is shown how it helps to understand the positive-to-one computads. In [Z2] part of the program of [Z1] was developed in the many-to-one context, i.e. the notion of an ordered face structure was introduced and related to the many-to-one computads. The first part of this paper is a sequel of [Z2] developing farther part of [Z1] in the many-to-one context. We show how the category of ordered face structures and monotone maps **oFs** can be used to show that the category of many-to-one computads $\mathbf{Comp}^{m/1}$ is equivalent to the presheaves category $Set^{\mathbf{pFs}^{op}}$, where **pFs** is the full subcategory of **oFs** whose objects are principal ordered face structures. In fact we show that both categories $\mathbf{Comp}^{m/1}$ and $Set^{\mathbf{pFs}^{op}}$ are equivalent to the category $Mod_{\otimes}(\mathbf{oFs}^{op}, Set)$ of Set-models of the graded tensor theory **oFs**. In [HMP] it was shown that the category $MltSet$ of multitopic set is a presheaf category on the category of multitopes **Mlt**. In [HMZ] it was shown that the categories $MltSet$ and $\mathbf{Comp}^{m/1}$ are equivalent. Thus as a corollary we get the statement from the title of this paper.

My main motivation to define the ordered face structures was to have an explicit combinatorial definition of multitopic sets[1] (or what comes to the same the many-to-one computads, c.f. [HMZ]) that allow fairly easy direct manipulation on cells. For this I wanted to describe not only the shapes of many-to-one indets (=indeterminates) but of all the cells build from them. To see some pictures and more explanations on this consult introduction to [Z2]. As there are several other structures that are serving a similar purpose, that I will discuss later, the anonymous referee asked to explain what is the role of the category **oFs** and why it is of an

[1]Recall that multitopic category, a weak ω-category in the sense of M.Makkai, is a multitopic set with a property, c.f. [M].

interest at all as its definition is not a simple one. It is not always easy to give a convincing answer to such questions. After my talk describing the ordered face structures in Patras (PSSL, April 2008) J. Kock suggested that the recent paper of M. Weber, c.f. [W], could provide a framework for a conceptual explanation what **oFs** is. The explanations I will present in the second part of the paper are very much inspired by the work of T. Leinster [Lei1] and M. Weber [W] but it also goes beyond that. The short answer is that the category **oFs** is the category of shapes of all cells, not only indeterminates (=indets), in many-to-one computads. The abstract properties of **oFs** are subsumed by the notion of a graded tensor theory. I can also make a broader but 'non-full' analogy concerning **oFs**. It is related to the ω-category monad on many-to-one computads in a similar way as the category of simple ω-graphs $s\omega\mathcal{G}$ (or globular cardinals) is related to the ω-category monad on one-to-one computads $\mathbf{Comp}^{1/1}$, i.e. the free ω-categories over ω-graphs with morphisms sending indets to indets. However the embedding $\mathbf{oFs} \to \mathbf{Comp}^{m/1}$ is not full and what is even worse it is not full on isomorphisms. The full image of **oFs** under this embedding is the category of ordered face structures and local maps $\mathbf{oFs}_{loc}$ which plays the role of the category of many-to-one cardinals in the Leinster-Weber approach. Thus we have here two different categories **oFs** and $\mathbf{oFs}_{loc}$ where T. Leinster and M. Weber have only one. What I mean by the category of shapes is a bit technical and the precise definition will be given in Section 8.

T. Leinster in [Lei2] explained that he started to love the nerve construction when he discovered that both category Δ and the nerve construction (for categories) arise canonically from the free category monad on graphs. This convinced him that the construction is *natural*. Before, he could only acknowledge that the nerve construction is just *useful*. I would consider even two earlier stages in the process of proving 'rights to exists' of a concept. One, when there is a *construction* of the object in question which is not totally unrelated to the purpose it serves. Then, if all else failed, there might be a *purity* of style behind the notion. The reason I explain all this is that I don't have a canonical simple construction that would make T. Leinster believe that the category **oFs** can be naturally derived from ω-category monad on the category of many-to-one computads $\mathbf{Comp}^{m/1}$ or possibly some other fundamental construction related $\mathbf{Comp}^{m/1}$. But I will argue about the three weaker claims.

1. *Purity: simple combinatorial data.* As I mentioned earlier, my main motivation to define the ordered face structures was to have an explicit combinatorial definition of multitopic sets that allow fairly easy direct manipulation on cells. I wanted to describe these shapes with the least possible structure. So in an ordered face structure we have functions γ, associating a *face* $\gamma(a)$ which is the codomain the face a, relations δ, associating a *set of faces* $\delta(a)$ that 'constitue' the domain to the face a, and strict order relations $<^{\sim}$ that will indicate in case of doubts in what order one should compose the faces. The structure is kept so simple at the expense of the axioms that are quite involved and do not look at first sight as something that have much to do with what it was designed for. To explain how ordered face structures describe many-to-one computads is a long story, see [Z2].

2. *Abstract construction: the category of shapes.* However there is an abstract definition of the category of shapes that in most considered cases gives the category which is equivalent to the category of T-cardinals considered by T. Leinster and M. Weber but in the case of many-to-one computads it is equivalent to **oFs** rather than the category of cardinals which is in this case $\mathbf{oFs}_{loc}$. This definition of the category of shapes is given in the section 8. It is at least related with the many-to-one computads from the very beginning but it is rather hard to believe that it might be of any practical use.

3. *Usefulness:* **oFs** *generates all the setup of Leinster and Weber.* In Sections 9 and 10 I will argue that **oFs** is useful as this category alone generates all the setup it is involved with. That includes the category of many-to-one computads $\mathbf{Comp}^{m/1}$, the ω-category monad on the category $\mathbf{Comp}^{m/1}$, the proof that that this monad is a parametric right adjoint, and that ω-categories can be considered as some presheaf satisfying an additional condition.

The notion of a graded tensor theory, GT-theory for short, is designed to describe the abstract features of the category **oFs**. Any model $M : \mathcal{C}^{op} \to \mathcal{A}$ of an GT-theory $\mathcal{C}$ in a category $\mathcal{A}$ gives rise to a functor $\bar{M} : \mathcal{A} \to \omega Cat$ from the category $\mathcal{A}$ to the category of strict ω-categories. This notion was inspired by and should be compared with the notion of a monoidal globular category, MG-category for short, of M. Batanin, c.f. [B]. Both notions deal with the k-domain and the k-codomain operations. Both notions have the k-tensor product operations that can be performed only if the k-codomain of the first object agrees with the k-domain of the second one. In GT-theories the cylinder operation is not given explicitly. However there are essential differences. An GT-theory $\mathcal{C}$ is a single (rigid) category together with a dimension functor $dim : \mathcal{C} \to \mathbf{N}$ into the linear order of natural numbers $\mathbf{N}$. The k-tensor operations are required to be functorial, as in MG-category, but the k-domain and the k-codomain operations are not functorial in general. Instead all these operations are given together with specified morphisms $\mathbf{d}^{(k)}_S : \mathbf{d}^{(k)}_S \to S$, $\mathbf{c}^{(k)}_S : \mathbf{c}^{(k)}_S \to S$ in $\mathcal{C}$, $\kappa^1_S : S \to S \otimes_k S'$, $\kappa^2_{S'} : S \to S \otimes_k S'$ that explain the relation of the domains and the codomains of objects to objects themselves and of the components of the tensor products to the tensor products. There are isomorphisms relating these operations as in MG-category but, as an GT-theory is a rigid category, the coherence conditions are satisfied automatically. Last but not least the category **oFs** is a GT-theory but the domain and the codomain operations are not functorial and the truncations of **oFs** do not form an MG-category, contrary to a public claim I have made. It is true that the isomorphism classes of objects in an GT-theory can be easily organized into a discrete MG-category. But this process when applied to **oFs** would destroy an essential information about the monotone morphisms. The notion of a model of GT-theory (a functor sending tensor square to pullbacks) is very important in this context but doesn't seem to have an analog in the context of MG-categories.

In Section 9 the setup developed by T. Leinster and M. Weber is recalled but not in the full generality of [W] and in a form that changes the emphasis. So I will not recall the setup here but only point out to the change in the emphasis. I will discuss only the parametric right adjoint monads on presheaf categories, called pra monads for short. The monadic functor inducing pra monad is called *pra monadic*. Among pra functors there are particularly simple ones that arise from factorizations system on small categories. If $(\mathcal{E}, \mathcal{M})$ is a factorization system on a category $\mathcal{C}$, and $\mathcal{C}_\mathcal{M}$ is the category with the objects from $\mathcal{C}$ and morphisms from $\mathcal{M}$ then the restriction functor $i^* : Set^{\mathcal{C}^{op}} \longrightarrow Set^{\mathcal{C}_\mathcal{M}{}^{op}}$ along the inclusion $i : \mathcal{C}_\mathcal{M} \to \mathcal{C}$ is pra monadic. Such functors I will call *presheaf pra monadic*. In this context the main conclusion of the work of T. Leinster and M. Weber, present in [W] in a sightly hidden form, is that any pra monadic functors, arise as pseudo-pullback of a presheaf pra monadic one along a full and faithful functor. Thus it can be thought of as a representation/completeness result for pra monads. In Section 10 an extension of the above setup is proposed and it is shown how the category **oFs** generates all its ingredients.

In the presheaf approaches to weak categories (as opposed to the algebraic ones) the weak categories are presheaves with some properties. If we believe that strict ω-categories should be special cases of weak ones we need to find the way how to interpret strict ω-categories as appropriate presheaves. The nerves of ω-categories are constructions that do exactly this and (should) provide abundance of examples of weak categories. In particular the many-to-one nerve functor sends strict ω-categories to multitopic categories.

The need to have a good description of higher many-to-one shapes was already clear at the conference *n-categories: Foundations and Applications* at IMA in Minneapolis, in June 2004. Now there is (at least) seven essentially different definitions that are attempting to describe shapes of indets of many-to-one computads or some supposedly equivalent entities. These definitions differ a lot in spirit and it is by far not clear that they are all equivalent. It seem that it is too early to call which one is better then the others and I think that all of them contribute to better understanding the concept they try to capture. So I will content myself by just listing them divided into four groups.

1. There are three kinds of opetopes [BD], [L], [KJBM] that describe the set of shapes of many-to-one indets without an attempt to make it into a category. The second and third kind of opetopes are proved in [KJBM] to be equivalent.

2. There are four categories describing the shapes of many-to-one indets: the category of multitopes, c.f. [HMP], the category of dendrotopes, c.f. [P], the category of opetopes [C] and the category of ordered face structures, c.f. [Z2]. The main purpose of this paper is to show that the categories first and last are equivalent.

3. The set of shapes of the, so called, pasting diagrams[2] is described in [HMP] as pasting diagrams and in [Z2] as normal ordered face structures.

4. The category of all the shapes of many-to-one cells is the category **oFs** described in [Z2].

The paper is organized as follows. In Section 2 we recall the definition of an ordered face structures and two kinds of maps between them: monotone and local. In section 3 we introduce the notion of a GT-theory that describes the abstract properties of **oFs**. Sections 4 to 7 establish the main goal of the paper. Through a sequence of three adjunctions we establish that the category of multitopes and the category of principal face structures are equivalent. The remaining three sections are exhibiting the properties of **oFs**. In Section 8 we define the category of shapes. In Section 9 we recall the relevant part of the work of T. Leinster and M. Weber in a way that is suitable for our context. Finally, in Section 10 we describe how the category **oFs** can generate all the ingredients involved in the definition of the many-to-one nerve construction for strict ω categories.

As this paper is a sequel to [Z2] we adopt here the notions and notation introduced there. This includes that we shall denote the compositions of morphisms both ways, i.e. the composition of two morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ may be denoted as either $g \circ f$ or $f; g$. But in any case we will write which way the composition is meant.

I would like to thank the anonymous referee for comments that encouraged me to simplify the exposition and to give a comprehensive explanation of the role the category of ordered face structures **oFs**. I also want to thank J. Kock for bringing [W] to my attention.

The diagrams for this paper were prepared with a help of *catmac* of Michael Barr.

2 Ordered face structures in a nutshell

This section is a quick introduction to ordered face structures. For more see [Z2].

A *hypergraph* S is

[2] n-pasting diagram are nothing but the shapes of domains of $n+1$ indets.

1. a family $\{S_k\}_{k\in\omega}$ of finite sets of faces; only finitely many among these sets are non-empty;

2. a family of functions $\{\gamma : S_{k+1} \sqcup 1_{S_k} \to S_k\}_{k\in\omega}$; where $1_{S_k} = \{1_u : u \in S_k\}$ is the set of *empty faces* of dimension k; the face 1_u is the empty $(k+1)$-dimensional face on a non-empty face u of dimension k.

3. a family of total relations $\{\delta : S_{k+1} \sqcup 1_{S_k} \to S_k \sqcup 1_{S_{k-1}}\}_{k\in\omega}$; for $a \in S_{k+1}$ we denote $\delta(a) = \{x \in S_k \sqcup 1_{S_{k-1}} : (a,x) \in \delta^S_k\}$; $\delta(a)$ is either singleton or it is non-empty subset of S_k[3]. Moreover $\delta : S_1 \sqcup 1_{S_0} \to S_0$ is a function ($S_{-1} = \emptyset$). We put $\dot{\delta}(a) = \delta(a) \cap S$.

A *morphism of hypergraphs* $f : S \longrightarrow T$ is a family of functions $f_k : S_k \longrightarrow T_k$ that preserves γ and δ i.e., for $k \in \omega$, $\gamma \circ f_{k+1} = f_k \circ \gamma$ and for $a \in S_{k+1}$ the restriction of f_k to $\delta(a)$: $f_a : \delta(a) \longrightarrow \delta(f(a))$ is a bijection (if $\delta(a) = 1_u$ we mean by that $\delta(f(a)) = 1_{f(u)}$).

Notation and conventions. If $a \in S_k$ we treat $\gamma(a)$ sometimes as an element of S_{k-1} and sometimes as a subset $\{\gamma(a)\}$ of S_{k-1}. Similarly $\delta(a)$ is treated sometimes as a set of faces or as a single face if this set of faces is a singleton. In particular, we say that a face a is a *loop* if $\gamma(a) = \delta(a)$ and by this we mean rather $\{\gamma(a)\} = \delta(a)$. If X is a set of faces in S then by $X^{-\lambda}$ we denote the set of faces in X that are not loops; $\dot{\delta}^{-\lambda}(a) = \dot{\delta}(a) \cap S^{-\lambda}$. The set of *internal faces* of a is $\iota(a) = \gamma\dot{\delta}^{-\lambda}(a) \cap \delta\dot{\delta}^{-\lambda}(a)$. The set $\theta(a) = \delta(a) \cup \gamma(a)$ ($\dot{\theta}(a) = \dot{\delta}(a) \cup \gamma(a)$) is the sets of (non-empty) faces of codimension 1 in a.

On each set S_k we introduce two binary relations $<^-$ and $<^+$. On S_0 the relation $<^-$ is empty. If $k > 0$, $<^-$ is the transitive closure of the relation $\lhd^-$ on S_k, such that $a \lhd^- b$ iff $\gamma(a) \in \delta(b)$. We write $a \perp^{S_k,-} b$ if either $a <^{S_k,-} b$ or $b <^{S_k,-} a$. $<^+$ is the transitive closure of the relation $\lhd^+$ on S_k, such that $a \lhd^+ b$ iff $a \neq b$ and there is $\alpha \in S^{-\lambda}_{k+1}$, such that $a \in \delta(\alpha)$ and $\gamma(\alpha) = b$. We write $a \perp^{S_k,+} b$ if either $a <^{S_k,+} b$ or $b <^{S_k,+} a$.

Let $A, B \subseteq S_k \cup 1_{S_{k-1}}$. We set that A is 1-*equal* B, notation $A \equiv_1 B$, iff $A \cup 1_{\theta(A\cap S)} = B \cup 1_{\theta(B\cap S)}$.

An *ordered face structure* $(S, <^{S_k,\sim})_{k\in\omega}$ (also denoted S) is a hypergraph S together with a family of $\{<^{S_k,\sim}\}_{k\in\omega}$ of binary relations ($<^{S_k,\sim}$ is a relation on S_k), if $S_0 \neq \emptyset$ and

1. *Globularity:* for $a \in S_{\geq 2}$: $\gamma\gamma(a) = \gamma\delta(a) - \delta\dot{\delta}^{-\lambda}(a)$, $\delta\gamma(a) \equiv_1 \delta\delta(a) - \gamma\dot{\delta}^{-\lambda}(a)$ and for any $x \in S$: $\delta(1_x) = x = \gamma(1_x)$.

2. *Local discreteness*: if $x, y \in \delta(a)$ then $x \not<^+ y$.

3. *Strictness*: for $k \in \omega$, the relations $<^+$ and $<^\sim$ are strict orders[4] on S_k; $<^+$ on S_0 is linear.

4. *Disjointness*: $\perp^\sim \cap \perp^+ = \emptyset$, and for any $a, b \in S_k$: if $a <^\sim b$ then $a <^- b$ moreover if $\theta(a) \cap \theta(b) = \emptyset$ then $a <^\sim b$ iff $a <^- b$.

5. *Pencil linearity*: for any $a, b \in S_{\geq 1}$, $a \neq b$,

$$\text{if } \dot{\theta}(\text{a}) \cap \dot{\theta}(\text{b}) \neq \emptyset \text{ then either a} \perp^\sim \text{b or a} \perp^+ \text{b}$$

for any $a \in S_{\geq 2}$such that $\delta(a) \in 1_S$, $b \in S_{\geq 2}$,

$$\text{if } \gamma\gamma(\text{a}) \in \iota(\text{b}) \text{ then either a} <^\sim \text{b or a} <^+ \text{b}$$

[3]In other words $\delta(a)$ is either equal to $\{1_x\}$ for some $x \in S_{k-1}$ or it is a non-empty subset of S_k.
[4]By *strict order* we mean an irreflexive and transitive relation.

6. *Loop-filling*: $S^\lambda \subseteq \gamma(S^{-\lambda})$ (where S^λ is the set of loops in S and $S^{-\lambda} = S - S^\lambda$).

The *monotone morphism* of ordered face structures $f : S \longrightarrow T$ is a hypergraph morphism that preserves the order $<^\sim$. The category of ordered face structures and monotone maps, is denoted by **oFs**.

The *size of an ordered face structure* S is the sequence natural numbers $size(S) = \{|S_n - \delta(S^{-\lambda}_{n+1})|\}_{n\in\omega}$, with almost all being equal 0. We have an order $<$ on such sequences, so that $\{x_n\}_{n\in\omega} < \{y_n\}_{n\in\omega}$ iff there is $k \in \omega$ such that $x_k < y_k$ and for all $l > k$, $x_l = y_l$. This order is well founded and many facts about ordered face structures can be proven by induction on the size. S is *principal* iff $size(S)_l \leq 1$, for $l \in \omega$. By **pFs** we denote full subcategory of **oFs** whose objects are principal ordered face structures. In [Z2] it was shown that either an ordered face structure is principal or there is a cut $\check{a}$ of S that is defining a proper decomposition of S into two ordered face structures $S^{\downarrow\check{a}}$ and $S^{\uparrow\check{a}}$ of smaller size than S such that their k-tensor product $S^{\downarrow\check{a}} \otimes_k S^{\uparrow\check{a}}$ is isomorphic to S, where k is the dimension of the cut. By $Sd(S)$ we denote the set of cuts of S defining proper decompositions of S.

The relation $<^\sim$ induces a binary relation $(\dot{\delta}(a), <^\sim_a)$ for each $a \in S_{>0}$ (where $<^\sim_a$ is the restriction of $<^\sim$ to the set $\dot{\delta}(a)$). The *local morphism* of ordered face structures $f : S \longrightarrow T$ is a hypergraph morphism that is a local isomorphism i.e. for $a \in S_{>1}$ the restricted map $f_a : (\dot{\delta}(a), <^\sim_a) \longrightarrow (\dot{\delta}(f(a)), <^\sim_{f(a)})$ is an order isomorphism, where f_a is the restriction of f to $\dot{\delta}(a)$. The category of ordered face structures and local morphisms is denoted by $\mathbf{oFs}_{loc}$.

3 Graded tensor theories

If we denote by $\mathbf{oFs}_n$ the full subcategory of **oFs** containing object of dimension at most n then for ordered face structures S, S' we have operations of the k-th domain $\mathbf{d}^{(k)}S$, the k-th codomain $\mathbf{c}^{(k)}S$ and k-tensor $S \otimes_k S'$ (whenever $\mathbf{c}^{(k)}S = \mathbf{d}^{(k)}S'$), see [Z2]. However the operations $\mathbf{d}^{(k)}S$ and $\mathbf{c}^{(k)}S$ are not functorial with respect to the monotone morphisms and **oFs** is not a monoidal globular category in the sense of Batanin, see [B], contrary to a public statement I have made. On the other hand the category of ordered face structures and inner maps, defined in Section 10 is a monoidal globular category.

To explain the essential abstract structure of the category **oFs** I introduce the notion of a graded tensor category. **N** is the poset of natural numbers.

Graded tensor theory $\mathcal{C}$ (GT-theory for short) is a category $\mathcal{C}$ equipped with

1. a *dimension* functor $dim : \mathcal{C} \to \mathbf{N}$; $\mathcal{C}_k$ is the full subcategory of $\mathcal{C}$ whose objects have dimension at most k;

2. the objects of $\mathcal{C}$ are rigid (i.e. no non-trivial automorphisms[5]);

3. for any object S of $\mathcal{C}$, such that $k \leq n = dim(S)$ there are domain and codomain morphisms

$$\begin{tikzcd} & S & \\ \mathbf{d}^{(k)}S \arrow[ur, "\mathbf{d}^{(k)}_S"] & & \mathbf{c}^{(k)}S \arrow[ul, "\mathbf{c}^{(k)}_S"'] \end{tikzcd}$$

[5]In particular if two objects are isomorphic in $\mathcal{C}$ the isomorphism is always unique. This allow us treating isomorphic objects as equal and morphism with isomorphic domains and codomains as parallel.

such that $dim(\mathbf{d}^{(k)}S) \leq k = dim(\mathbf{c}^{(k)}S)$ and, for $k < l \leq n$,

$$\mathbf{d}^{(l)}_S \circ \mathbf{c}^{(k)}_{\mathbf{d}^{(l)}S} = \mathbf{c}^{(l)}_S \circ \mathbf{c}^{(k)}_{\mathbf{c}^{(l)}S} = \mathbf{c}^{(k)}_S$$

$$\mathbf{c}^{(l)}_S \circ \mathbf{d}^{(k)}_{\mathbf{c}^{(l)}S} = \mathbf{d}^{(l)}_S \circ \mathbf{d}^{(k)}_{\mathbf{d}^{(l)}S} = \mathbf{d}^{(k)}_S$$

in particular the diagram

$$\begin{tikzcd}
& S & \\
\mathbf{d}^{(l)}S \arrow[ur, "\mathbf{d}^{(l)}_S"] & & \mathbf{c}^{(l)}S \arrow[ul, "\mathbf{c}^{(l)}_S"'] \\
\mathbf{d}^{(k)}S \arrow[u, "\mathbf{d}^{(k)}_{\mathbf{d}^{(l)}S}"] \arrow[urr, "\mathbf{d}^{(k)}_{\mathbf{c}^{(l)}S}"' near end] & & \mathbf{c}^{(k)}S \arrow[u, "\mathbf{c}^{(k)}_{\mathbf{c}^{(l)}S}"'] \arrow[ull, "\mathbf{c}^{(k)}_{\mathbf{d}^{(l)}S}" near end]
\end{tikzcd}$$

commutes. If $k \geq n$, we put $\mathbf{d}^{(k)}S$ and $\mathbf{c}^{(k)}S$ to be identities on S.

4. For $k < n$, the category $\mathcal{C}_n \times_k \mathcal{C}_n$ is the category whose objects consists of three objects, two in $\mathcal{C}_n$ and one in $\mathcal{C}_k$ and two maps as follows

$$\begin{tikzcd}
S & & S' \\
& \mathbf{d}^{(k)}S \cong \mathbf{c}^{(k)}S' \arrow[ul, "\mathbf{c}^{(k)}_S"] \arrow[ur, "\mathbf{d}^{(k)}_{S'}"'] &
\end{tikzcd}$$

Clearly the objects of $\mathcal{C}_n \times_k \mathcal{C}_n$ can be thought of as pairs (S, S') of objects of $\mathcal{C}_n$ satisfying an obvious compatibility condition. The morphisms in $\mathcal{C}_n \times_k \mathcal{C}_n$ are triples of morphisms in $\mathcal{C}$ commuting with the morphisms $\mathbf{c}^{(k)}$ and $\mathbf{d}^{(k)}$. We have three functors

$$\otimes_k, \pi_1, \pi_2 : \mathcal{C}_n \times_k \mathcal{C}_n \longrightarrow \mathcal{C}_n$$

where π_1 and π_2 are the obvious projections and two natural transformations

$$\pi_1 \xrightarrow{\kappa^1} \otimes_k \xleftarrow{\kappa^2} \pi_2$$

so that the squares

$$\begin{tikzcd}
S \arrow[r, "\kappa^1_S"] & S \otimes_k S' \\
\mathbf{c}^{(k)}S \arrow[u, "\mathbf{c}^{(k)}_S"] \arrow[r, "\mathbf{d}^{(k)}_{S'}"'] & S' \arrow[u, "\kappa^2_{S'}"']
\end{tikzcd}$$

commute, for any (S, S') in $\mathcal{C}_n \times_k \mathcal{C}_n$. Such squares, or squares isomorphic to them, are called *k-tensor squares* or simply *tensor squares.*

5. These data are related by the existence of some isomorphisms between objects and morphisms. But, as $\mathcal{C}$ is rigid, these isomorphisms are necessarily unique and hence there are no coherence conditions since any diagram of isomorphisms in $\mathcal{C}$ commutes. Thus we will put no names on those isomorphisms. R, R', S, S', S'' are assumed to be ordered face structures.

(a) *Domains and codomains of compositions.* For $k > l$, there are isomorphism making the triangles

$$\begin{tikzcd} & S \otimes_l S' & \\ \mathbf{d}^{(k)} S \otimes_l \mathbf{d}^{(k)} S' \arrow[ur, "\mathbf{d}^{(k)}_S \otimes_l \mathbf{d}^{(k)}_{S'}"] \arrow[rr, "\cong"'] & & \mathbf{d}^{(k)}(S \otimes_l S') \arrow[ul, "\mathbf{d}^{(k)}_{S\otimes_l S'}"'] \end{tikzcd} \qquad \begin{tikzcd} & S \otimes_l S' & \\ \mathbf{c}^{(k)} S \otimes_l \mathbf{c}^{(k)} S' \arrow[ur, "\mathbf{c}^{(k)}_S \otimes_l \mathbf{c}^{(k)}_{S'}"] \arrow[rr, "\cong"'] & & \mathbf{c}^{(k)}(S \otimes_l S') \arrow[ul, "\mathbf{c}^{(k)}_{S\otimes_l S'}"'] \end{tikzcd}$$

commute. For $k \leq l$, there are isomorphism making the triangles

$$\begin{tikzcd} & S \otimes_l S' & \\ \mathbf{d}^{(k)} S \arrow[ur, "\kappa^1_S \circ \mathbf{d}^{(k)}_S"] \arrow[rr, "\cong"'] & & \mathbf{d}^{(k)}(S \otimes_l S') \arrow[ul, "\mathbf{d}^{(k)}_{S\otimes_l S'}"'] \end{tikzcd} \qquad \begin{tikzcd} & S \otimes_l S' & \\ \mathbf{c}^{(k)} S' \arrow[ur, "\kappa^2_{S'} \circ \mathbf{c}^{(k)}_S"] \arrow[rr, "\cong"'] & & \mathbf{c}^{(k)}(S \otimes_l S') \arrow[ul, "\mathbf{c}^{(k)}_{S\otimes_l S'}"'] \end{tikzcd}$$

commute.

(b) *Units.* The following diagrams

$$\begin{tikzcd} X \arrow[r, "1_X"] & X \\ \mathbf{d}^{(k)} X \arrow[u, "\mathbf{d}^{(k)}_X"] \arrow[r, "1_{\mathbf{d}^{(k)}X}"'] & \mathbf{d}^{(k)} X \arrow[u, "\mathbf{d}^{(k)}_X"'] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "1_X"] & X \\ \mathbf{c}^{(k)} X \arrow[u, "\mathbf{c}^{(k)}_X"] \arrow[r, "1_{\mathbf{c}^{(k)}X}"'] & \mathbf{c}^{(k)} X \arrow[u, "\mathbf{c}^{(k)}_X"'] \end{tikzcd}$$

are k-tensor squares.

(c) *Associativity.* Whenever any of the expressions make sense we have an isomorphism

$$S \otimes_k (S' \otimes_k S'') \cong (S \otimes_k S') \otimes_k S''$$

(d) *Middle exchange.* For $k < l$, if we have a diagram

$$\begin{tikzcd} R & & S \\ \mathbf{c}^{(l)} R \arrow[u, "\mathbf{c}^{(l)}"] \arrow[d, "\mathbf{d}^{(l)}"'] & \mathbf{c}^{(k)} R \arrow[l, "\mathbf{c}^{(k)}"'] \arrow[r, "\mathbf{d}^{(k)}"] & \mathbf{c}^{(l)} S \arrow[u, "\mathbf{c}^{(l)}"'] \arrow[d, "\mathbf{d}^{(l)}"] \\ R' & & S' \end{tikzcd}$$

then the two objects we can form out of it

$$(R \otimes_l R') \otimes_k (S \otimes_l S') \cong (R \otimes_k S) \otimes_l (R' \otimes_k S')$$

are isomorphic.

A *model* of a GT-theory $\mathcal{C}$ in a category $\mathcal{A}$, or $\mathcal{A}$-model of $\mathcal{C}$ for short, is a functor from $\mathcal{C}^{op}$ to $\mathcal{A}$ which sends tensor squares to pullbacks and the distinguished isomorphisms to the canonical isomorphisms. By $Mod_\otimes(\mathcal{C}^{op}, \mathcal{A})$ w denote the category of $\mathcal{A}$-models of $\mathcal{C}$ and natural transformations. A model $\mathcal{G}_\mathcal{C} : \mathcal{C}^{op} \to \mathcal{A}$ is *generic* iff for any other model $M : \mathcal{C}^{op} \to \mathcal{B}$ there is a unique (up to an iso) functor $\overline{M} : \mathcal{A} \to \mathcal{B}$ making the triangle

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathcal{G}_\mathcal{C}"] \arrow[dr, "M"'] & \mathcal{A} \arrow[d, "\overline{M}"] \\ & \mathcal{B} \end{tikzcd}$$

commute up to an isomorphism, and for any natural transformation $\sigma : M \to M'$ between models there is a unique natural transformation $\overline{\sigma} : \overline{M} \to \overline{M'}$ such that $\overline{\sigma}_{\mathcal{G}_\mathcal{C}} = \sigma$. If identity functor on GT-theory $\mathcal{C}$ is a model then it is the generic model of $\mathcal{C}$ and $\mathcal{C}$ is a called a *realized* GT-theory.

Examples.

1. The category Δ_0 of finite linear graphs is clearly a GT-theory. It contains one object $[0]$ of dimension 0 and all the other objects are of dimension 1. The domain and codomain maps $\mathbf{d}^{(0)}_{[n]}, \mathbf{c}^{(0)}_{[n]} : [0] \to [n]$ send the only vertex of $[0]$ to the first and last vertex in the in $[n]$, respectively. The 0-tensor of any two objects is defined and we have $[n] \otimes_0 [m] = [n+m]$ with the obvious inclusions. In this case the tensor squares are actual pushouts in Δ_0, i.e. Δ_0 is a realized GT-theory. Note that this tensor operation does not make Δ_0 a monoidal category as $+$ is not 'sufficiently functorial'.

2. The category $s\omega Gr$ of simple ω-graphs (or globular cardinals) is also a realized GT-theory. If we look at the objects of $s\omega Gr$ as one-to-one pasting diagrams then the domain and the codomain operations are the pasting diagrams of the k-th domain and the k-th codomain of this diagram.

3. The whole GT-theory structure of the category $\mathbf{Fs}^{+/1}$ of positive face structures is described in [Z1] and even in this case the tensor squares are pushouts, i.e. $\mathbf{Fs}^{+/1}$ is a realized GT-theory, as well.

4. The GT-theory structure of the category $\mathbf{oFs}$ of ordered face structures is described in [Z2] but in this case the tensor squares are not pushouts in general. This is because only part of the order $<^\sim$ in the tensor is determined by the components, see [Z2] details. The embedding functor $\mathcal{G}_{\mathbf{oFs}} : \mathbf{oFs}^{op} \to \mathbf{oFs}^{op}_{loc}$ is the generic model of $\mathbf{oFs}$.

5. In Section 10 we shall define still another GT-theory $\mathbf{oFs}_\mu$ of ordered face structures and monotone ω-maps which has a non-identity generic model $\mathcal{G}_{\mathbf{oFs}_\mu} : \mathbf{oFs}_\mu \to \mathbf{oFs}_\omega$.

We have the following

Proposition 3.1 *Let $\mathcal{C}$ be an essentially small GT-theory and $\mathcal{A}$ be a locally small category, and $M : \mathcal{C} \longrightarrow \mathcal{A}$ be a functor such that its dual $M^{op} : \mathcal{C}^{op} \longrightarrow \mathcal{A}^{op}$ is an $\mathcal{A}^{op}$-model of $\mathcal{C}$. Then M induces a functor $\bar{M} : \mathcal{A} \longrightarrow \omega Cat$.*

Proof. I will give the construction of $\bar{M}$ only. Let A be an object of $\mathcal{A}$. The n-cells of the ω-category $\bar{M}(A)$ are given by

$$\bar{M}(A)_n = \coprod_S \mathcal{A}(M(S), A)$$

where the coproduct is taken over all (up to isomorphism[6]) objects of $\mathcal{C}$ of dimension at most n. If S has dimension lower than n than the morphism $M(S) \to A$ is considered as the identity at dimension n of a lower dimension cell. In particular, the identity operations in the ω-category $\bar{M}$ are inclusions. The k-th domain and k-th codomain operations $d^{(k)}, c^{(k)} : \bar{M}(A)_n \longrightarrow \bar{M}(A)_k$ are defined by composition as follows. For $a : M(S) \to A \in \bar{M}(A)_n$ we put

$$d^{(k)}(a) = a \circ M(\mathbf{d}^{(k)}_S), \qquad c^{(k)}(a) = a \circ M(\mathbf{c}^{(k)}_S).$$

[6]In practice we think that we take the coproduct over all objects of $\mathcal{C}$ of dimension at most n and we identify two maps $a : M(S) \to A$ with $a' : M(S') \to A$ if there is a (necessarily unique) isomorphism $i : S :\to S'$ such that $a' \circ M(i) = a$.

If $b : S' \to A \in \bar{M}(A)_n$ so that $c^{(k)}(a) = d^{(k)}(b)$ the outer square in the following diagram

$$\begin{tikzcd}
 & & A \\
M(S) \arrow[rr] \arrow[urr, "a"] & & M(S \otimes_k S') \arrow[u, "a;_k b"'] \\
M(\mathbf{c}^{(k)}S) \arrow[u, "M(\mathbf{c}^{(k)}_S)"] \arrow[rr, "M(\mathbf{d}^{(k)}_{S'})"'] & & M(S') \arrow[u] \arrow[uu, bend right, "b"']
\end{tikzcd}$$

commutes. Since M is a model of $\mathcal{C}$ the inner square is a pushout and we have a morphism $a;_k b : M(S \otimes S') \to A$ making the remaining two triangles commute. $\Box$

Remark. It is convenient to thing about **oFs** in terms of the abstract notion of a graded tensor category. In fact more than the above Lemma can be stated for an abstract graded tensor category not only **oFs**. But I think that the general theory of GT-theories should wait until more non-trivial GT-theories that are not a realized GT-theories are found.

4 The category $\mathbf{oFs}_{loc}$

The following Lemma subsume some properties of the category of ordered face structures and local maps $\mathbf{oFs}_{loc}$ that are essentially in [Z2].

Lemma 4.1 *Let $f : S \to T$ and $g : P \to T$ be morphisms in $\mathbf{oFs}^{op}_{loc}$, with P being a principal ordered face structure of dimension n, $P_n = \{\mathbf{m}_P\}$, $a \in S_n$. If $f(a) = g(\mathbf{m}_P)$ then there is a unique map $\bar{g} : P \to S$ such that $\bar{g}(\mathbf{m}_P) = a$ and hence $f \circ \bar{g} = g$. In particular, any principal ordered face structure is projective in $\mathbf{oFs}^{op}_{loc}$.*

Proof. The first statement follows from Lemmas 11.1 and 11.2 from [Z2]. To see that this imply that principal ordered face structures are projective in $\mathbf{oFs}^{op}_{loc}$ it is enough to note that the local maps in $\mathbf{oFs}_{loc}$ are epi iff they are onto. $\Box$

Let S be an ordered face structure. We have an obvious projection functor

$$\Sigma^S : \mathbf{pFs} \downarrow S \longrightarrow \mathbf{pFs} \longrightarrow \mathbf{oFs}_{loc}$$

such that

$$\Sigma^S(f : P \to S) = P$$

and the *principal cocone over* S

$$\sigma^S : \Sigma^S \longrightarrow S$$

such that

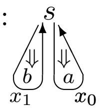
$$\sigma^S_{(f:P\to S)} = f : \Sigma^S(f : P \to S) = P \longrightarrow S$$

We have

Lemma 4.2 *The cocone $\sigma^S : \Sigma^S \stackrel{\cdot}{\longrightarrow} S$ is a colimiting cocone in $\mathbf{oFs}_{loc}$.*

Proof. Simple check. $\Box$

The colimits in $\mathbf{oFs}_{loc}$, as described above, are called *principal colimits* (and when considered in $\mathbf{oFs}_{loc}^{op}$ are called *principal limits*). Note that we are not saying that the above cocone is a colimit in $\mathbf{oFs}$ i.e. in the category of ordered face structures and monotone maps. For example if S is

$S :$


then clearly the principal cocone over S is not a colimiting cocone in $\mathbf{oFs}$ as it does not determine the order $<^{\sim}$ between faces x_1 and x_0. In fact the ordered face structures S for which the cocone $\sigma^S : \Sigma^S \stackrel{\cdot}{\longrightarrow} S$ is a colimiting cocone in $\mathbf{oFs}$ have several good properties that are going to be studied elsewhere.

The following Lemma states some properties of principal cocones over tensors.

Lemma 4.3 *Let S and S' be ordered face structures such that $\mathbf{c}^{(k)}(S) = \mathbf{d}^{(k)}(S')$, P a principal ordered face structure, and $f : P \longrightarrow S \otimes_k S'$ a map in $\mathbf{oFs}_{loc}$. Then*

1. *either f factorizes (uniquely) via κ^1_S or f factorizes (not necessarily uniquely) via $\kappa^2_{S'}$;*

$$\begin{tikzcd}
S \arrow[r, "\kappa^1_S"] & S \otimes_k S' & S' \arrow[l, "\kappa^2_{S'}"'] \\
& P \arrow[ul, "g"] \arrow[u, "f"] \arrow[ur, "h"'] &
\end{tikzcd}$$

2. *if there are both g and h factorizations of f then there is a factorization l making the diagram*

$$\begin{tikzcd}
S & P \arrow[l, "g"'] \arrow[r, "h"] \arrow[d, "l"] & S' \\
& \mathbf{c}^{(k)}S \arrow[ul, "\mathbf{c}^{(k)}_S"] \arrow[ur, "\mathbf{d}^{(k)}_{S'}"'] &
\end{tikzcd}$$

   commute;

3. *finally, if there are two factorizations h and h' of f via $\kappa^2_{S'}$ then there a factorization g via κ^1_S.*

Proof. This easily follows from the explicite description of the tensors in [Z2]. $\Box$

5 Simple adjunction

The proof of the main theorem proceeds by establishing three adjoint equivalences.

By Lemma 11.1 of [Z2], the inclusion functor $\mathbf{i} : \mathbf{pFs} \longrightarrow \mathbf{oFs}_{loc}$ is full and faithful. It induces the adjunction

$$\begin{tikzcd}
Set^{\mathbf{pFs}^{op}} \arrow[r, shift left, "Ran_{\mathbf{i}}"] & Set^{\mathbf{oFs}_{loc}^{op}} \arrow[l, shift left, "\mathbf{i}^*"]
\end{tikzcd}$$

where $\mathbf{i}^*$ is the functor of composing with $\mathbf{i}$ and $Ran_{\mathbf{i}}$ is the right Kan extension along $\mathbf{i}$. Recall that for F in $Set^{\mathbf{pFs}^{op}}$, S in $\mathbf{oFs}$, it is defined as the following limit

$$(Ran_{\mathbf{i}}F)(S) = LimF \circ (\Sigma^S)^{op}$$

where $(\Sigma^S)^{op} : S \downarrow \mathbf{pFs}^{op} \to \mathbf{pFs}^{op}$. Clearly $Ran_{\mathbf{i}}F$ preserves principal limits. As $\mathbf{i}$ is full and faithful the right Kan extension $Ran_{\mathbf{i}}(F)$ is an extension, i.e. the counit of this adjunction

$$\varepsilon_F : (Ran_{\mathbf{i}}\, F) \circ \mathbf{i} \longrightarrow F$$

is an isomorphism. In particular, $Ran_{\mathbf{i}}$ is full and faithful. It is easy to see, that for G in $Set^{\mathbf{oFs}^{op}_{loc}}$ the unit of adjunction

$$\eta_G : G \longrightarrow Ran_{\mathbf{i}}(G \circ \mathbf{i})$$

is an isomorphism iff G preserves principal limits. Thus we have proved

Proposition 5.1 *The above adjunction restricts to the following equivalence of categories*

$$\begin{tikzcd} Set^{\mathbf{pFs}^{op}} \arrow[r, shift left, "Ran_{\mathbf{i}}"] & pLim(\mathbf{oFs}^{op}_{loc}, Set) \arrow[l, shift left, "\mathbf{i}^*"] \end{tikzcd}$$

where $pLim(\mathbf{oFs}^{op}_{loc}, Set)$ is the category of principal limits preserving functors and natural transformation. $\Box$

6 Tensor squares vs principal limits

We shall define an adjunction

$$\begin{tikzcd} Set^{\mathbf{oFs}^{op}_{loc}} \arrow[r, shift right, "\mathcal{L}"'] & Mod_{\otimes}(\mathbf{oFs}^{op}, Set) \arrow[l, shift right, "\mathbf{e}"'] \end{tikzcd}$$

The functor $\mathbf{e}$ is sending Set-models of $\mathbf{oFs}$ to the presheaves on $\mathbf{oFs}_{loc}$ along the generic model $\mathcal{G}_{\mathbf{oFs}}$ and natural transformations to the same natural transformations. Thus $\mathbf{e}$ can be thought of as an embedding that is extending the models of $\mathbf{oFs}$ by defining them on all the local maps in $\mathbf{oFs}_{loc}$. We tend to omit $\mathbf{e}$ in formulas writing for example the unit of this adjunction as $\eta_F : F \to \mathcal{L}(F)$, understanding that it is a morphism in $Set^{\mathbf{oFs}^{op}_{loc}}$ rather than in $Mod_{\otimes}(\mathbf{oFs}^{op}, Set)$.

Let $F : \mathbf{oFs}^{op}_{loc} \to Set$ be a presheaf. Then

$$\mathcal{L}(F) = Lim(F \circ (\Sigma^{(-)})^{op}) : \mathbf{oFs}^{op}_{loc} \longrightarrow Set$$

i.e. $\mathcal{L}(F)(S)$ is the limit of the following functor

$$(\mathbf{pFs} \downarrow S)^{op} \xrightarrow{(\Sigma^S)^{op}} \mathbf{oFs}^{op}_{loc} \xrightarrow{F} Set$$

with the limiting cone $\sigma^{F,S} : \mathcal{L}(F)(S) \longrightarrow F \circ (\Sigma^S)^{op}$. For a monotone map $f : S \to S'$ the function $\mathcal{L}(F)(f)$ is so defined that for any $h : P \to S$ in $(\mathbf{pFs} \downarrow S)^{op}$ the triangle

$$\begin{tikzcd} \mathcal{L}(F)(S) \arrow[rr, "\mathcal{L}(F)(f)"] \arrow[dr, "\sigma^{F,S}_h"'] & & \mathcal{L}(F)(S') \arrow[dl, "\sigma^{F,S'}_{f\circ h}"] \\ & F(P) & \end{tikzcd}$$

commutes. For a natural transformation $\tau : F \longrightarrow F'$ and $S \in \mathbf{oFs}$, $\mathcal{L}(\tau)_S$ is the unique map making the squares

$$
\begin{tikzcd}
\mathcal{L}(F)(S) \arrow[r, "\mathcal{L}(\tau)_S"] \arrow[d, "\sigma_h^{F,S}"'] & \mathcal{L}(F')(S) \arrow[d, "\sigma_h^{F',S}"] \\
F(P) \arrow[r, "\tau_P"'] & F'(P)
\end{tikzcd}
$$

commutes, for any $h : P \to S \in \mathbf{oFs}$.

Proposition 6.1 *$\mathcal{L}$ is well defined functor and $\mathcal{L} \dashv \mathbf{e}$.*

Proof. The fact that $\mathcal{L}(F)$ is a functor and $\mathcal{L}(\tau)$ is a natural transformation is left to the reader. We shall verify that $\mathcal{L}(F)$ is a model of $\mathbf{oFs}$, i.e. sends tensor squares to pullbacks.

Let S and S' be ordered face structures such that $\mathbf{c}^{(k)}S = \mathbf{d}^{(k)}S'$. As $\mathcal{L}$ is a functor it sends commuting squares to commuting squares and hence we have a unique function φ making the diagram

$$
\begin{tikzcd}
& & \mathcal{L}(F)(S \otimes_k S') \arrow[dll, "\mathcal{L}(F)(\kappa^1_S)"'] \arrow[d, "\varphi"] \arrow[drr, "\mathcal{L}(F)(\kappa^2_{S'})"] & & \\
\mathcal{L}(F)(S) \arrow[drr, "\mathcal{L}(F)(\mathbf{c}^{(k)}_S)"'] & & \mathcal{L}(F)(S) \times_{\mathcal{L}(F)(\mathbf{c}^{(k)}S)} \mathcal{L}(F)(S') \arrow[ll, "\pi^{F,S}"'] \arrow[rr, "\pi^{F,S'}"] & & \mathcal{L}(F)(S') \arrow[dll, "\mathcal{L}(F)(\mathbf{d}^{(k)}_{S'})"] \\
& & \mathcal{L}(F)(\mathbf{c}^{(k)}S) & &
\end{tikzcd}
$$

commute. We shall define a function

$$\psi : \mathcal{L}(F)(S) \times_{\mathcal{L}(F)(\mathbf{c}^{(k)}S)} \mathcal{L}(F)(S') \longrightarrow \mathcal{L}(F)(S \otimes_k S')$$

the inverse of φ, by defining a cone ξ from $\mathcal{L}(F)(S) \times_{\mathcal{L}(F)(\mathbf{c}^{(k)}S)} \mathcal{L}(F)(S')$ to the functor

$$(\mathbf{pFs} \downarrow (S \otimes_k S'))^{op} \xrightarrow{(\Sigma^{S\otimes_k S'})^{op}} \mathbf{oFs}^{op}_{loc} \xrightarrow{\quad F \quad} Set$$

Let $f : P \to S \otimes_k S'$ be a map in $\mathbf{oFs}_{loc}$

$$\xi_f = \begin{cases} \pi^{F,S}; \sigma^{F,S}_g & \text{if } f = g; \kappa^1_S \text{ for some } g : P \to S, \\ \pi^{F,S'}; \sigma^{F,S'}_h & \text{if } f = h; \kappa^2_{S'} \text{ for some } h : P \to S', \end{cases}$$

The Lemma 4.3 guarantee that this definition gives in fact a cone ξ over $F \circ (\Sigma^{S\otimes_k S'})^{op}$. Thus we have ψ as in the diagram

$$
\begin{tikzcd}
\mathcal{L}(F)(S) \times_{\mathcal{L}(F)(\mathbf{c}^{(k)}S)} \mathcal{L}(F)(S') \arrow[rr, "\psi"] \arrow[dr, "\pi^{F,S'}"] \arrow[dd, "\pi^{F,S}"'] & & \mathcal{L}(F)(S \otimes_k S') \arrow[dd, "\sigma^{F,S\otimes_k S'}_f"] \\
& \mathcal{L}(F)(S') \arrow[dr, "\sigma^{F,S'}_h"] & \\
\mathcal{L}(F)(S) \arrow[rr, "\sigma^{F,S}_g"'] & & F(Q) = F \circ (\Sigma^{S\otimes_k S'})^{op}(f)
\end{tikzcd}
$$

As both φ and ψ are defined using universal properties of limits, it is easy to see that they are mutually inverse, i.e. $\mathcal{L}(F)$ preserves special pullbacks.

As the image under F of the principal cocone σ^S from Σ^S to S is a cone from $F(S)$ to $F \circ (\Sigma^S)^{op}$ we have a unique map $(\eta_F)_S : F(S) \longrightarrow \mathcal{L}(F)(S)$ making the triangles

$$
\begin{tikzcd}
F(S) \arrow[rr, "(\eta_F)_S"] \arrow[dr, "F(h)"'] & & \mathcal{L}(F)(S) \arrow[dl, "\sigma^{F,S}_h"] \\
& F(P) &
\end{tikzcd}
$$

commute, for any $h : P \to S \in \mathbf{oFs}_{loc}$. That defines the unit of adjunction $\mathcal{L} \dashv \mathbf{e}$. For any principal ordered face structure P, the category $\mathbf{pFs} \downarrow P$ has the terminal object $1_P : P \to P$. Thus any P-component of the unit of adjunction $(\eta_F)_P : F(P) \longrightarrow \mathcal{L}(F)(P)$ is an isomorphism.

The counit of adjunction $\varepsilon_G : \mathcal{L}(G) \to G$ is defined using the fact that both G and $\mathcal{L}(G)$ are models of $\mathbf{oFs}$. The map $(\varepsilon_G)_S : \mathcal{L}(G)(S) \to G(S)$ is defined by induction on the size of S. If $S = P$ is a principal ordered face structure then we put $(\varepsilon_G)_P = (\eta_F)^{-1}_P$. If S is not principal than with $\check{a} \in Sd(S)_k$, and we have $S = S^{\downarrow\check{a}} \otimes_k S^{\downarrow\check{a}}$ we put

$$(\varepsilon_G)_S = (\varepsilon_G)_{S^{\downarrow\check{a}}} \times_{(\varepsilon_G)_{\mathbf{c}^{(k)}(S^{\downarrow\check{a}})}} (\varepsilon_G)_{S^{\uparrow\check{a}}}$$

To verify the triangular equalities it is enough to show that the triangles

$$
\begin{tikzcd}
& \mathcal{L}(\mathcal{L}(F))(S) \arrow[dr, "(\varepsilon_{\mathcal{L}(F)})_S"] & \\
\mathcal{L}(F)(S) \arrow[ur, "\mathcal{L}((\eta_F)_S)"] \arrow[rr, "1_{\mathcal{L}(F)(S)}"'] & & \mathcal{L}(F)(S)
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathcal{L}(G)(S) \arrow[dr, "(\varepsilon_G)_S"] & \\
G(S) \arrow[ur, "(\eta_G)_S"] \arrow[rr, "1_{G(S)}"'] & & G(S)
\end{tikzcd}
$$

commute, for each ordered face structure S separately. The commutation of the left triangle can be shown using the fact that all functors involved preserves principal limits and the commutation of the right triangle can be shown by induction on the size of S using the fact that all the involved functors are models of $\mathbf{oFs}$. The remaining details are left for the reader. $\Box$

Proposition 6.2 *The above adjunction restricts to the following equivalence of categories*

$$
\begin{tikzcd}
pLim(\mathbf{oFs}^{op}_{loc}, Set) \arrow[r, shift right, "\mathcal{L}"'] & Mod_{\otimes}(\mathbf{oFs}^{op}, Set) \arrow[l, shift right, "\mathbf{e}"']
\end{tikzcd}
$$

Proof. As $\mathbf{e}$ is full and faithful the counit of the adjunction $\mathbf{e} \dashv \mathcal{L}$ is an isomorphism. From the description of the functor $\mathcal{L}$ it is clear that, for any functor $F : \mathbf{oFs}^{op}_{loc} \to Set$, $\mathcal{L}(F)$ preserves principal limits and that $\eta_F : F \to \mathcal{L}(F)$ is an isomorphism iff F preserves principal limits. $\Box$

7 Third adjunction

Recall that in [Z2] we have defined a functor $(-)^* : \mathbf{oFs} \to \mathbf{Comp}^{m/1}$ associating to any ordered face structure S the many-to-one computad S^* generated by S. The n-cells of S^* are (equivalence classes of) local maps $a : R \to S$ from ordered face structures R of dimension at most n. The k-domain and the k-codomain of a are $d^{(k)}(a) = a \circ \mathbf{d}^{(k)}_R$ and $c^{(k)}(a) = a \circ \mathbf{c}^{(k)}_R$, respectively. If $b : R' \to S$ is another local morphisms such that $c^{(k)}(a) = d^{(k)}(b)$ then the unique local map $a;_k b : R \otimes_k R' \to S$ such, that $a;_k b \circ \kappa^1 = a$ and $a;_k b \circ \kappa^2 = b$, is the composition of a and b in S^*. For more details on functor $(-)^*$ see [Z2].

Now we will set up the adjunction

$$\begin{tikzcd}[column sep=huge] Mod_{\otimes}(\mathbf{oFs}^{op}, Set) \arrow[r, shift left, "\widetilde{(-)}"] & \mathbf{Comp}^{m/1} \arrow[l, shift left, "\widehat{(-)} = \mathbf{Comp}^{m/1}((\simeq)^*, -)"] \end{tikzcd}$$

which will turn out to be an equivalence of categories. The functor $\widehat{(-)}$ is sending a many-to-one computad $\mathcal{P}$ to a functor

$$\widehat{\mathcal{P}} = \mathbf{Comp}^{m/1}((-)^*, \mathcal{P}) : \mathbf{oFs}^{op} \longrightarrow Set$$

$\widehat{(-)}$ is defined on morphism in the obvious way, by composition. We have

Lemma 7.1 *Let $\mathcal{P}$ be a many-to-one computad. Then $\widehat{\mathcal{P}}$ defined above sends tensor squares to pullbacks.*

Proof. This is an immediate consequence of the fact that the functor $(-)^* : \mathbf{oFs} \to \mathbf{Comp}^{m/1}$ sends tensor squares to pushouts, Corollary 13.3 in [Z2]. $\Box$

The functor $\widetilde{(-)}$ that we describe below is the induced functor described in Proposition 3.1 for the model $(-)^* : \mathbf{oFs}^{op} \to (\mathbf{Comp}^{m/1})^{op}$. As we need to establish some properties of $\widetilde{(-)}$ we give here a more detailed description.

Suppose we have a model $F : \mathbf{oFs}^{op} \longrightarrow Set$. We shall define a many-to-one computad $\widetilde{F}$. As the set of n-cells of $\widetilde{F}$ we take

$$\widetilde{F}_n = \coprod_S F(S)$$

where the coproduct[7] is taken over all (up to a monotone isomorphisms) ordered face structures S of dimension at most n. By $\kappa_n^{F,S} : F(S) \longrightarrow \widetilde{F}_n$ we denote the coprojection into the coproduct. For $k \leq n$, the identity map

$$1^{(n)} : \widetilde{F}_k \longrightarrow \widetilde{F}_n$$

is the obvious embedding induced by identity maps on the components of the coproducts. For $k \leq n$, we define the k-domain and the k-codomain functions in $\widetilde{F}$

$$d^{(k)}, c^{(k)} : \widetilde{F}_n \longrightarrow \widetilde{F}_k.$$

Abstractly, $d^{(k)}$ is the unique map, that makes the diagram

$$\begin{tikzcd}[column sep=huge, row sep=large] \widetilde{F}_n \arrow[r, "d^{(k)}"] & \widetilde{F}_k \\ F(S) \arrow[u, "\kappa_n^{F,S}"] \arrow[r, "F(\mathbf{d}_S^{(k)})"'] & F(\mathbf{d}^{(k)}S) \arrow[u, "\kappa_n^{F,d^{(k)}S}"'] \end{tikzcd}$$

commute, for any ordered face structure S. $c^{(k)}$ is defined similarly. In more concrete terms $d^{(k)}$ and $c^{(k)}$ are defined as follows. Let S be an ordered face structure of dimension at most n, $a \in F(S) \longrightarrow \widetilde{F}_n$ an n-cell in $\widetilde{F}$. We have in $\mathbf{oFs}$ the morphisms of the k-th domain and the k-th codomain introduced in [Z2]:

[7] In fact, we think about such a coproduct $\coprod_S F(S)$ as if it were to be taken over sufficiently large (so that each isomorphism type of ordered face structures is represented) set of ordered face structures S of dimension at most n. Then, if ordered face structures S and S' are isomorphic via (necessarily unique) monotone isomorphism $h : S' \to S$, then the cells $x \in F(S)$ and $x' \in F(S')$ are considered equal iff $F(h)(x) = x'$.

$$\begin{tikzcd} & S & \\ \mathbf{d}^{(k)}S \arrow[ur, "\mathbf{d}^{(k)}_S"] & & \mathbf{c}^{(k)}S \arrow[ul, "\mathbf{c}^{(k)}_S"'] \end{tikzcd}$$

We put

$$d^{(k)}(a) = F(\mathbf{d}^{(k)}_S)(a) \in F(\mathbf{d}^{(k)}S) \longrightarrow \widetilde{F}_k,$$
$$c^{(k)}(a) = F(\mathbf{c}^{(k)}_S)(a) \in F(\mathbf{c}^{(k)}S) \longrightarrow \widetilde{F}_k.$$

Finally, we define the compositions in $\widetilde{F}$. Again we shall do it first abstractly and then in concrete terms. Note that the pullback

$$\begin{tikzcd} \widetilde{F}_n \times_{\widetilde{F}_k} \widetilde{F}_n \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & \widetilde{F}_n \arrow[d, "d^{(k)}"] \\ \widetilde{F}_n \arrow[r, "c^{(k)}"'] & \widetilde{F}_k \end{tikzcd}$$

can be describe as a coproduct

$$\widetilde{F}_n \times_{\widetilde{F}_k} \widetilde{F}_n = \coprod_{S,S'} F(S) \times_{F(\mathbf{c}^{(k)}S)} F(S') \quad (\cong \coprod_{S,S'} F(S \otimes_k S'))$$

where the coproduct is taken over all (up to monotone isomorphisms) pairs of ordered face structures S and S' of dimension at most n such that $\mathbf{c}^{(k)}S = \mathbf{d}^{(k)}S'$. The coprojections are denoted by

$$\kappa^{F,S,S'}_{n,k} : F(S) \times_{F(\mathbf{c}^{(k)}S)} F(S') \longrightarrow \coprod_{S,S'} F(S) \times_{F(\mathbf{c}^{(k)}S)} F(S') = \widetilde{F}_n \times_{\widetilde{F}_k} \widetilde{F}_n$$

Then the composition morphism

$$;_k : \widetilde{F}_n \times_{\widetilde{F}_k} \widetilde{F}_n \longrightarrow \widetilde{F}_n$$

is the unique map that for any pair S, S' as above makes the square

$$\begin{tikzcd} \widetilde{F}_n \times_{\widetilde{F}_k} \widetilde{F}_n \arrow[r, ";_k"] & \widetilde{F}_n \\ F(S) \times_{F(\mathbf{d}^{(k)}S)} F(S') \arrow[u, "\kappa^{F,S,S'}_{n,k}"] \arrow[r, "\zeta_{S,S'}"'] & F(S \otimes_k S') \arrow[u, "\kappa^{F,S\otimes_k S'}_{n}"'] \end{tikzcd}$$

commute, where $\zeta_{S,S'}$ is the inverse of the canonical isomorphism

$$F(S \otimes_k S') \longrightarrow F(S) \times_{F(\mathbf{c}^{(k)}S)} F(S')$$

that exists as F preserves special pullbacks. In concrete terms, the composition in $\widetilde{F}$ can be described as follows. Let $k < n$, $dim(S), dim(S') \leq n$, $\mathbf{c}^{(k)}S = \mathbf{d}^{(k)}S'$,

$$a \in F(S) \longrightarrow \widetilde{F}_n \qquad b \in F(S') \longrightarrow \widetilde{F}_n,$$

such that

$$c^{(k)}(a) = F(\mathbf{c}^{(k)}_S)(a) = F(\mathbf{d}^{(k)}_{S'})(b) = d^{(k)}(b).$$

We shall define the cell $a;_k b \in \widetilde{F}_n$. We have a tensor square in **oFs**:

$$\begin{tikzcd} S \arrow[r, "\kappa_S"] & S \otimes_k S' \\ \mathbf{c}^{(k)} S \arrow[u, "\mathbf{c}^{(k)}_S"] \arrow[r, "\mathbf{d}^{(k)}_{S'}"'] & S' \arrow[u, "\kappa_{S'}"'] \end{tikzcd}$$

As F is a model of **oFs** the square

$$\begin{tikzcd} F(S) \arrow[r, "F(\kappa_S)"] & F(S \otimes_k S') \\ F(\mathbf{c}^{(k)} S) \arrow[u, "F(\mathbf{c}^{(k)}_S)"] \arrow[r, "F(\mathbf{d}^{(k)}_{S'})"'] & F(S') \arrow[u, "F(\kappa_{S'})"'] \end{tikzcd}$$

is a pullback in Set. Thus there is a unique element

$$x \in F(S \otimes_k S') \longrightarrow \widetilde{F}_n$$

such that

$$F(\kappa_S)(x) = a, \qquad F(\kappa_{S'})(x) = b.$$

We put

$$a;_k b = x.$$

This ends the definition of $\widetilde{F}$.

For a morphism $\alpha : F \longrightarrow G$ in $Mod_\otimes(\mathbf{oFs}^{op}, Set)$ we put

$$\widetilde{\alpha} = \{\widetilde{\alpha}_n : \widetilde{F}_n \longrightarrow \widetilde{G}_n\}_{n\in\omega}$$

such that

$$\widetilde{\alpha}_n = \coprod_S \alpha_S : \widetilde{F}_n \longrightarrow \widetilde{G}_n$$

where the coproduct is taken over all (up to monotone isomorphism) ordered face structures S of dimension at most n. This ends the definition of the functor $\widetilde{(-)}$.

We have

Proposition 7.2 *The functor*

$$\widetilde{(-)} : Mod_\otimes(\mathbf{oFs}^{op}, Set) \longrightarrow \mathbf{Comp}^{m/1}$$

is well defined.

Proof. The verification that $\widetilde{(-)}$ is a functor into ωCat is left for the reader. We shall verify that, for model $F : \mathbf{oFs}^{op} \longrightarrow Set$ of **oFs**, $\widetilde{F}$ is a many-to-one computad, whose n-indets are

$$|\widetilde{F}|_n = \coprod_{P\in\mathbf{pFs}, dim(P)=n} F(P) \;\longrightarrow\; \coprod_{S\in\mathbf{oFs}, dim(S)\leq n} F(S) = \widetilde{F}_n.$$

Let $\mathcal{P}$ be the n-truncation of $\widetilde{F}$ in $\mathbf{Comma}^{m/1}_n$, i.e. $\mathcal{P} = \widetilde{F}^{\natural,n}$ in the notation from Appendix of [Z2]. We shall show that $\widetilde{F}_n$ is in a bijective correspondence with $\overline{\mathcal{P}}_n$ described in [Z2]. We define a function

$$\varphi : \overline{\mathcal{P}}_n \longrightarrow \widetilde{F}_n$$

so that for a cell $f : S^{\sharp,n} \longrightarrow \mathcal{P}$ in $\overline{\mathcal{P}}_n$ we put

$$\varphi(f) = \begin{cases} 1_{f_{n-1}(S)} & \text{if } dim(S) < n, \\ f_n(m_S) & \text{if } dim(S) = n,\ S \text{ principal},\ S_n = \{m_S\} \\ \varphi(f^{\downarrow\check{a}});_k \varphi(f^{\uparrow\check{a}}) & \text{if } dim(S) = n,\ \check{a} \in Sd(S)_k. \end{cases}$$

and the morphisms in $\varphi(f^{\downarrow\check{a}})$ and $\varphi(f^{\uparrow\check{a}})$ in $\mathbf{Comma}_n^{m/1}$ are obtained by compositions so that the diagram

$$\begin{tikzcd}
(S^{\uparrow\check{a}})^{\sharp,n} \arrow[dr] \arrow[drr, "f^{\uparrow\check{a}}"] & & \\
& S^{\sharp,n} \arrow[r, "f"] & \mathcal{P} \\
(S^{\downarrow\check{a}})^{\sharp,n} \arrow[ur] \arrow[urr, "f^{\downarrow\check{a}}"'] & &
\end{tikzcd}$$

commutes. We need to verify, by induction on n, that φ is well defined, bijective and that it preserves compositions, identities, domains, and codomains.

We shall only verify (partially) that φ is well defined, i.e. that the definition of φ for any non-principal ordered face structure S of dimension n does not depend on the choice of the saddle point of S. Let $\check{a}, \check{b} \in Sd(S)$. We shall show, in case $dim(a) = dim(b) = k$ and $a <_l b$, that we have

$$\varphi(f^{\downarrow\check{a}});_k \varphi(f^{\uparrow\check{a}}) = \varphi(f^{\downarrow\check{b}});_k \varphi(f^{\uparrow\check{b}})$$

Using Lemma 12.6 of [Z2] and the fact that $(-)^{\sharp,n}$ preserves special pushouts (Corollary 13.2 of [Z2]), we have

$$\begin{aligned}
&\varphi(f^{\downarrow\check{a}});_k \varphi(f^{\uparrow\check{a}}) = \\
&= \varphi(f^{\downarrow a});_k (\varphi(f^{\uparrow\check{a}\downarrow b});_k \varphi(f^{\uparrow\check{a}\uparrow\check{b}})) = \\
&= (\varphi(f^{\downarrow a});_k \varphi(f^{\uparrow\check{a}\downarrow b}));_k \varphi(f^{\uparrow\check{b}\uparrow\check{a}}) = \\
&= \varphi(f^{\downarrow a};_k f^{\uparrow\check{a}\downarrow b});_k \varphi(f^{\uparrow\check{b}\uparrow\check{a}}) = \\
&= \varphi(f^{\downarrow b};_k f^{\uparrow\check{b}\downarrow a});_k \varphi(f^{\uparrow\check{b}\uparrow\check{a}}) = \\
&= (\varphi(f^{\downarrow b});_k \varphi(f^{\uparrow\check{b}\downarrow a}));_k \varphi(f^{\uparrow\check{b}\uparrow\check{a}}) = \\
&= \varphi(f^{\downarrow b});_k (\varphi(f^{\uparrow\check{b}\downarrow a});_k \varphi(f^{\uparrow\check{b}\uparrow\check{a}})) = \\
&= \varphi(f^{\downarrow b});_k \varphi(f^{\uparrow\check{b}})
\end{aligned}$$

The reader can compare these calculations with the those, in the same case, of Proposition 13.1 of [Z2] (f replaces φ and φ replaces F). So there is no point to repeat the other calculations here. $\Box$

For $\mathcal{P}$ in $\mathbf{Comp}^{m/1}$ we define a computad map

$$\eta_{\mathcal{P}} : \mathcal{P} \longrightarrow \widetilde{\widehat{\mathcal{P}}}$$

so that for $x \in \mathcal{P}_n$ we put

$$\eta_{\mathcal{P},n}(x) = \tau_x : T_x^* \longrightarrow \mathcal{P} \in \widehat{\mathcal{P}}(T_x) \xrightarrow{\kappa_n^{T_x}} \widetilde{\widehat{\mathcal{P}}}_n$$

such that $\tau_x(1_{T_x}) = x$.

For F in $Mod_{\otimes}(\mathbf{oFs}^{op}, Set)$ we define a natural transformation

$$\varepsilon_F : \widehat{\widetilde{F}} \longrightarrow F,$$

such that, for an ordered face structure S of dimension n,

$$(\varepsilon_F)_S : \widehat{\widetilde{F}}(S) \longrightarrow F(S)$$

and $g : S^* \to \widetilde{F} \in \widehat{\widetilde{F}}(S)$ we put

$$(\varepsilon_F)_S(g) = g_n(1_S).$$

Proposition 7.3 *The functors*

$$\begin{tikzcd} Mod_{\otimes}(\mathbf{oFs}^{op}, Set) \arrow[r, shift left, "\widetilde{(-)}"] & \mathbf{Comp}^{m/1} \arrow[l, shift left, "\widehat{(-)} = \mathbf{Comp}^{m/1}((\simeq)^*, -)"] \end{tikzcd}$$

together with the natural transformations η *and* ε *defined above form an adjunction* $(\widehat{(-)} \dashv \widetilde{(-)})$. *It establishes an equivalence of categories* $Mod_{\otimes}(\mathbf{oFs}^{op}, Set)$ *and* $\mathbf{Comp}^{m/1}$.

Proof. The fact that both η and ε are bijective on each component follows immediately from Proposition 15.1 of [Z2]. So we shall verify the triangular equalities only.

Let $\mathcal{P}$ be a computad, and F be a functor in $Mod_{\otimes}(\mathbf{oFs}^{op}, Set)$. We need to show that the triangles

$$\begin{tikzcd} & \widehat{\widetilde{\widehat{\mathcal{P}}}} \arrow[dr, "\varepsilon_{\widehat{\mathcal{P}}}"] & \\ \widehat{\mathcal{P}} \arrow[ur, "\widehat{\eta_{\mathcal{P}}}"] \arrow[rr, "1_{\widehat{\mathcal{P}}}"'] & & \widehat{\mathcal{P}} \end{tikzcd} \qquad \begin{tikzcd} & \widetilde{\widehat{\widetilde{F}}} \arrow[dr, "\widetilde{\varepsilon_F}"] & \\ \widetilde{F} \arrow[ur, "\eta_{\widetilde{F}}"] \arrow[rr, "1_{\widetilde{F}}"'] & & \widetilde{F} \end{tikzcd}$$

commute. So let $f : S^* \to \mathcal{P} \in \widehat{\mathcal{P}}(S)$. Then, we have

$$\varepsilon_{\widehat{\mathcal{P}}} \circ \widehat{\eta_{\mathcal{P}}}(f) = \varepsilon_{\widehat{\mathcal{P}}}(\eta_{\mathcal{P}} \circ f) = (\eta_{\mathcal{P}} \circ f)_n(1_S) =$$

$$= (\eta_{\mathcal{P}})_n(f_n(1_S)) = \tau_{f_n(1_S)} = f$$

Last equation follows from the fact that $(\tau_{f_n(1_S)})_n(1_S) = f_n(1_S)$ and Proposition 15.1 of [Z2]. Now let $x \in F(S) \longrightarrow \widetilde{F}_n$. Then we have

$$\widetilde{\varepsilon_F} \circ \eta_{\widetilde{F}}(x) = \widetilde{\varepsilon_F}(\tau_x) = (\tau_x)_n(1_{T_x}) = x$$

So both triangles commute, as required. □

If we compose the three established adjoint equivalences we get from Propositions 5.1, 6.2, and 7.3

Corollary 7.4 *The functor*

$$\widehat{(-)} : \mathbf{Comp}^{m/1} \longrightarrow Set^{\mathbf{pFs}^{op}}$$

such that for a many-to-one computad X,

$$\widehat{X} = \mathbf{Comp}^{m/1}((-)^*, X) : \mathbf{pFs}^{op} \longrightarrow Set$$

is an equivalence of categories.

The fact that the category $\mathbf{Comp}^{m/1}$ is a presheaf category was first established in [HMZ] using an earlier result from [HMP]. From this we know that the category of $\mathbf{Comp}^{m/1}$ is equivalent the category of presheaves on the category of multitopes **Mlt** introduced in [HMP].

Theorem 7.5 *The category* **pFs** *of principal ordered face structure is equivalent to the category of multitopes* **Mlt**.

Proof. The categories of presheaves on both categories are equivalent to the category of many-to-one computads. As these categories have no nontrivial idempotents they must be equivalent. $\Box$

8 The shapes of cells in computads

Let $\mathbf{Comp}^{?/?}$ be a full subcategory of the category of computads $\mathbf{Comp}$[8] of some kind of computads. The particular examples we have in mind and we will be refereing to later are free categories over graphs $\mathbf{Comp}^{1/1}_1$, one-to-one computads $\mathbf{Comp}^{1/1}$, positive-to-one computads $\mathbf{Comp}^{+/1}$, many-to-one computads $\mathbf{Comp}^{m/1}$, or even all computads $\mathbf{Comp}$.

One of the ways to think about the shape of a cell α in a computad C from the category $\mathbf{Comp}^{?/?}$ is the following. We consider the category of pointed computads $\mathbf{Comp}^{?/?}_*$ whose objects are computads with chosen cells and morphisms are computad maps preserving the distinguished cells. Then the *shape* of a cell $\alpha \in C$ (if exists) can be identified with the initial object of the slice category $\mathbf{Comp}_* \downarrow (C, \alpha)$. It is obvious that the computad maps preserve so understood shapes i.e. if $f : C \to D$ is a computad map, $\tau_\alpha : (S, s) \to (C, \alpha)$ is the initial object of $\mathbf{Comp}_* \downarrow (C, \alpha)$ then $F \circ \tau_\alpha : (S, m) \to (D, f(\alpha))$ is the initial object of $\mathbf{Comp}_* \downarrow (D, f(\alpha))$. Unfortunately not every cell has a shape. For example if we take two 2-indets α and β whose domain and codomain is 1_x the identity of a 0-cell x then $\beta \circ_0 \alpha$ does not have a shape. This 'innocent' problem is responsible for very serious complications and it is one of the reasons for the restriction of shapes of cells in weak ω-categories to more manageable shapes like one-to-one, many-to-one, etc. Note that the shape (S, m) of the cell α is not necessarily determined by what we can call the (pure) shape S.

Now assume that all cells in all computads in the given category of computads $\mathbf{Comp}^{?/?}$ have shapes. To define the category $\mathbf{Shape}^{?/?}$ of shapes of cells for $\mathbf{Comp}^{?/?}$ we could just take all the computad maps between shapes of all cells in computads from $\mathbf{Comp}^{?/?}$. But such morphisms can identify different shapes by making them isomorphic. This is why we shall take a longer route by specifying some of the morphisms that we definitely want in the category $\mathbf{Shape}^{?/?}$ and then we shall generate all the other morphisms inside $\mathbf{Comp}^{?/?}$ via composition and graded tensor operation. The closure under the later operation is to ensure that the graded tensor operation is functorial.

First kind of morphisms we shall consider comes from the fact that we have in computads the k-domain $d^{(k)}$ and the k-codomain $c^{(k)}$ operations that associate the domain and the codomain of dimension k, respectively. Thus if (S, m) is a shape, m is a cell in S of dimension n and $k \leq n$, then $(S, d^{(k)}(m))$ and $(S, c^{(k)}(m))$ are pointed computads in $Comp^{?/?}$. Thus the cells $d^{(k)}(m)$ and $c^{(k)}(m)$ in S have shapes which we denote

$$\mathbf{d}^{(k)}_S : (\mathbf{d}^{(k)}S, d^{(k)}(m)) \longrightarrow (S, d^{(k)}(m))$$

$$\mathbf{c}^{(k)}_S : (\mathbf{c}^{(k)}S, c^{(k)}(m)) \longrightarrow (S, c^{(k)}(m))$$

In particular, we have the computad maps

$$\mathbf{d}^{(k)}_S : \mathbf{d}^{(k)}S \longrightarrow S, \quad \mathbf{c}^{(k)}_S : \mathbf{c}^{(k)}S \longrightarrow S$$

[8] By a computad we mean here an ω-category C that is levelwise free, i.e. if we truncate it to an $n+1$-category C_{n+1} then it arises as an n-category C_n with freely added $n+1$-indeterminate cells (=indets). Morphisms of computads are ω-functors that are required to send indets to indets.

such that $\mathbf{d}_S^{(k)}(d^{(k)}(m)) = d^{(k)}(m)$ and $\mathbf{c}_S^{(k)}(c^{(k)}(m)) = c^{(k)}(m)$. Note that both $d^{(k)}(m)$ and $c^{(k)}(m)$ name two different cells in two different computads.

The second kind of morphism comes from the fact that we can (de)compose cells in computads. Suppose (S, m) is a shape such that the cell m can be decomposed as $m = m_1;_k m_2$. Then we have shapes of m_1 and m_2 in S:

$$\kappa^1 : (S_1, m_1) \longrightarrow (S, m_1), \qquad \kappa^2 : (S_1, m_2) \longrightarrow (S, m_2)$$

so that $m = m_1;_k m_2 = \kappa^1(m_1);_k \kappa^2(m_2)$ in S. Here again both m_1 and m_2 name two different cells in two different computads. If we denote by (S_3, m_3) the shape of $c^{(k)}(m_1) = d^{(k)}(m_2)$ in S then we obtain a commuting square

$$\begin{tikzcd}
(S_1, m_1) \arrow[r, "\kappa^1"] & (S, m) \\
(S_3, m_3) \arrow[u, "\mathbf{d}_{S_1}^{(k)}"] \arrow[r, "\mathbf{c}_{S_2}^{(k)}"'] & (S_2, m_2) \arrow[u, "\kappa^2"']
\end{tikzcd}$$

called the tensor square.

Note that it is very likely that the computad S will turn out to be the pushout $S_1 +_{S_3} S_2$ in $Comp^{?/?}$ but this doesn't mean that (S, m) will be the pushout $(S_1, m_1) +_{(S_3, m_3} (S_2, m_2)$ in $Shape^{?/?}$.

The graded tensor operation is defined as follows. Suppose we have tensor squares defined form decompositions of cells $m = m_1;_k m_2$ and $m' = m'_1;_k m'_2$ in shapes (S, m) and (S', m'), respectively, and for some morphisms f_1, f_2, f_3, the squares

$$\begin{tikzcd}
(S_1, m_1) \arrow[d, "f_1"'] & (S_3, m_3) \arrow[l, "\mathbf{c}_{S_1}^{(k)}"'] \arrow[r, "\mathbf{d}_{S_2}^{(k)}"] \arrow[d, "f_3"] & (S_2, m_2) \arrow[d, "f_2"] \\
(S'_1, m'_1) & (S'_3, m'_3) \arrow[l, "\mathbf{c}_{S'_1}^{(k)}"] \arrow[r, "\mathbf{d}_{S'_2}^{(k)}"'] & (S'_2, m'_2)
\end{tikzcd}$$

commute. Then we require to exist a unique morphism $f_1 \otimes_k f_1 : S \to S'$, called a *graded tensor* of f_1 and f_1, making the squares

$$\begin{tikzcd}
(S_1, m_1) \arrow[r, "\kappa_{S_1}^1"] \arrow[d, "f_1"'] & (S, m) \arrow[d, "f_1 \otimes_k f_2"'] & (S_2, m_2) \arrow[l, "\kappa_{S_2}^2"'] \arrow[d, "f_2"] \\
(S'_1, m'_1) \arrow[r, "\kappa_{S'_1}^1"'] & (S', m') & (S'_2, m'_2) \arrow[l, "\kappa_{S'_2}^2"]
\end{tikzcd}$$

commute.

The category of ?/?-shapes $Shape^{?/?}$ has as objects shapes (S, m) of cells in $Comp^{?/?}$ and as morphisms the least class of computad morphism containing $\mathbf{d}^{(k)}$, $\mathbf{d}^{(k)}$, κ^1, κ^2, identities closed under composition and graded tensor operation.

Examples.

1. For the category of free categories over graphs i.e. $Comp_1^{1/1}$ the category of shapes defined above in (equivalent to) Δ_0. Recall that Δ_0 is the full subcategory of the category of graphs whose objects are linear graphs $[n]$ with n edges and $n+1$ vertices. The morphism $\mathbf{d}_{[n]}^{(0)} : [0] \to [n]$ is the inclusion sending the unique vertex of $[0]$ to the first vertex of $[n]$ and $\mathbf{c}_{[n]}^{(0)} : [0] \to [n]$ is the inclusion sending the unique

vertex of [0] to the last vertex of $[n]$. The morphisms $\kappa^1_{[n]} : [n] \to [n+m]$ and $\kappa^2_{[m]} : [n] \to [n+m]$ are the inclusions onto the first n and the last m edges of the graph $[n+m]$, respectively. The tensor morphisms are also obvious. Note that in this case we generate all the graph morphisms between the objects of Δ_0.

2. The shapes of cells for the category of one-to-one computads $Comp^{1/1}$ are determined by what is called in different terminologies globular cardinals, simple ω-graphs $s\omega Gr$, T-cardinals for the free category monad on ω-graphs. By this I mean that for every n-cell α in every one-to-one computad C there is a unique (up to iso) simple ω-graph S and a unique cell S in the ω-category S^* generated[9] by S, and a unique pointed computad morphism $\tau_\alpha : (S^*, S) \to (C, \alpha)$. Moreover this map τ_α is the initial object in $Comp^{1/1}_* \downarrow (C, \alpha)$. The shape (S^*, S) is uniquely determined by ω-graph S and even by the ω-category S^*.

3. The shapes of cells for the category of positive-to-one computads $\mathbf{Comp}^{+/1}$, see [Z1], are determined by positive face structures. The category $\mathbf{Fs}^{+/1}$ of positive face structures is the category of shapes for $\mathbf{Comp}^{+/1}$. Despite the fact that it is considerably more complicated than $s\omega Gr$, it shares some good properties of $s\omega Gr$. For example the embedding $(-)^* : \mathbf{Fs}^{+/1} \to \mathbf{Comp}^{+/1}$ is full.

4. The shapes of cells for the category of many-to-one computads $\mathbf{Comp}^{m/1}$, see [Z2], are determined by ordered face structures. The category $\mathbf{oFs}$ of positive face structures and monotone maps is the category of shapes for $\mathbf{Comp}^{m/1}$. Here however the theory changes considerably. The main reason is that the $(-)^* : \mathbf{oFs} \to \mathbf{Comp}^{m/1}$ is not full. The full image of $(-)^*$ in this case is the category of ordered face structures and local maps.

5. As not all the cells in arbitrary computads have shape, there is no category of shapes for the category of all computads $Comp$.

9 Pra monads and nerves

The idea that algebras can be presented as a full subcategory of the category of free algebras preserving some limits goes back to the thesis of our jubilee. In [Law] F.W. Lawvere have shown that finitary algebras can be presented as a full subcategory of presheaves on the finitely generated free algebras that preserve some finite products. The next step was made by F.E.Linton, c.f. [Lin], when he has shown that for any[10] monad T on Set it is true that the category $Alg(T)$ of the Eilenberg-Moore algebras for T is equivalent to the category of product preserving functors from the dual of the category $\mathcal{K}(T)$ of Kleisli algebras[11]. He also noticed that under further size restrictions on T one can take an essentially small full subcategory of $\mathcal{K}(T)$.

The recent development due to T. Leinster, c.f. [Lei1], and then to M. Weber, c.f. [W], brought some new light on this construction. Below I will describe briefly the theory developed by them but not in full generality of M. Weber and changing slightly the perspective occasionally.

T. Leinster's setup consists of a parametric right adjoint monad (T, ν, μ), pra monad for short, on a presheaf category $Set^{\mathcal{C}^{op}}$. By this he means that both natural transformations are cartesian and that functor T a parametric right adjoint i.e. that the functor $T_1 : Set^{\mathcal{C}^{op}} \longrightarrow Set^{\mathcal{C}^{op}} \downarrow T(1)$ induced by T has a left adjoint. He has shown that in this case there is a canonical choice for a small category θ_T a full subcategory of $\mathcal{K}(T)$ and a canonical choice of the limits in θ_T so that the category of presheaves preserving those limits is equivalent to $Alg(T)$.

[9] The n-cells of S^* can be identified with simple subgraphs of S of dimension at most n

[10] In fact F.E.Linton needed some minor size restricting condition called *tractable* saying that operation of any (possibly infinitary) arity form a set.

[11] At that time it was not expressed in these terms.

A functor T defined on a presheaf category is pra iff it preserves wide pullbacks iff it is family representable, c.f. [Lei1], [W]. Recall that a functor on a presheaf category $Set^{\mathcal{C}^{op}}$ is a family representable iff for every object $c \in \mathcal{C}$ there is a set of objects $\{T_{c,i}\}_{i\in I_c}$ of $Set^{\mathcal{C}^{op}}$ such that we have an isomorphism of functors $ev_c \circ T \cong \coprod_{i\in I_c} Y(T_{c,i})$, where $ev_c : Set^{\mathcal{C}^{op}} \to Set$ is the evaluation on c, and $Y(T_{c,i})$ is the covariant functor representable by $T_{c,i}$. The category θ_T is the full subcategory of $Alg(T)$ whose objects are the free T-algebras over the representing objects $\{T_{c,i}\}_{i\in I_c, c\in\mathcal{C}}$.

In M. Weber's terminology the objects of form $T_{c,i}$ for $c \in C$, $i \in I_c$ are called *T-cardinals*. In order to make the distinction I will call his category Θ_0 a full subcategory of $Set^{\mathcal{C}^{op}}$ as *the category of T-cardinals* and the full image of it in $Alg(T)$ denoted by him Θ_T as the *T-cardinal algebras.*

M. Weber is considering a more general setup than T. Leinster. The monad (T, ν, μ) is defined on a cocomplete category $\mathcal{A}$. In this more general situation the choice of the category of T-cardinals, called there the category of arities, does not need to be canonical and is given explicitly as full dense subcategory of $\mathcal{A}$. M. Weber also requires η and μ to be cartesian but the condition on T is slightly more technical and I will not recall it here. The more general setup covers some cases not covered by T. Leinster approach but the additional level of generality has in the present context only restricted and negative application to which I will come back later. On the other hand, the Theorem 4.10 in [W] seems to be more informative even when applied to the original setup of T. Leinster. In fact I will state it in combination with other results from [W] and [Lei1] in the form that is relevant to the present context.

Now let $p\Theta$ be a small category (T, η, μ) be a pra monad on a presheaf category $Set^{p\Theta}$, Θ_0 the full subcategory of $Set^{p\Theta}$ whose objects are T-cardinals, Θ_T the full image of the category Θ_0 in $Alg(T)$, $\mathcal{M}$ the class of morphisms in Θ_0. Then, following M. Weber, we can conclude that there is a class $\mathcal{E}$ orthogonal to $\mathcal{M}$ so that $(\mathcal{E}, \mathcal{M})$ form a factorization system in Θ_T moreover we have a commuting square of categories and functors

$$\begin{tikzcd} Alg(T) \arrow[r, "\mathcal{N}_T"] \arrow[d, "U"'] & Set^{\Theta_T^{op}} \arrow[d, "i^*"] \\ Set^{p\Theta^{op}} \arrow[r] & Set^{\Theta_0^{op}} \end{tikzcd}$$

which is a pseudo-pullback, where the horizontal maps are the obvious maps generated by the (full) embeddings $\Theta_0 \longrightarrow Set^{p\Theta}$ and $\Theta_T \longrightarrow Alg(T)$. As the first embedding is full (and faithful) so is the nerve functor $\mathcal{N}_T$. We can think about this result as saying that all the monadic functors (like $U : Alg(T) \longrightarrow Set^{p\Theta^{op}}$) for pra monads can be obtained via pseudo-pullbacks along full and faithful functors from particularly simple monadic functors namely those coming from presheaf pra monads, (like $i^* : Set^{\Theta_T^{op}} \longrightarrow Set^{\Theta_0^{op}}$), see Example 1 below.

Examples.

1. There is a whole class of simple examples of pra monads. Let Ξ be a small category with a factorization system $(\mathcal{E}, \mathcal{M})$ on Ξ. Let $\Xi_{\mathcal{M}}$ be the subcategory for Ξ consisting of all objects of Ξ and morphisms from the class $\mathcal{M}$ only. We have a non-full bijective on objects embedding $i : \Xi_{\mathcal{M}} \to \Xi$. Then the functor $i^* : Set^{\Xi^{op}} \to Set^{\Xi_{\mathcal{M}}^{op}}$ is a monadic and the monad (T^i, η^i, μ^i) on $Set^{\Xi_{\mathcal{M}}^{op}}$ induced by i is pra. For $X \in Set^{\Xi_{\mathcal{M}}^{op}}$ and $S \in \Xi_{\mathcal{M}}$ the functor T^i is given by

$$T^i(X)(S) = \coprod_{e:S\to S'\in\mathcal{E}} X(S') \xleftarrow{\kappa_e^{X,S}} X(S')$$

where the coproduct is taken over all (up to iso) morphisms in $\mathcal{E}$ with domain S. For $X \in Set^{\Xi^{op}_{\mathcal{M}}}$, $\eta_X : X \to T^i(X)$ is given by

$$X(S) \xrightarrow{(\eta_X)_S = \kappa^{X,S}_{1_S}} \coprod_{e:S\to S'\in\mathcal{E}} X(S') = T^i(X)(S)$$

for $S \in \mathbf{oFs}$, and $\mu_X : (T^i)^2(X) \longrightarrow T^i(X)$ is so defined that for any morphisms $e : S \to S'$ and $e' : S' \to S''$ in $\mathcal{E}$ the triangle

$$\begin{tikzcd}
(T^i)^2(X)(S) = \coprod_{e:S\to S'}\coprod_{e':S'\to S''} X(S'') \arrow[rr, "(\mu_X)_S"] & & \coprod_{f:S\to S''} X(S'') = T^i(X)(S) \\
& X(S'') \arrow[ul, "\kappa^{X,S}_{e,e'}"] \arrow[ur, "\kappa^{X,S}_{e'\circ e}"'] &
\end{tikzcd}$$

commutes, where

$$X(S'') \xrightarrow{\kappa^{X,S}_{e,e'}} \coprod_{e:S\to S'\in\mathcal{E}}\coprod_{e':S'\to S''\in\mathcal{E}} X(S'') = (T^i)^2(X)(S)$$

is the obvious embedding. Bothη and μ are cartesian since for any $\tau : X \to Y \in Set^{\Xi^{op}_{\mathcal{M}}}$ and $S \in \mathbf{oFs}$ the squares

$$\begin{tikzcd}
X(S) \arrow[r, "(\eta_X)_S"] \arrow[d, "\tau_S"'] & \coprod_{f:S\to S''} X(S'') \arrow[d, "\coprod_{f:S\to S''}\tau_{S''}"] \\
Y(S) \arrow[r, "(\eta_Y)_S"'] & \coprod_{f:S\to S''} X(S'')
\end{tikzcd}$$

and

$$\begin{tikzcd}
\coprod_{e:S\to S'}\coprod_{e':S'\to S''} X(S'') \arrow[r, "(\mu_X)_S"] \arrow[d, "\coprod_{e:S\to S'}\coprod_{e':S'\to S''}\tau_{S''}"'] & \coprod_{f:S\to S''} X(S'') \arrow[d, "\coprod_{f:S\to S''}\tau_{S''}"] \\
\coprod_{e:S\to S'}\coprod_{e':S'\to S''} Y(S'') \arrow[r, "(\mu_Y)_S"'] & \coprod_{f:S\to S''} X(S'')
\end{tikzcd}$$

are pullbacks. As for such monad T^i not only the base category $Set^{\Xi^{op}_{\mathcal{M}}}$ but also the category of algebras $Alg(T)$ is a presheaf category $Set^{\Xi^{op}}$, I will call such monads *presheaf pra monads.*

2. As it was pointed out in [Lei1] and [W] this framework fits well the free category monad over graphs and the free ω-category monad over simple ω-graphs. I will elaborate on the first case as both cases are in a sense quite similar, well known and the first is simpler. In this case $p\Theta$ is the full subcategory of the category of graphs containing two graphs [0] and [1]. The category of T-cardinals Θ_0 is Δ_0 and the category of T-cardinal algebras Θ_T is Δ. The free category monad T on $Set^{\Theta_0^{op}}$ is pra and the left adjoint L_T to the functor $T_1 : Set^{\Theta_0^{op}} \longrightarrow Set^{\Theta_0^{op}} \downarrow T(1)$ can be described explicitly[12]. We shall sketch this definition to show the role of Δ_0 in it. In many-to-one case the role of Δ_0 will be taken by **oFs**.

Let $(G, |-|)$ be an object of the slice category of graphs $Graph \downarrow T(1)$. Thus we have a pair of function $d, c : E \to V$ from the set edges to the set of vertices and a function $|-| : E \to N$ from the set of edges to the set of natural numbers. We define the diagram

$$\Gamma_{(G,|-|)} : \tilde{G} \longrightarrow \Delta_0 \longrightarrow Graph$$

[12]In Example 2.5 of [W] the functor L_T is correctly described in words but L_T is not given by the left Kan extension contrary to what was claimed there. I will came back to this point later.

whose colimit is $L_T(G, |-|)$. The second functor is the usual embedding. The set $\tilde{V}$ of vertices of $\tilde{G}$ contains both vertices and edges of G as disjoint sets. The set $\tilde{E}$ of edges of $\tilde{G}$ has two edges

$$d(e) \xrightarrow{s_e} e, \quad c(e) \xrightarrow{t_e} e$$

for each edge $e \in E$ with the domain and codomain as displayed. The functor $\tilde{G} \longrightarrow \Delta_0$ sends vertices from V to $[0]$ and the vertex $e \in E \subset \tilde{V}$ to the linear graph $[|e|]$. Moreover it sends the edges s_e and t_e to

$$\mathbf{d}^0_{[|e|]} : [0] \longrightarrow [|e|], \text{ and } \mathbf{c}^0_{[|e|]} : [0] \longrightarrow [|e|]$$

respectively. In particular, here and in all the other cases considered to get the formula for L_T we use the domain and the codomain maps.

3. The case of positive-to-one computads also fits this setup and seems to be new. The category of positive face structures $\mathbf{Fs}^{+/1}$ is both the category of shapes for many-to-one computads $\mathbf{Shape}^{+/1}$ and the category of $T^{+/1}$-cardinals for the free ω-category monad $T^{+/1}$ on positive-to-one computads $\mathbf{Comp}^{+/1}$. Its image in ωCat is the category of $T^{+/1}$-cardinal algebras. The left adjoint $L_{T^{+/1}}$ to the $T_1^{+/1}$ functor can be described much as in the previous case.

4. For the many-to-one computads the above setup does not seem to be sufficient. This is the first case where the category of many-to-one shapes $\mathbf{Shape}^{m/1}$ exists but it is not a full subcategory of the category $\mathbf{Comp}^{m/1}$ of many-to-one computads. The category $\mathbf{Shape}^{m/1}$ is equivalent to the category $\mathbf{oFs}$ of ordered face structures and monotone maps. But the category of $T^{m/1}$-cardinals, for the free ω-category monad $T^{m/1}$ on many-to-one computads $\mathbf{Comp}^{m/1}$ is equivalent to the category $\mathbf{oFs}_{loc}$ of ordered face structures and local maps. The category $\mathbf{oFs}_\omega$ of $T^{m/1}$-cardinal algebras will be described in the next section. Note that in the previous examples the categories of cardinals were GT-theories but this time only the category of shapes $\mathbf{oFs}$ is a GT-theory and the category $\mathbf{oFs}_{loc}$ is not.

5. Finally let me point out one non-example namely the category of all computads $\mathbf{Comp}$. It is still true, by a beautiful argument of V. Harnik [H], that ωCat is monadic over $\mathbf{Comp}$ via right adjoint to the inclusion functor. However $\mathbf{Comp}$ is not a presheaf category, c.f. [MZ2], the free ω-category monad on $\mathbf{Comp}$ is not pra as can be easily shown using Proposition 2.6 from [W]. This adds to the long list of reasons why we don't get a good theory of weak categories when considering all possible shapes of cells.

10 GT-theories and nerves

I formulate below the general setup to show where it modifies the previous one. After stating the abstract pattern I shall make a case study on the many-to-one computads to show the usefulness of this approach. However I do not provide any general results concerning this abstract setup as I prefer to collect more than one true example ($\mathbf{oFs}$) before developing this theory any farther. The present approach is in a sense much more modest than the one from previous section. We deal here exclusively with the monads whose categories of algebras are equivalent to the category of strict ω-categories ωCat only (or its truncations). Moreover we want these monad to be defined on various reflective in ωCat subcategories of the category of computads $\mathbf{Comp}$. On the other hand taking advantage of this more specific situation one may hope to get a more convenient description of concrete cases as in case of the category of many-to-one computads $\mathbf{Comp}^{m/1}$. Still a word why all sorts of nerves of strict ω-categories might be of interest. In the presheaf approach to weak categories (as opposed to the algebraic approach) the weak categories are presheaves with some properties. If we believe that strict ω-categories should be special cases of weak ones, we need to study various nerves of strict ω-categories as they will provide abundance of examples.

The setup consists of GT-theory Φ with a full subcategory $p\Phi$ so that the obvious induced functor

$$Mod_{\otimes}(\Phi^{op}, Set) \longrightarrow Set^{p\Phi^{op}}$$

is an equivalence of categories. With this data we want to get the following.

1. The generic model $\mathcal{G}_\Phi : \Phi \to \Phi_l$ induces an equivalence of categories
$$Mod_{\otimes}(\Phi^{op}, Set) \simeq pLim(\Phi_l^{op}, Set)$$
where $pLim(\Phi_l^{op}, Set)$ is the category of functors that preserves the principal limits, i.e. the canonical limits defined over the diagrams consisting objects from $p\Phi$ only.

2. There is a dense embedding
$$\Phi \longrightarrow Mod_{\otimes}(\Phi^{op}, Set)$$
$$S \mapsto \bar{S} = \Phi_l(-, S)$$

3. Φ induces a pra monad (T, η, μ) on $Mod_{\otimes}(\Phi^{op}, Set)$.

4. There is an explicit formula for the left adjoint L_T to T_1.

5. Let Φ_T denote the full image of Φ in $Alg(T)$ and $i : \Phi \to \Phi_T$ the embedding. Then pra monadic functor $Alg(T) \longrightarrow Mod_{\otimes}(\Phi^{op}, Set)$ is a pseudo-pullback of the presheaf pra monadic functor $i^* : Set^{\Phi_T{}^{op}} \longrightarrow Set^{\Phi^{op}}$, i.e. we have a pseudo-pullback
$$\begin{tikzcd} Alg(T) \arrow[r] \arrow[d, "U"'] \arrow[dr, phantom, "\cong"] & Set^{\Phi_T^{op}} \arrow[d, "i^*"] \\ Mod_{\otimes}(\Phi^{op}, Set) \arrow[r] & Set^{\Phi_l^{op}} \end{tikzcd}$$

6. In particular, the image of the full nerve functor $Alg(T)$ in $Set^{\Phi_T{}^{op}}$ consists of those functors that send $\otimes$-squares in Φ_T to pullbacks.

7. The category of T-algebras $Alg(T)$ is equivalent to ωCat.

Now I will show how all this can be produced from the GT-theory **oFs** playing the role of Φ, together with its full subcategory of principal ordered face structures **pFs** playing the role of $p\Phi$.

The generic model of **oFs** is the inclusion functor $\mathbf{oFs}^{op} \to \mathbf{oFs}^{op}_{loc}$. Thus the first two points are clear. In order to define the monad T on $Mod_{\otimes}(\Phi^{op}, Set)$ it is convenient to have already the full image of the category **oFs** in $Alg(T)$ defined. This category $\mathbf{oFs}_\omega$, playing the role of Φ_T, can be defined directly from **oFs** alone. The objects of $\mathbf{oFs}_\omega$ are the objects of **oFs**. A morphism $\xi : R \to S$ is $\mathbf{oFs}_\omega$ is an ω-*map* that is a transformation between presheaves $\xi : \mathbf{oFs}_{loc}(-, R) \longrightarrow \mathbf{oFs}_{loc}(-, S)$ which associate to a morphism $a : V \to R$ in $\mathbf{oFs}_{loc}(-, R)$ a morphism $\xi_a : V_a \to S$ in $\mathbf{oFs}_{loc}(-, S)$ so that

1. $dim(V_a) \leq dim(V)$;

2. $\xi(a\circ \mathbf{d}^{(k)}_V) = \xi(a)\circ \mathbf{d}^{(k)}_{V_a}$, $\quad \xi(a\circ \mathbf{c}^{(k)}_V) = \xi(a)\circ \mathbf{c}^{(k)}_{V_a}$,

3. if $V = V^1\otimes_k V^2$ then $V_a = V^1_{a\circ\kappa^1_{V^1}} \otimes_k V^2_{a\circ\kappa^2_{V^2}}$ and moreover
$$\xi(a\circ\kappa^1) = \xi(a)\circ\bar{\kappa}^1, \qquad\qquad \xi(a\circ\kappa^2) = \xi(a)\circ\bar{\kappa}^2$$
where $\kappa^i = \kappa^i_{V^i}$, $\bar{\kappa}^i = \kappa^i_{V^i_{a\circ\kappa^i}}$ for $i=1,2$.

We identify two ω-maps ξ and ξ' iff for every $a : V\to R$ there is an isomorphism σ_a making the triangle

$$\begin{tikzcd} V_a \arrow[rr, "\sigma_a"] \arrow[dr, "\xi_a"'] & & V'_a \arrow[dl, "\xi'_a"] \\ & S & \end{tikzcd}$$

commute.

Let $\xi : R\to S$ be an ω-map. ξ is an *inner ω-map* iff $\xi(1_R) = 1_S$. ξ is a *monotone ω-map* iff $\xi(1_R)$ a monotone morphism. Its is easy to see that ω-maps, inner ω-maps and monotone ω-maps do compose. We have categories $\mathbf{oFs}_\omega$ of ordered face structures and ω-map and $\mathbf{oFs}_\mu$ of ordered face structures and monotone ω-map. We write $f : R \multimap S$ to indicate that f is an inner map. Clearly the ω-maps and the monotone ω-maps do compose. Thus we have categories $\mathbf{oFs}_\omega$ of ordered face structures and ω-maps, and $\mathbf{oFs}_\mu$ of ordered face structures and monotone ω-maps. We have an embedding
$$\iota_\omega : \mathbf{oFs}_{loc} \longrightarrow \mathbf{oFs}_\omega$$
sending the local map $f : R\to S$ to the ω-map $\iota_\omega(f) : R\to S$ such that for $h : V\to R$ in $\mathbf{oFs}_{loc}(-,R)$ we have $\iota_\omega(f)(h) = f\circ h$. We shall identify $\iota_\omega(f)$ and f. ι_ω restricts to the embedding
$$\iota_\mu : \mathbf{oFs} \longrightarrow \mathbf{oFs}_\mu$$

Thus we have a commuting square of categories and functors

$$\begin{tikzcd} \mathbf{oFs} \arrow[r, "\mathcal{G}_{\mathbf{oFs}}"] \arrow[d, "\iota_\mu"'] & \mathbf{oFs}_{loc} \arrow[d, "\iota_\omega"] \\ \mathbf{oFs}_\mu \arrow[r, "\mathcal{G}_{\mathbf{oFs}_\mu}"'] & \mathbf{oFs}_\omega \end{tikzcd}$$

Note that the generic models $\mathcal{G}_{\mathbf{oFs}}$ and $\mathcal{G}_{\mathbf{oFs}_\mu}$ can be considered as a process symmetrization of the tensor as the pushouts tend to be more symmetric than tensors. Thus one cannot expect these functors to be full even on isomorphisms. On the other hand, both ι_μ, ι_ω are full on isomorphisms and they fall under the scheme of the Example 1 from the previous section. In particular, the composition functors induced by them $Set^{\mathbf{oFs}_\mu^{op}} \longrightarrow Set^{\mathbf{oFs}^{op}}$ and $Set^{\mathbf{oFs}_\omega^{op}} \longrightarrow Set^{\mathbf{oFs}_{loc}^{op}}$ are pra monadic.

Now the monad T can be defined as follows. Let X be a model in $Mod_\otimes(\Phi^{op}, Set)$ and $k : R\to S$ a monotone morphism. For $S\in\mathbf{oFs}$ we put
$$T(X)(S) = \coprod_{g:S\multimap S'} X(S')$$
where the coproduct is taken over all (up to iso) inner ω-maps g with the domain S. For a monotone morphism $k : R\to S$, the function $T(X)(k)$ is so defined that the square

$$\begin{tikzcd}
T(X)(R) = \coprod_{f:R \circ\!\!\!\rightarrow R'} X(R') & & X(R') \arrow[ll, "\kappa_f^{X,R}"'] \\
T(X)(S) = \coprod_{g:S \circ\!\!\!\rightarrow S'} X(S') \arrow[u, "T(X)(k)"] & & X(S') \arrow[ll, "\kappa_g^{X,S}"] \arrow[u, "X(\bar{k})"']
\end{tikzcd}$$

commutes, for every inner ω-map $g : S \circ\!\!\!\rightarrow S'$ so that f and $\bar{k}$ is the inner-monotone factorization of $g \circ k$, i.e. the square

$$\begin{tikzcd}
R \arrow[r, "f"] \arrow[d, "k"'] & R' \arrow[d, "\bar{k}"] \\
S \arrow[r, "g"'] & S'
\end{tikzcd}$$

commutes. Clearly both f and $\bar{k}$ are unique up an isomorphism. For the natural transformation $\alpha : X \to Y$ in $Mod_{\otimes}(\Phi^{op}, Set)$ we define the natural transformation $T(\alpha)$ so that the square

$$\begin{tikzcd}
T(X)(S) \arrow[r, "T(\alpha)_S"] & T(Y)(S) \\
X(S') \arrow[u, "\kappa_g^{X,S}"] \arrow[r, "\alpha_{S'}"'] & Y(S') \arrow[u, "\kappa_g^{Y,S}"']
\end{tikzcd}$$

commutes, for any $S \in \mathbf{oFs}$ and any inner ω-map $g : S \to S'$. This ends the definition of T.

The transformations η and μ are defined like in Example 1 in Section 9. For $X \in Mod_{\otimes}(\Phi^{op}, Set)$ the unit

$$\eta_X : X \to T(X)$$

is so define that for any $S \in \mathbf{oFs}$, we have $(\eta_X)_S = \kappa_{1_S}^{X,S}$. The multiplication

$$\mu_X : T^2(X) \to T(X)$$

is so define that for any $R \in \mathbf{oFs}$ and a pair of inner ω-maps $f : R \to R'$ and $g : R' \to R''$ the triangle

$$\begin{tikzcd}
T^2(X)(R) \arrow[rr, "(\mu_X)_S"] \arrow[d, equal] & & T(X)(R) \arrow[d, equal] \\
\coprod_{f:R \circ\!\!\!\rightarrow R'} \coprod_{g:R' \circ\!\!\!\rightarrow R''} X(R'') & & \coprod_{h:R \circ\!\!\!\rightarrow R''} X(R'') \\
& X(R'') \arrow[ul, "\kappa_{f,g}^{X,S}"] \arrow[ur, "\kappa_{g\circ f}^{X,S}"'] &
\end{tikzcd}$$

commutes. The fact that both η and μ are cartesian can be easily checked as in Example 1 in Section 9.

To show that (T, η, μ) is a pra monad, it remains to show that the functor T is a parametric right adjoint functor. We shall construct this left adjoint L_T explicitly, again with a help of the category $\mathbf{oFs}$. First let us describe $T(1)$. We can assume that for $R \in \mathbf{oFs}$

$$T(1)(R) = \{f : R \circ\!\!\!\rightarrow R' : f \in \mathbf{oFs},\ f \,\text{inner}\} \cong \coprod_{f:R \circ\!\!\!\rightarrow R'} 1(R')$$

Moreover, for $k : R \to S$ in $\mathbf{oFs}$ and $g : S \to S'$ an inner ω-map we have $T(1)(k)(g) = f$ if we have a square in $\mathbf{oFs}_{\omega}$

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "k"'] & R' \arrow[d, "k'"] \\ S \arrow[r, "g"'] & S' \end{tikzcd}$$

with f an inner ω-map and k' monotone morphism.

$$T_1 : Mod_{\otimes}(\Phi^{op}, Set) \longrightarrow Mod_{\otimes}(\Phi^{op}, Set) \downarrow T(1)$$

$$X \quad \mapsto \quad T(!_X) : T(X) \longrightarrow T(1)$$

has a left adjoint

$$L_T : Mod_{\otimes}(\Phi^{op}, Set) \downarrow T(1) \longrightarrow Mod_{\otimes}(\Phi^{op}, Set)$$

The construction of L_T we give below is very similar in spirit to the construction from Example 1 in Section 9. Let $|-| : Z \longrightarrow T(1)$ be an object of $Mod_{\otimes}(\Phi^{op}, Set) \downarrow T(1)$. We shall define a diagram

$$\Gamma_{(Z,|-|)} : \mathcal{D}_{(Z,|-|)} \longrightarrow Mod_{\otimes}(\Phi^{op}, Set)$$

whose colimit is $L_T(Z, |-|)$. The set of objects of $\mathcal{D}_{(Z,|-|)}$ is $\coprod_{S\in\mathbf{oFs}} Z(S)$, where as usual we take the coproduct over isomorphism classes of objects of **oFs**. In other words an object of $\mathcal{D}_{(Z,|-|)}$ is a pair (y, S) so that $S \in \mathbf{oFs}$ $y \in Z(S)$. We identify to such pairs (y, S) and (y', S') if there is a monotone isomorphism $f : S \to S'$ such that $Z(h)(y') = y$. For any object (y, S) and $k < dim(S)$ there are two arrows $d^{(k)}_{y,S}$ and $c^{(k)}_{y,S}$ in $\mathcal{D}_{(Z,|-|)}$ with codomain (y, S). To describe the domains of $d^{(k)}_{y,S}$ and $c^{(k)}_{y,S}$ let us note that since $|y| : S \to R$ is an inner ω-map we can form a diagram

$$\begin{tikzcd} \mathbf{d}^{(k)}S \arrow[r, "\mathbf{d}^{(k)}_S"] \arrow[d, "d^{(k)}(|y|)"'] & S \arrow[r, "\mathbf{c}^{(k)}_S"] \arrow[d, "|y|"'] & \mathbf{c}^{(k)}S \arrow[d, "c^{(k)}(|y|)"] \\ \mathbf{d}^{(k)}R \arrow[r, "\mathbf{d}^{(k)}_R"'] & R \arrow[r, "\mathbf{c}^{(k)}_R"'] & \mathbf{d}^{(k)}R \end{tikzcd}$$

with $\mathbf{d}^{(k)}_S$, $\mathbf{c}^{(k)}_S$, $\mathbf{d}^{(k)}_R$, $\mathbf{c}^{(k)}_R$, monotone and $d^{(k)}(|y|)$, $c^{(k)}(|y|)$ inner ω-maps. Then $d^{(k)}_{y,S}$ and $c^{(k)}_{y,S}$ have the domains as displayed in the diagram:

$$(Z(\mathbf{d}^{(k)}_S)(y), \mathbf{d}^{(k)}S) \xrightarrow{d^{(k)}_{(y,S)}} (y, S) \xleftarrow{c^{(k)}_{(y,S)}} (Z(\mathbf{c}^{(k)}_S)(y), \mathbf{c}^{(k)}S)$$

We have a projection functor

$$\mathcal{D}_{(Z,|-|)} \xrightarrow{\quad \pi_{(Z,|-|)} \quad} \mathbf{oFs}$$

which, in the notation as above with $|y| : S \to R$, is given by

$$\begin{tikzcd}[column sep=small] & (y, S) & & & & R & \\ (Z(\mathbf{d}^{(k)}_S)(y), \mathbf{d}^{(k)}S) \arrow[ur, "d^{(k)}_{(y,S)}"] & & (Z(\mathbf{c}^{(k)}_S)(y), \mathbf{c}^{(k)}S) \arrow[ul, "c^{(k)}_{(y,S)}"'] & \longmapsto & \mathbf{d}^{(k)}R \arrow[ur, "\mathbf{d}^{(k)}_R"] & & \mathbf{c}^{(k)}R \arrow[ul, "\mathbf{c}^{(k)}_R"'] \end{tikzcd}$$

so that the composition of $\pi_{(Z,|-|)}$ with the embedding $\mathbf{oFs} \longrightarrow Mod_{\otimes}(\Phi^{op}, Set)$ is the required diagram $\Gamma_{(Z,|-|)}$ for which we have $L_T(Z, |-|) = Colim\Gamma_{(Z,|-|)}$.

Next I will describe explicitly the adjunction:

$$
\begin{tikzcd}
Mod_{\otimes}(\Phi^{op}, Set) \arrow[r, shift left, "F^{m/1}"] & \omega Cat \arrow[l, shift left, "U^{m/1}"]
\end{tikzcd}
$$

For $X \in Mod_{\otimes}(\Phi^{op}, Set)$ the set of n-cells is give by the coproduct

$$F^{m/1}(X)_n = \coprod_{S \in \mathbf{oFs}_n} \coprod_{S \mathrel{\circ\!\!\!\rightarrow} R} X(R)$$

The k-domain of n-cell operation is the unique morphism that for any $S \in \mathbf{oFs}_n$, and $f : S \to R$ inner ω-map makes the square

$$
\begin{tikzcd}
F^{m/1}(X)_n \arrow[r, "d^{(k)}"] \arrow[d, "\kappa^{X,S}_{n,f}"'] & F^{m/1}(X)_k \arrow[d, "\kappa^{X,\mathbf{d}^{(k)}S}_{k,\mathbf{d}^{(k)}(f)}"] \\
X(S) \arrow[r, "X(\mathbf{d}^{(k)}_S)"'] & X(\mathbf{d}^{(k)}S)
\end{tikzcd}
$$

commute. The k-codomain of n-cell operation is defined analogously and the identity operation is the obvious embedding. Then the set of k-composable n-cells is

$$F^{m/1}(X)_{n,k,n} = \coprod_{S,S' \in \mathbf{oFs}_n,\ \mathbf{c}^{(k)}S = \mathbf{d}^{(k)}S'} \quad \coprod_{f : S \mathrel{\circ\!\!\!\rightarrow} R,\ f' : S' \mathrel{\circ\!\!\!\rightarrow} R'} X(R \otimes_k R')$$

The composition morphism is the unique morphism making the triangles

$$
\begin{tikzcd}
F^{m/1}(X)_{n,k,n} \arrow[rr, "m_{n,k,n}"] & & F^{m/1}(X)_n \\
& X(R \otimes_k R') \arrow[ul, "\kappa^{X,S,S'}_{n,k,f,f'}"] \arrow[ur, "\kappa^{X,S \otimes_k S'}_{n,f \otimes_k f'}"'] &
\end{tikzcd}
$$

commute, for any inner ω-maps $f : S \to R$, $f' : S' \to R'$ so that $\mathbf{c}^{(k)}S = \mathbf{d}^{(k)}S'$. The ω-map $f \otimes_k f' : S \otimes_k S' \longrightarrow R \otimes_k R'$ is well defined as as both f and f' are inner ω-maps. This ends the definition of the ω-category $F^{m/1}(X)$. The definition of $F^{m/1}(X)$ on morphism is obvious.

The right adjoint to $F^{m/1}$ is induced by the embedding $\varepsilon : \mathbf{oFs} \to \omega Cat$. For $C \in \omega Cat$ and $S \in \mathbf{oFs}$ we have

$$U^{m/1}(C)(S) = \omega Cat(\varepsilon(S), C)$$

The adjunction $F^{m/1} \dashv U^{m/1}$ induces the above pra monad (T, η, μ). Moreover, by the Harnik argument c.f. [H][13], $U^{m/1}$ is monadic. Then from the theory developed by T. Leinster and M. Weber the left square is a pseudo-pullback

$$
\begin{tikzcd}
\omega Cat \arrow[r, "\mathcal{N}_\omega"] \arrow[d, "U^{m/1}"'] \arrow[rr, bend left, "\mathcal{N}_\mu"] & Set^{\mathbf{oFs}_\omega^{op}} \arrow[r] \arrow[d, "\iota_\omega^*"] & Set^{\mathbf{oFs}_\mu^{op}} \arrow[d, "\iota_\mu^*"] \\
Mod_{\otimes}(\mathbf{oFs}^{op}, Set) \arrow[r] & Set^{\mathbf{oFs}_{loc}^{op}} \arrow[r] & Set^{\mathbf{oFs}^{op}}
\end{tikzcd}
$$

[13]This argument was presented in [MZ1] for the case of positive-to-one computads but it works without changes for the many-to-one computads as well.

Thus the pra monadic functor $U^{m/1}$ is a pseudo-pullback of pra monadic functor ι_ω^*. As the left bottom functor, induced by the generic model $\mathbf{oFs} \to \mathbf{oFs}_{loc}$, is full and faithful so is the many-to-one nerve functor $\mathcal{N}_\omega$ whose essential image contains those presheaves whose restriction to $\mathbf{oFs}_{loc}^{op}$ preserves principal limits. It is not true that the right hand square in the above diagram is a pseudo-pullback but the outer square still is and the composition of the bottom functors is full and faithful. Hence the pra monadic functor $U^{m/1}$ is a pseudo-pullback of pra monadic functor ι_μ^* as well. Thus we have another full nerve functor $\mathcal{N}_\mu$ whose essential image consists of those functors whose restriction to $\mathbf{oFs}^{op}$ are models of $\mathbf{oFs}$.

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