On ordered face structures and many-to-one computads

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Abstract

We introduce the notion of an ordered face structure. The ordered face structures to many-to-one computads are like positive face structures, c.f. [Z], to positive-to-one computads. This allow us to give an explicit combinatorial description of many-to-one computads in terms of ordered face structures.

Contents

1 Introduction

The definition of multitopic categories the weak ω-categories in the sense of Makkai contains two ingredients. The first constitutes a description of shapes of cells that are considered (this includes the relation between cells and their domains and codomains), c.f. [HMP] and the second constitutes a mechanism of composition, c.f. [M]. This paper is a contribution to a better understanding of the first ingredient of the M.Makkai's definition of multitopic categories, and we provide a relatively simple combinatorial description of the category many-to-one computads. The paper goes much along with [Z] except it deals with all many-to-one computads rather than positive-to-one computads. This generates some substantial complications and the structure of cells turns out to be much richer.

Ordered face structures

Our main combinatorial device introduced and studied in this paper is the *ordered face structure*. The ordered face structures correspond to all possible 'shapes' of cells (not only indeterminates) in many-to-one computads[1]. In order to relate them to our previous work [MZ], [Z] we can draw an analogy in the following table.

type of computads	*shapes of*			
	indeterminates		*arbitrary cells*	
	described in terms of			
	graph−like structures	*computads*	*graph−like structures*	*computads*
one−to−one [MZ]	α^n	$(\alpha^n)^*$	*simple ω−graphs*	*simple categories*
positive−to−one [Z]	*principal positive face structures*	*positive computopes*	*positive face structures*	*positive computypes*
many−to−one [this paper]	*principal ordered face structures*	*computopes*	*ordered face structures*	*pointed computypes*

Now are going to explain it in an intuitive way. In the table we describe cells in computads of three kinds. The later being strictly more general than the former. The one-to-one computads are the simplest. They are free ω-categories over ω-graphs[2]. The positive-to-one computads are computads in which the indeterminates (or indets) on the higher dimension have as codomains indeterminates and as domains cells that are not identities. Finally, the many-to-one computads are computads in which the indets have as codomains indets again but there is no specific restriction for the domains (other than that they must be parallel to codomains).

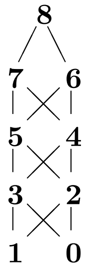
Fix $n \in \omega$. The ω-graph (also called globular set) α^n, has one n-face and exactly two faces of lower dimensions than n, i.e.

$$\alpha^n_l \;=\; \begin{cases} \emptyset & \text{if } l > n \\ \{2n\} & \text{if } l = n \\ \{2l+1,\, 2l\} & \text{if } 0 \leq l < n \end{cases}$$

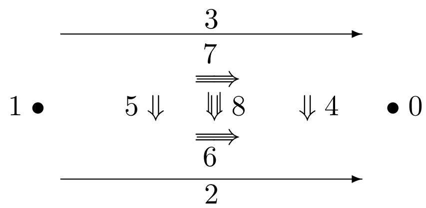
with domain and codomain given by $d,\, c : \alpha^n_l \longrightarrow \alpha^n_{l-1}$, $d(x) = \{2l-1\}$, $c(x) = 2l-2$ for $x \in \alpha^n_l$, and $1 \leq l \leq n$. For example α^4 can be pictured as follows:

[1]For the definition of many-to-one computad see the appendix.

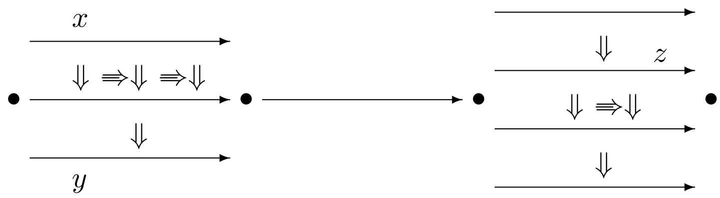
[2]In the literature ω-graphs are sometimes called globular sets.



i.e. 8 is the unique face of dimension 4 in α^4 that has 7 as its domain and 6 as its codomain, 7 and 6 have 5 as its domain and 4 as its codomain, and so on. More visually we can draw α^4 as follows



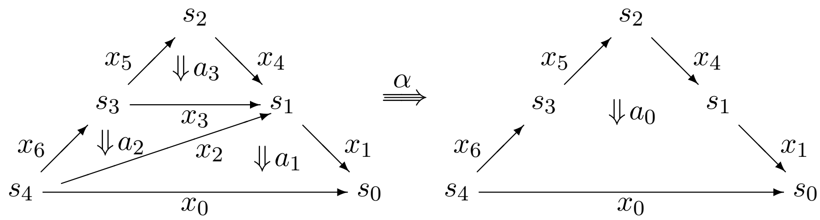
The free category $(\alpha^n)^*$ generated by α^n has the property that for any ω-category C, the set $\omega Cat((\alpha^n)^*, C)$ of ω-functors from $(\alpha^n)^*$ to C correspond naturally to the set C_n of n-cells of C. Thus in one-to-one computads the shapes of indets are particularly simple and this is why the ω-graphs describing them are called *simple*. Simple ω-graphs are some 'special' pushouts of α's. Instead of trying to repeat the definition from [MZ] we rather show an example:



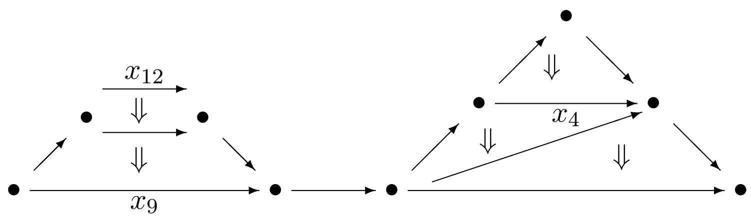
So indets have still indets as domains and codomains and even if there is no one indet that generates all the ω-graph, as in α^n's, the domains and codomains of indets so fit together that they could be (uniquely) composed 'if they were placed in an ω-category'. Simple ω-categories, c.f. [MZ], are ω-categories generated by such ω-graphs. The category of simple ω-categories is dual to the category of disks introduced in [J] . Note that there are two definite ways the indets of the same dimension can be compared. The face x is smaller from y in one way and from z the other way. We write $x <^+ y$ and $x <^- z$. The first order[3] is called *upper* and the second is called *lower*. More formally, the upper order on cells of dimension n is the least transitive relation such that $d(a) <^+ c(a)$ for any face a of dimension $n+1$ (d and c are operations of domain and codomain, respectively). Similarly, the lower order on cells of dimension n is the least transitive relation such that if $d(x) = c(z)$ then $x <^- z$. In this case both orders are definable using d and c. For more on this see [MZ].

The shapes of indeterminates in positive-to-one face structures are more complicated. We again use drawing to explain what principal positive face structures are. The one below has dimension 3.

[3] Here and later by order we mean *strict order* i.e. irreflexive and transitive relation.



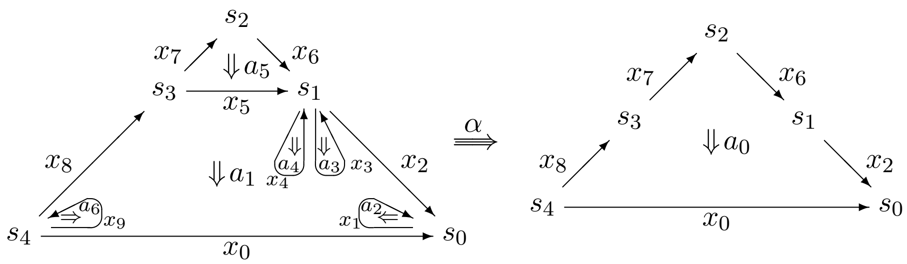
Thus in positive face structures the codomains of indets are still indets but the domains are so called *pasting diagrams* of indets, i.e. domains contains indets that 'suitably fit together so that we could compose them'. In these structures we have the usual operation of taking codomain but the 'operation' of taking domain of a face returns a non-empty set of faces rather than a single face. To emphasize this change we use for these operations the Greek letters γ and δ instead of c and d. Thus $\gamma(\alpha) = a_0$, $\gamma(a_3) = x_3$, $\delta(\alpha) = \{a_1, a_2, a_3\}$, $\delta(a_0) = \{x_1, x_4, x_5, x_6\}$, $\delta(a_2) = \{x_3, x_6\}$. From the table we have that positive face structures to principal positive face structures are like simple ω-graphs to ω-graphs of form α^n, for some n. Thus it should be not surprising that positive face structures looks like this:



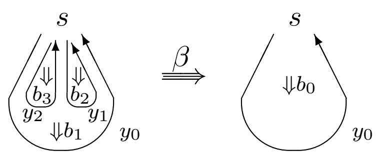
Different points, arrows etc. denote necessarily different cells, and if we omit their names in figures it is for making it less baroque. Note that in this case the indets of the same dimension can be compared much the same way as indets in simple ω-graphs in two definite ways. The face x_{12} is smaller than x_9 in one way and than x_4 the other way, and again we write $x_{12} <^+ x_9$ and $x_{12} <^- x_4$. Again the first order is called *upper* and the second is called *lower*. More formally, the upper order on faces of dimension n is the least transitive relation such that $x <^+ y$ whenever there is a face a of dimension $n+1$ such that $x \in \delta(a)$ and $\gamma(a) = y$. Similarly, the lower order on faces of dimension n is the least transitive relation such that $x <^- y$ whenever $\gamma(x) \in \delta(y)$. Any positive face structure T generates a computads T^*. The cells of dimension n of such a computad are positive face substructures of T of dimension at most n. These computads are called *positive computypes*. If T is a principal positive face structure then T^* is a *positive computope*[4]. In this case T^* determines T up to an isomorphism. For more on this see [Z].

The shapes of indeterminates in many-to-one face structures are even more complicated as this time the domains of indets might be identities (='empty on something'). This generates a lot of complications as we have three new kinds of faces. Apart from *positive faces* like in previous case we have *empty-domain faces* and then as a consequence we have *loops* (=faces with domain equal codomain) and we also need to deal with *empty faces*. The last kind of faces is not indicated in the pictures. On each face x of dimension n there is an empty face 1_x of dimension $n+1$. They are much like with identities whose role they play. We again use drawing to explain intuitively what principal ordered face structures are:

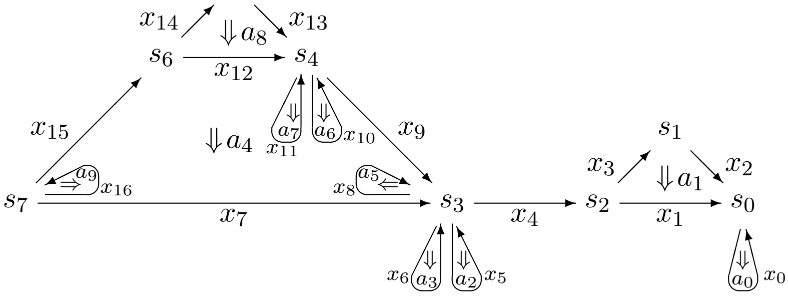
[4]The word 'positive' is used here more like a shorthand and in presence of 'other positive' notions this one should be named properly as 'positive-to-one'.



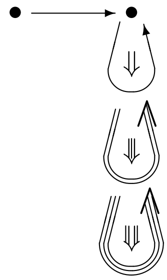
and a bit more fancy



In these structures we also use the Greek letters γ and δ for domains and codomains, respectively. Similarly as in positive face structures the codomain is an operation associating faces to faces. But the domain operation is still more involved as it may associate to a face a non-empty set of faces or a single empty face. Thus we have $\delta(\alpha) = \{a_1, \ldots, a_6\}$, $\gamma(\alpha) = a_0$ but $\delta(a_2) = 1_{s_0}$, $\delta(b_0) = 1_s$. Note that we should write $\delta(x_1) = \{s_0\}$ instead of $\delta(x_0) = s_0$ but we will, as we did in [Z], mix singletons with elements when dealing with faces or sets of faces e.g. both conditions $\gamma(x_0) \in \delta(x_0)$ and $\gamma(x_0) = \delta(x_0)$ are meaningful in this convention and in fact, as we will see later, due to this 'double meaning' they are equivalent in all ordered face structures saying that x_0 is a loop. This time the relations between faces and their domains and codomains does not encode all the needed data. The upper order $<^+$ can be defined like in positive face structures from γ and δ. However, due to existence of loops, the relation $<^-$ defined as before is not a strict order in general. In the above examples we have $x_3 <^- x_4$, $x_4 <^- x_3$ and similarly $y_2 <^- y_1$, $y_1 <^- y_2$. But we definitely need to know that x_4 comes before x_3 and that y_2 comes before y_1. This is why we need as a separate additional data a strict order $<^\sim$ that is contained in $<^-$ telling us that $x_4 <^\sim x_3$ and $y_2 <^\sim y_1$ but not that $x_3 <^\sim x_3$ and $y_1 <^\sim y_2$. As we need to have the strict order $<^\sim$ as an additional piece of data we call those face structures *ordered*. Note however that in the above cases we could solve our problem of ordering the faces locally that is having just restriction of the order $<^\sim$ to sets that are domains of other faces. But to describe all the cells of many-to-one computads we need more than just that. Below we have some examples of ordered face structure



and



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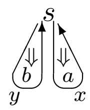
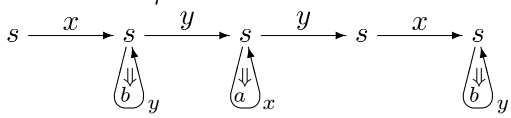
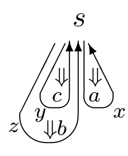
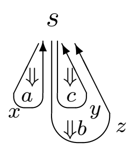


We see that faces x_6 and x_5 must be comparable via $<^\sim$ but they are not in domain of any other face. Thus a kind of global order $<^\sim$ is needed. Note however that the fact that x_{11} comes before both x_6 and x_4 and that x_0 comes after all of them could be deduced in a different way. The way the ordered face structure T generate a many-to-one computad T^* is more involved then in case of positive face structures. An n-cell in T^*_n is a *local morphism* $\varphi : X \to T$ where X is an ordered face structure of dimension at most n and φ is a map that preserves γ, δ but the order $<^\sim$ is preserved only locally i.e. for $a \in X$ $\varphi : (\delta(a), <^\sim_a) \to (\delta(\varphi(a)), <^\sim_{\varphi(a)})$ is an order isomorphism, where $<^\sim_a$, $<^\sim_{\varphi(a)}$ are restrictions of orders $<^\sim$ to $\delta(a)$ and $\delta(\varphi(a))$, respectively. Thus we have a cell φ:



in the computad generated by the ordered face structure T^* (where T is the last example of an ordered face structure above). The faces of the above ordered face structures are labelled by the faces they are sent to by the local morphism φ. Clearly, in this case the local preservation of the order $<^\sim$ does not impose any restriction on the map $\varphi : X \to T$ other than preservation of γ and δ. From this it should be clear that we cannot in general determine T having just T^*. For example the ordered face structures





are not isomorphic, as $y <^\sim x$ and $z <^\sim x$ in the left one and $x <^\sim y$ and $x <^\sim z$ in the right one, but they generate isomorphic computads. In other word passing from T to T^* we are loosing part of data and this is why T^* is not sufficient, in general (unlike T), to determine the shape of a cell in a many-to-one computad. To keep this information we need to choose one cell in T^* with the natural choice being the identity on T, $id_T : T \to T$. The ω-categories T^* together with a distinguished cells id_T are pointed computypes which are the computad-like descriptions of types of all cells in many-to-one computads. The pointed computypes can be defined abstractly but we are going to explain it elsewhere. If T is a principal ordered face structure then this distinguished cell can be chosen in a unique way and hance it does not to be chosen at all as we know anyway which one we were to choose. This is why the computopes, the computad-like descriptions of types of indets in many-to-one computads are the ω-categories generated by principal ordered face structures (without an additional cell chosen).

Primitive notions and axioms

Thus we related ordered face structures to simple ω-graphs and positive face structures and we have described the primitive notions γ, δ, $<^{\sim}$ that we had chosen to axiomatize them. Now we shall describe some intuitions behind the axioms of ordered faces structures. Even if they are more involved they are quite close in the spirit to the axioms of positive face structures.

As in case of positive face structures, the most important axiom is the axiom of *globularity*. In case of ω-graphs it is just $cc = cd$ and $dc = dd$ which, if we rebaptize c as γ and d as δ, take form

$$\gamma\gamma(\alpha) = \gamma\delta(\alpha), \quad \delta\gamma(\alpha) = \delta\delta(\alpha). \tag{1}$$

As it was pointed out in [Z] this equations cannot hold even for positive-to-one faces as the right hand sides might be much bigger the left hand sides. In the example of a principal positive computad from page 4, we have

$$\gamma\gamma(\alpha) = x_0 \neq \{x_0, x_2, x_3\} = \gamma\delta(\alpha),$$

$$\delta\gamma(\alpha) = \{x_1, x_4, x_5, x_6\} \subsetneq \{x_1, x_2, x_3, x_4, x_5, x_6\} = \delta\delta(\alpha).$$

Thus we corrected the formula (1) by subtracting some faces from the right side getting

$$\gamma\gamma(\alpha) = \gamma\delta(\alpha) - \delta\delta(\alpha), \quad \delta\gamma(\alpha) = \delta\delta(\alpha) - \gamma\delta(\alpha). \tag{2}$$

Now it works for positive-to-one faces but if we allow loops in the domains of faces, and we must if we allow empty-domain faces, these formulas still doesn't work as we can see for the face a_1 in positive ordered face structure on page 5. We have

$$\gamma\gamma(a_1) = s_0 \neq \emptyset = \gamma\delta(a_1) - \delta\delta(a_1), \quad \delta\gamma(a_1) = s_4 \neq \emptyset = \delta\delta(a_1) - \gamma\delta(a_1)$$

Thus we see that we subtracted too much. Correcting this we drop these loops and we get

$$\gamma\gamma(\alpha) = \gamma\delta(\alpha) - \delta\delta^{-\lambda}(\alpha), \quad \delta\gamma(\alpha) = \delta\delta(\alpha) - \gamma\delta^{-\lambda}(\alpha). \tag{3}$$

where $\delta^{-\lambda}(\alpha)$ means the set of those faces in $\delta(\alpha)$ that are not loops. Now the formula (3) works for the face a_1 and even for the face α on page 5. But there is still a problem with empty-domain faces, as we have for b_0 in the same ordered face structure.

$$\gamma\gamma(b_0) = s \neq \emptyset = \gamma\delta(b_0) - \delta\delta^{-\lambda}(b_0),$$

$$\delta\gamma(b_0) = s \neq \emptyset = \delta\delta(b_0) - \gamma\delta^{-\lambda}(b_0).$$

As a remedy for this we shall still diminish the set that we subtract by dropping empty faces which might be there. So we drop these empty-faces and we get

$$\gamma\gamma(\alpha) = \gamma\delta(\alpha) - \delta\dot{\delta}^{-\lambda}(\alpha), \quad \delta\gamma(\alpha) = \delta\delta(\alpha) - \gamma\dot{\delta}^{-\lambda}(\alpha). \tag{4}$$

where $\dot{\delta}^{-\lambda}(\alpha)$ means the set of those faces in $\delta(\alpha)^{-\lambda}$ that are not empty faces[5]. We are almost there but if in the domain $\delta(\alpha)$ of a face α we have both empty-domain faces and faces with positive domains as we have in $\delta(\alpha)$ in on page 5, then the set $\delta\delta(\alpha)$ may contain both empty and non-empty faces whereas $\delta\gamma(\alpha)$ definitely

[5] This means that this set is either empty if $\delta(\alpha)$ is an empty face or it is $\delta(\alpha)^{-\lambda}$.

contain just one kind of faces either one single empty faces or a non-empty set of non-empty faces. However if we have faces of both kinds in $\delta\delta(\alpha)$ the empty faces must be empty-faces on domains or codomains of the non-empty faces in this set. And this is the final modification that we do to our equation:

$$\gamma\gamma(\alpha) = \gamma\delta(\alpha) - \delta\dot{\delta}^{-\lambda}(\alpha), \quad \delta\gamma(\alpha) \equiv_1 \delta\delta(\alpha) - \gamma\dot{\delta}^{-\lambda}(\alpha). \tag{5}$$

where $A \equiv_1 B$ is equivalence of two set of faces of the same dimension modulo empty faces which means that

1. if one set contains only empty faces then the other also contains only empty faces and these sets are equal,
2. or else both sets contain the same non-empty faces and any empty face in either set is an empty face on domain or codomain of a non-empty face is those sets.

In other words if $\dot{A}$ denote non-empty faces in A, $\ddot{A}$ denote empty faces in A we have $A \equiv_1 B$ iff $\dot{A} = \dot{B}$ and $\ddot{A} \subseteq \ddot{B} \cup 1_{\gamma(\dot{B})\cup\delta(\dot{B})}$ and $\ddot{B} \subseteq \ddot{A} \cup 1_{\gamma(\dot{A})\cup\delta(\dot{A})}$. Still in other words A and B are sets of faces that generates, via γ and δ, the same substructures.

The last axiom, *loop filling* is the only other axiom that does not mention order explicitly, it says that there are no empty loops, i.e. if there is a loop it must be a codomain of at least one face which is not a loop.

The remaining four axioms talk about orders $<^+$ and $<^\sim$. *Local discreteness* says that faces in a domain of any other face cannot be comparable via the upper order $<^+$. The *strictness*, *disjointness* together with *pencil linearity* say in a sense that $<^\sim$ is the maximal strict order order relation that is contained in the relation $<^-$ and disjoint from $<^+$.

Note that as $<^+$, $<^-$ are the transitive closures of elementary relations so these axioms are not first order axiom and in fact they are expressed in the transitive closure logic.

Future work

This paper covers only part of the program developed in [Z] for positive-to-one computads. We end this here as it is already very long paper. But the remaining parts of the program from [Z] for many-to-one computads and the application of this to the multitopic categories, c.f. [HPM],[M], will be presented soon.

Content of the paper

Section 2 contains the definition of a hypergraph and some notation needed to introduce the notion of an ordered face structure. In section 3 we introduce the main notion of this paper the notion of an ordered face structure. In section 4 we develop most of the needed elementary theory of ordered face structures. This section should be more consulted when needed than read through. The monotone morphism, is the stricter of two kind of morphisms between ordered face structures, it preserves the order $<^\sim$ globally. The image of such a morphism is a convex set. In section 5 we show that from the convex set we can recover the whole morphism up to an isomorphism. The domain of such a morphism is recovered via cuts of empty loops in the convex subset. The next two sections show the connection between positive and ordered face structures. In section 6 we describe how we can divide a positive face structure by an ideal to get an ordered face structure. In section 7 we show that for any ordered face structure S there is a positive face structure $S^\dagger$ and ideal so that $S^\dagger$ divided by this ideal is isomorphic to S. The positive face structure $S^\dagger$ is defined with the help of cuts of so called initial faces. In sections 8 and 9 we describe some abstract structure of the category **oFs** and show some of their properties. This allow us to define in section 10 a free functor $(-)^* : \mathbf{lFs} \longrightarrow \omega Cat$ from the category of local face structures **lFs** to the category of ω-categories. Local face structures are structures that have operations γ and δ as in ordered face structures but with the order $<^\sim$ (in fact a binary relation) restricted to domains of faces only. The section 11 discusses basic properties of principal and normal ordered face structures the face structures that are generated by a single face and such that can be domains (in the sense of monoidal globular category **oFs**) of such structures. In section 12 we study decompositions of ordered face structures. In [Z] we have defined the decompositions of positive face structures along some faces. Here we decompose ordered face structure S along a cut of initial faces $\breve{a}$ rather than a face a as this decomposition in more like a decomposition of the positive cover $S^\dagger$ and then after decomposition divided to get the decomposition of S. Doing it this way we can deduce most of the properties of this decomposition from the corresponding decomposition of positive face structures. In section 13 we show that the ω-category T^*, for T being an ordered face structure, is in fact a many-to-one computad. The next two sections 14 and 15 describe with the help of ordered face structures the terminal many-to-one computad and all the cells in an arbitrary many-to-one computad.

Notation and conventions

As we already indicated we will intensionally confuse singletons with elements when dealing with faces in ordered face structures. In the paper we often will be using cells of different but neighboring dimensions. As it is a bit confusing anyway we try to make it a bit easier to follow by a careful use of the following convention. α, β are faces of the same dimension, say n, then a, b are the faces of the same dimension $n-1$, x, y, z are the faces of the same dimension $n-2$, t, s are the faces of the same dimension $n-3$, u, v are the faces of the same dimension $n-4$. We may use occasionally A, B to denote faces of dimension $n+1$. These faces may appear with indices but these letter should be a direct hint which dimension we are working on. The above examples were already using this convention. Last but not least the composition of two morphisms

$$X \xrightarrow{\ f\ } Y \xrightarrow{\ g\ } Z$$

may be denoted as either $g \circ f$ or more often $f; g$. In any case we will write which way we mean the composition.

Acknowledgements

I like to thank Bill Boshuck, Victor Harnik, and Mihaly Makkai for the conversation we had at the early stage of this work.

2 Hypergraphs

A *hypergraph* S is

1. a family $\{S_k\}_{k\in\omega}$ of finite sets of faces; only finitely many among these sets are non-empty;

2. a family of functions $\{\gamma^S_k : S_{k+1} \sqcup \mathbf{1}_{S_k} \to S_k\}_{k\in\omega}$; where $\mathbf{1}_{S_k} = \{1_u : u \in S_k\}$ is the set of empty faces of dimension k; the face 1_u is the empty k-dimensional face on a non-empty face u of dimension $k-1$.

3. a family of total relations $\{\delta^S_k : S_{k+1} \sqcup \mathbf{1}_{S_k} \to S_k \sqcup \mathbf{1}_{S_{k-1}}\}_{k\in\omega}$; for $a \in S_{k+1}$ we denote $\delta^S_k(a) = \{x \in S_k \sqcup \mathbf{1}_{S_{k-1}} : (a,x) \in \delta^S_k\}$; $\delta^S_k(a)$ is either singleton or it is non-empty subset of S_k. Moreover $\delta^S_0 : S_1 \sqcup \mathbf{1}_{S_0} \to S_0 \sqcup \mathbf{1}_{S_{-1}}$ is a function (for this expression to make sense we assume that $S_{-1} = \emptyset$). We put $\dot{\delta}(a) = \delta(a) \cap S$ and $\ddot{\delta}(a) = \delta(a) \cap \mathbf{1}_S$.

$$\begin{tikzcd}
 & S_{k+1} \sqcup \mathbf{1}_{S_k} \arrow[dl, "\delta^S_k"'] \arrow[dr, "\gamma^S_k"] & \\
S_k \sqcup \mathbf{1}_{S_{k-1}} & & S_k
\end{tikzcd}$$

A *morphism of hypergraphs* $f : S \longrightarrow T$ is a family of functions $f_k : S_k \longrightarrow T_k$, for $k \in \omega$, such that the diagrams

$$\begin{tikzcd}
S_{k+1} \arrow[r, "f_{k+1}"] \arrow[d, "\gamma"'] & T_{k+1} \arrow[d, "\gamma"] \\
S_k \arrow[r, "f_k"'] & T_k
\end{tikzcd}
\qquad
\begin{tikzcd}
S_{k+1} \arrow[r, "f_{k+1}"] \arrow[d, "\delta"'] & T_{k+1} \arrow[d, "\delta"] \\
S_k \sqcup \mathbf{1}_{S_{k-1}} \arrow[r, "f_k + \mathbf{1}_{f_{k-1}}"'] & T_k \sqcup \mathbf{1}_{T_{k-1}}
\end{tikzcd}$$

commute (where $\mathbf{1}_{f_{k-1}}(1_x) = 1_{f_{k-1}(x)}$, for $x \in S_{k-2}$), for $k \in \omega$.

The commutation of the left hand square is the commutation of the diagram of sets an functions but in case of the right hand square we mean more than commutation of a diagram of relations, i.e. we demand that for any $a \in S_{\geq 1}$, $f_a : \delta(a) \longrightarrow \delta(f(a))$ be a bijection, where f_a is the restriction of f to $\dot{\delta}(a)$ (if $\delta(a) = 1_u$ we mean by that $\delta(f(a)) = 1_{f(u)}$). The category of hypergraphs is denoted by **Hg**.

Convention. If $a \in S_k$ we treat $\gamma(a)$ sometimes as an element of S_{k-1} and sometimes as a subset $\{\gamma(a)\}$ of S_{k-1}. Similarly $\delta(a)$ is treated sometimes as a set of faces or as a single face if this set of faces is a singleton. In particular, when we write $\gamma(a) = \delta(b)$ we mean rather $\{\gamma(a)\} = \delta(b)$ or in other words that $\delta(b)$ has one element this element is a face (not an empty face) and that this face is $\gamma(a)$. We can also write $\gamma(a) \in \delta(b)$ to mean that $\delta(b) \subset S$ and that $\gamma(a)$ is one (of possibly many) elements of $\delta(b)$. This convention simplifies the formulas considerably.

Notation. Before we go on, we need some notation. Let S be an ordered hypergraph.

1. The dimension of S is $\max\{k \in \omega : S_k \neq \emptyset\}$, and it is denoted by $dim(S)$.

2. The sets of faces of different dimensions are assumed to be disjoint (i.e. $S_k \cap S_l = \emptyset$, for $k \neq l$); S is also used to mean the set of all faces of S i.e. $\bigcup_{k=0}^n S_k$; the notation $A \subseteq S$ mean that A is a set of some faces of S; $A_k = A \cap S_k$, for $k \in \omega$.

3. If $a \in S_k$ then the face a has dimension k and we write $dim(a) = k$.

4. $S_{\geq k} = \bigcup_{i \geq k} S_i$, $\qquad S_{\leq k} = \bigcup_{i \leq k} S_i$. The set $S_{\leq k} = \bigcup_{i \leq k} S_i$ is closed under δ and γ so it is a sub-hypergraph of S, called *k-truncation* of S.

5. $\delta(A) = \bigcup_{a \in A} \delta(a)$ is the image of $A \subseteq S$ under δ;
$\gamma(A) = \{\gamma(a) : a \in A\}$ is the image of A under γ.
Following the convention mentioned above if either $\gamma(A)$ or $\delta(A)$ is a singleton we may treat them as a (possibly empty) single face.

6. For $a \in S_{\geq 1}$, the set $\theta(a) = \delta(a) \cup \gamma(a)$ is *the set of codimension 1 faces* in a. We put $\dot{\theta}(a) = \theta(a) \cap S$.

7. Let $x, \alpha \in S$. We define the following subsets of faces of S:

   (a) *empty domain faces*: $S^{\varepsilon} = \{a \in S : \delta(a) \in \mathbf{1}_S\}$;

   (b) *non-empty domain faces*: $S^{-\varepsilon} = S - S^{\varepsilon}$; we write $\delta^{-\varepsilon}(A)$ for $\delta(A) \cap S^{-\varepsilon}$;

   (c) *loops*: $S^{\lambda} = \{a \in S : \delta(a) = \gamma(a)\}$;

   (d) *non-loops*: $S^{-\lambda} = S - S^{\lambda}$; we also write $\delta^{-\lambda}(A)$ for $\delta(A) \cap S^{-\lambda}$;

   (e) *unary faces*: $S^{u} = \{a \in S : |\dot{\delta}(a)| = 1\}$;

   (f) for $\alpha \in S_{\geq 2}$ we define the set of *internal faces* of α;

   $$\iota(\alpha) = \{x \in S : \exists a, b \in \dot{\delta}^{-\lambda}(\alpha) : \gamma(a) = x \in \delta(b)\} = \gamma\dot{\delta}^{-\lambda}(\alpha) \cap \delta\dot{\delta}^{-\lambda}(\alpha)$$

   (g) *internal faces*: $\iota(S)$;

   (h) *initial faces*: $\mathcal{I} = \mathcal{I}^S = S^{\varepsilon} - \gamma(S^{-\lambda})$;

   (i) *$x-$cluster (of initial faces)*: $\mathcal{I}_x = \mathcal{I}^S_x = \{\alpha \in \mathcal{I}^S : \gamma\gamma(\alpha) = x\}$;

   (j) *initial faces over α*: $\mathcal{I}^{\leq^+\alpha} = \{\beta \in \mathcal{I} : \beta \leq^+ \alpha\}$;

   (k) *$x-$cluster (of initial faces) over α*: $\mathcal{I}^{\leq^+\alpha}_x = \mathcal{I}^{\leq^+\alpha} \cap \mathcal{I}_x$.

8. On each set S_k we introduce two binary relations $<^{S_k,-}$ and $<^{S_k,+}$. We usually omit k in the superscript and sometimes even S.

   (a) $<^{S_0,-}$ is the empty relation. For $k > 0$, the relation $<^{S_k,-}$ is the transitive closure of the relation $\lhd^{S_k,-}$ on S_k, such that $a \lhd^{S_k,-} b$ iff $\gamma(a) \in \delta(b)$. We write $a \perp^{S_k,-} b$ if either $a <^{S_k,-} b$ or $b <^{S_k,-} a$, and we write $a \leq^- b$ iff $a = b$ or $a <^- b$;

   (b) $<^{S_k,+}$ is the transitive closure of the relation $\lhd^{S_k,+}$ on S_k, such that $a \lhd^{S_k,+} b$ iff there is $\alpha \in S^{-\lambda}_{k+1}$, such that $a \in \delta(\alpha)$ and $\gamma(\alpha) = b$. We write $a \perp^{S_k,+} b$ if either $a <^{S_k,+} b$ or $b <^{S_k,+} a$, and we write $a \leq^+ b$ if either $a = b$ or $a <^+ b$.

   (c) $a \not\perp b$ if both conditions $a \not\perp^+ b$ and $a \not\perp^- b$ hold.

9. Let $a, b \in S_k$. A *lower path from* a to b in S is a sequence of faces $a_0, \ldots, a_m$ in S_k such that $a = a_0$, $b = a_m$ and, $\gamma(a_{i-1}) \in \delta(a_i)$, for $i = 1, \ldots, m$.

   A lower path is a *flat lower path* if it contains no loops other than a or b.

10. Let $x, y \in S_k$. An *upper path from* x to y in S is a sequence $a_0, \ldots, a_m$ in S_{k+1} such that $x \in \delta(a_0)$, $y = \gamma(a_m)$ and, $\gamma(a_{i-1}) \in \delta(a_i)$, for $i = 1, \ldots, m$.

    An upper path is a *flat upper path* if it contains no loops.

11. The iterations of γ, δ and θ will be denoted in two different ways. By γ^k, δ^k and θ^k we mean k applications of γ and δ, respectively. By $\gamma^{(k)}$, $\delta^{(k)}$ and $\theta^{(k)}$ we mean the application as many times γ, δ and θ, respectively, to get faces of dimension k. For example if $a \in S_5$ then $\delta^3(a) = \delta\delta\delta(a) \subseteq S_2$ and $\delta^{(3)}(a) = \delta\delta(a) \subseteq S_3$.

3 Face structures

To simplify the notation, we treat both δ and γ as functions acting on faces as well as on sets of faces, which means that sometimes we confuse elements with singletons. Clearly, both δ and γ when considered as functions on sets are monotone.

We need the following relation. Let S be a hypergraph. We introduce an 'equality' relations $\equiv_1$ on subsets of $S_k \cup \mathbf{1}_{S_{k-1}}$, for $k \in \omega$, that may ignore the $\mathbf{1}_S$-part of the sets in presence of faces from S. Let $A, B \subseteq S_k \cup \mathbf{1}_{S_{k-1}}$. We set that A is 1-*equal* B, notation $A \equiv_1 B$, iff $A \cup 1_{\theta(A\cap S)} = B \cup 1_{\theta(B\cap S)}$.

An *ordered face structure* $(S, <^{S_k,\sim})_{k\in\omega}$ (also denoted S) is a hypergraph S together with a family of $\{<^{S_k,\sim}\}_{k\in\omega}$ of binary relations ($<^{S_k,\sim}$ is a relation on S_k), if it is non-empty, i.e. $S_0 \neq \emptyset$ and

1. *Globularity:* for $a \in S_{\geq 2}$:
$$\gamma\gamma(a) = \gamma\delta(a) - \delta\dot{\delta}^{-\lambda}(a), \qquad \delta\gamma(a) \equiv_1 \delta\delta(a) - \gamma\dot{\delta}^{-\lambda}(a).$$
and for any $a \in S$:
$$\delta(1_a) = a = \gamma(1_a).$$

2. *Local discreteness*: if $x, y \in \delta(a)$ then $x \not<^{+} y$.

3. *Strictness*: for $k \in \omega$, the relations $<^{S_k,+}$ and $<^{S_{k+1},\sim}$ are strict orders[6]; $<^{S_0,+}$ is linear; (i.e. no flat path is a cycle).

4. *Disjointness*: for $k \in \omega$, the relation $<^{S_k,\sim}$ is a maximal strict order relation contained in $<^{S_k,-}$ that is disjoint from $<^{S_k,+}$, i.e. for $k > 0$,
$$\perp^{S_k,\sim} \cap \perp^{S_k,+} = \emptyset$$
for any $a, b \in S_k$:
$$\text{if } \ a <^{\sim} b \ \text{ then } \ a <^{-} b$$
$$\text{if } \ \theta(\mathrm{a}) \cap \theta(\mathrm{b}) = \emptyset \ \text{ then } \ a <^{\sim} b \ \text{ iff } \ a <^{-} b$$
(i.e. if faces are not incident then $<^{\sim}$ is the same as $<^{-}$).

5. *Pencil linearity*: for any $a, b \in S_{\geq 1}$, $a \neq b$,
$$\text{if } \ \dot{\theta}(\mathrm{a}) \cap \dot{\theta}(\mathrm{b}) \neq \emptyset \ \text{ then either } \ a \perp^{\sim} b \ \text{ or } \ a \perp^{+} b$$
for any $a \in S^{\varepsilon}_{\geq 2}$, $b \in S_{\geq 2}$,
$$\text{if } \ \gamma\gamma(\mathrm{a}) \in \iota(\mathrm{b}) \ \text{ then either } \ a <^{\sim} b \ \text{ or } \ a <^{+} b$$
(i.e. if faces are incident then they are comparable).

6. *Loop-filling*: $S^{\lambda} \subseteq \gamma(S^{-\lambda})$; (i.e. no empty loops).

The relation $<^{+}$ is called the *upper order* and $<^{\sim}$ is called *lower order*.

The morphism of ordered face structures, the *monotone morphism*, $f : S \longrightarrow T$ is a hypergraph morphism that preserves the order $<^{\sim}$. The category of ordered face structures, is denoted by **oFs**.

The relation $<^{\sim}$ in an ordered face structure S induces a binary relation $(\dot{\delta}(a), <^{\sim}_{a})$ for each $a \in S_{>0}$ (where $<^{\sim}_{a}$ is the restriction of $<^{\sim}$ to the set $\dot{\delta}(a)$). In the construction of the free ω-categories over an ordered face structure we need

[6]By *strict order* we mean an irreflexive and transitive relation.

to consider hypergraph morphisms that preserves only this induced structure (not the whole relation $<^{\sim}$). This is why we introduce the category of local face structures.

A *local face structure* $(S, <_a^{S_k,\sim})_{a\in S}$ is a hypergraph S together with a family of $\{(\delta(a), <_a^{S_k,\sim})\}_{a\in S}$ of binary relations. The morphism of local face structures, the *local morphism*, $f : S \longrightarrow T$ is a hypergraph morphism that is a local isomorphism i.e. for $a \in S_{>1}$ the restricted map $f_a : (\dot{\delta}(a), <_a^{\sim}) \longrightarrow (\dot{\delta}(f(a)), <_{f(a)}^{\sim})$ is an order isomorphism, where f_a is the restriction of f to $\dot{\delta}(a)$. The category of local face structures, is denoted by **lFs**.

Clearly we have a 'forgetful' functor:

$$| - | : \mathbf{oFs} \longrightarrow \mathbf{lFs}$$

sending $(S, <^{S_k,\sim})_{k\in\omega}$ to $(S, <_a^{\sim})_{a\in S_{>1}}$, where $<_a^{\sim}$ is the restriction of $<^{\sim}$ to $\dot{\delta}(a)$, for $a \in S_{>1}$.

Remarks. Before we go on, we shall comment on the notions introduced above.

1. The reason why we call the first condition 'globularity' is that it will imply the usual globularity condition in the ω-categories generated by ordered face structures. The word 'local' in 'Local discreetness' as anywhere else in the paper refers to the fact that this property concerns sets of faces constituting the domain of a face rather than the set of all faces.

   The property of 'pencil linearity' is strongly connected with the property of positive face structures with the same name, c.f. [Z]. There it means that the set of faces with a fixed codomain x, γ-*pencil*, (as well as the set of faces whose domains contain a fixed face x, δ-*pencil*,) are linearly ordered by $<^{+}$. For ordered face structures the same is true about the faces that are not loops. The full condition also has some implications for loops in pencils.

2. The relation $\equiv_1$, needed to express the δ-globularity, is a way to say that two sets of faces, that may contain empty faces, are essentially equal, even if they differ by some empty faces. We identify via $\equiv_1$ two such sets if those empty faces are morally there anyway. $A, B \subseteq S_k \cup \mathbf{1}_{S_{k-1}}$. Then the following conditions are equivalent

   (a) $A \equiv_1 B$;
   (b) the subhypergraphs of S generated by A and B are equal;
   (c) $\dot{A} = \dot{B}$, and $\ddot{A} \cup \mathbf{1}_{\theta(\dot{A})} = \ddot{B} \cup \mathbf{1}_{\theta(\dot{B})}$.

3. We shall analyze in details γ-globularity and δ-globularity but some easier observations first:

   (a) $\delta\delta^{-\varepsilon}(a) = \dot{\delta}\delta(a)$, $\delta\delta^{\varepsilon}(a) = \ddot{\delta}\delta(a)$.
   (b) If $x \in T^{\varepsilon}$ then $\dot{\delta}^{-\lambda}(x) = \emptyset$ and $\gamma\gamma(x) = \gamma\delta(x) = \gamma(1_u) = \delta(1_u) = \delta\delta(x) = \gamma\delta(x)$. In particular, $\gamma(x)$ is a loop and $\delta(x) = 1_{\gamma\gamma(x)}$, (i.e. $u = \gamma\gamma(x)$).
   (c) If $\gamma(a) \in T^{\varepsilon}$ then $\delta\gamma(a) = 1_{\gamma\gamma\gamma(a)}$.

   For δ-globularity we distinguish two cases $\gamma(a) \in T^{\varepsilon}$ and $\gamma(a) \in T^{-\varepsilon}$, and each has two parts, for faces, and for empty-faces (the condition for empty faces is translated to the condition about faces one dimension lower).

   Case $\gamma(a) \in T^{-\varepsilon}$:

   faces: $\delta\gamma(a) = \dot{\delta}\delta(a) - \gamma\dot{\delta}^{-\lambda}(a)$;

e-faces: $\gamma\gamma\delta^{\varepsilon}(a) \subseteq \theta\delta\gamma(a)$;
this is because we must have $\delta\delta^{\varepsilon}(a) \subseteq \mathbf{1}_{\theta\delta\gamma(a)}$.

Case $\gamma(a) \in T^{\varepsilon}$:

faces: $\dot{\delta}\delta(a) \subseteq \gamma\dot{\delta}^{-\lambda}(a)$;

e-faces: $\gamma\gamma\gamma(a) = \gamma\gamma\delta^{\varepsilon}(a)$;
this is because we must have $1_{\gamma\gamma\gamma(a)} = \delta\gamma(a) = \delta\delta^{\varepsilon}(a) = 1_{\gamma\gamma\delta^{\varepsilon}(a)}$.

The γ-globularity is much easier. We notice that if $a \in T^{\varepsilon}$ then the condition is still slightly simpler, empty faces play no role. We consider again two cases:

Case $a \in T^{\varepsilon}$**:** $\gamma\gamma(a) = \gamma\delta(a)$.

Case $a \in T^{-\varepsilon}$**:** $\gamma\gamma(a) = \gamma\delta(a) - \delta\dot{\delta}^{-\lambda}(a)$;
i.e. all elements of $\gamma\delta(a)$ but $\gamma\gamma(a)$ are in $\delta\dot{\delta}^{-\lambda}(a)$. So we have $x_0 \in \dot{\delta}(a)$ $\sim$-maximal in $\dot{\delta}(a)$ such that $\gamma\gamma(a) = \gamma(x_0)$. x_0 might be a loop in which case, if $\gamma(a)$ is not a loop, there is another (unique) element $x_1 \in \dot{\delta}^{-\lambda}(a)$, such that $\gamma(x_1) = \gamma\gamma(a)$.

4. If S has dimension n, as a hypergraph, then we say that S is *ordered n-face structure.*

5. A k-truncation of an ordered n-face structure S is not in general an an ordered k-face structure. However k-truncation of a local n-face structure is a local k-face structure. This will be important later, in the description of the many-to-one computads.

6. The *size of an ordered face structure* S is the sequence natural numbers $size(S) = \{|S_n - \delta(S^{-\lambda}_{n+1})|\}_{n\in\omega}$, with almost all being equal 0. We have an order $<$ on such sequences, so that $\{x_n\}_{n\in\omega} < \{y_n\}_{n\in\omega}$ iff there is $k \in \omega$ such that $x_k < y_k$ and for all $l > k$, $x_l = y_l$. This order is well founded and many facts about ordered face structures will be proven by induction on the size.

7. Let S be an ordered face structure. S is k-normal iff $dim(S) \leq k$ and $size(S)_l = 1$, for $l < k$. S is k-principal iff $size(S)_l = 1$, for $l \leq k$. S is principal iff $size(S)_l \leq 1$, for $l \in \omega$. S is principal of dimension k iff S is principal and $dim(S) = k$. By **pFs** (**nFs**) we denote full subcategories of **oFs** whose objects are principal (normal) ordered face structures, respectively.

4 Combinatorial properties of ordered face structures

Local properties

Lemma 4.1 *Let S be an ordered face structure, $x, a \in S$. Then*

1. *if $\delta(a) = 1_x$ then $x = \gamma\gamma(a)$;*
2. *if $a \in S^{\varepsilon}$ then $\gamma(a) \in S^{\lambda}$;*
3. *if $\gamma(a) \in S^{\varepsilon}$ then $\delta^{\varepsilon}(a) \neq \emptyset$;*
4. *if $a \in S^{-\lambda}$ then $\gamma(a) \not\in \delta(a)$;*
5. *$\ddot{\theta}\theta(a) = \delta\delta^{\varepsilon}(a)$;*
6. *if $x <^{+} y$ then $y \not\in \mathcal{I}$;*
7. *if $x <^{\sim} y$ then $y \not\in S^{-\varepsilon}$.*

Proof. Ad 1. Assume that $\delta(a) = 1_x$ for some $x \in S$. Then $\dot{\delta}^{-\lambda}(a) = \emptyset$ and using γ-globularity we get

$$\gamma\gamma(a) = \gamma\delta(a) - \delta\dot{\delta}^{-\lambda}(a) = \gamma\delta(a) = \gamma(1_x) = x$$

Ad 2. Suppose $a \in S^{\varepsilon}$. By 1. we have $\delta(a) = 1_{\gamma\gamma(a)}$. Then $\dot{\delta}^{-\lambda}(a) = \emptyset$. So using δ-globularity we have

$$\delta\gamma(a) = \delta\delta(a) - \gamma\dot{\delta}^{-\lambda}(a) = \delta(1_{\gamma\gamma(a)}) = \gamma\gamma(a)$$

i.e. $\gamma(a)$ is a loop.

Ad 3. Assume $\gamma(a) \in S^{\varepsilon}$. The using 2. and globularity we obtain

$$1_{\gamma\gamma\gamma(a)} = \delta\gamma(a) = \delta\delta(a) - \gamma\dot{\delta}^{-\lambda}(a).$$

Thus there is $x \in \delta(a)$ such that $\delta(x) = 1_{\gamma\gamma\gamma(a)}$, i.e. $x \in \delta^{\varepsilon}(a)$.

Ad 4. If we were to have $\gamma(a) \in \delta(a)$ then we would have $\gamma(a) <^{+} \gamma(a)$ contradicting strictness of $<^{+}$.

Ad 5. As we have $\gamma\gamma(a), \gamma\delta(a) \subseteq S$ and $\delta\gamma(a) \subseteq \delta\delta(a)$ we have

$$\ddot{\theta}\theta(a) = (\gamma\gamma(a) \cap \delta\gamma(a) \cap \gamma\delta(a) \cap \delta\delta(a)) \cap 1_S = \ddot{\delta}\delta(a) = \delta\delta^{\varepsilon}(a).$$

6. and 7. are obvious. $\Box$

Lemma 4.2 *Let S be an ordered face structure, $t, a, b, \alpha \in S$.*

1. *If $a \neq b$, $a, b \in \dot{S}^{-\lambda}$, and either $\gamma(a) = \gamma(b)$ or $\dot{\delta}(a) \cap \dot{\delta}(b) \neq \emptyset$ then $a \perp^{+} b$.*
2. *If $a, b \in \dot{\delta}^{-\lambda}(\alpha)$, and either $\gamma(a) = \gamma(b)$ or $\dot{\delta}(a) \cap \dot{\delta}(b) \neq \emptyset$ then $a = b$.*
3. *Let $t \in \dot{\delta}\delta(a)$. Then there is a unique flat upper $\dot{\delta}^{-\lambda}(a)$-path from t to $\gamma\gamma(\alpha)$.*
4. *If $\alpha \in S^{-\varepsilon}$ then there is the $\sim$-largest element $a \in \dot{\delta}(\alpha)$. For this a we have $\gamma(a) = \gamma\gamma(\alpha)$. All other elements of $\dot{\delta}(\alpha)$ have a well-defined $\sim$-successor.*
5. *If $\gamma(\alpha) \in S^{-\lambda}$ then there is the $\sim$-largest element $a \in \dot{\delta}^{-\lambda}(\alpha)$. For this a we have $\gamma(a) = \gamma\gamma(\alpha)$. All other elements of $\dot{\delta}^{-\lambda}(\alpha)$ have a well-defined $\sim$-successor in $\dot{\delta}^{-\lambda}(\alpha)$.*
6. *If $\gamma(\alpha) \in S^{-\lambda}$ and $x \in \dot{\delta}\gamma(\alpha)$ then there is $a \in \dot{\delta}^{-\lambda}(\alpha)$ such that $x \in \delta(a)$.*
7. *If $a <^{+} b$ then $\gamma(a) \leq^{+} \gamma(b)$.*

Proof. Ad 1. Let a, b be as in the Lemma. By pencil linearity, as $\dot{\theta}(a) \cap \dot{\theta}(b) \neq \emptyset$, it is enough to show that $a \perp^{\sim} b$ does not hold. Suppose contrary that $a <^{\sim} b$. Then, by disjointness, $a <^{-} b$. Thus we have a flat lower path $a = a_0, \ldots a_k = b$, with $k > 0$. Now if $\gamma(a) = \gamma(b)$ then we have a flat upper path $\gamma(a), a_1, \ldots, a_k, \gamma(a)$ showing that $\gamma(a) <^{+} \gamma(a)$. This contradicts strictness. on the other hand, if some $x \in \delta(a) \cap \delta(b)$ then we have a flat upper path $x, a_0, \ldots, a_{k-1}, \gamma(a_{k-1})$. Hence $x <^{+} \gamma(a_{k-1})$. As $x, \gamma(a_{k-1}) \in \delta(b)$ we get a contradiction with local discreetness.

Ad 2. This is immediate consequence of 1. and local discreetness.

Ad 3. Fix $t \in \dot{\delta}\delta(a)$. Let $t, x_1, \ldots, x_k, \gamma(x_k)$ be the longest flat upper $\dot{\delta}\delta(a)$-path starting from t (it might be empty). Such a path exists by strictness. If $\gamma(x_k) = \gamma\gamma(a)$ we are done. So assume that $\gamma(x_k) \neq \gamma\gamma(a)$. We have $\gamma(x_k) \in \gamma\delta(a)$. So by globularity $\gamma(x_k) \in \delta\dot{\delta}^{-\lambda}$ and hence there is $x_{k+1} \in \dot{\delta}^{-\lambda}(a)$ such that $\gamma(x_k) \in \delta(x_{k+1})$, i.e. $t, x_1, \ldots, x_k, x_{k+1}, \gamma(x_{k+1})$ is a longer flat path starting from t and we get a contradiction. Thus $\gamma(x_k) = \gamma\gamma(a)$ and we have $\dot{\delta}\delta(a)$-path from t to $\gamma\gamma(a)$. The uniqueness of this path follows from 2.

Ad 4. As $\alpha \in S^{-\varepsilon}$, by globularity, there is $a \in \delta(\alpha)$ such that $\gamma(a) = \gamma\gamma(\alpha)$. If there is a loop $a \in \delta(a)$ such that $\gamma(a) = \gamma\gamma(a)$ then by local discreetness there is $\sim$-largest such loop. Let a_0 be the $\sim$-largest loop in $\delta(\alpha)$ such that $\gamma(a_0) = \gamma\gamma(\alpha)$ if such a loop exists or else the unique $a_0 \in \delta^{-\lambda}(\alpha)$ such that $\gamma(a_0) = \gamma\gamma(\alpha)$. We shall show that a_0 is the $\sim$-largest element in $\delta(\alpha)$.

We consider two cases. If $\gamma(b) = \gamma\gamma(\alpha)$ by pencil linearity, 2. and definition of a_0 we have $b \leq^{\sim} a_0$. If $\gamma(b) \neq \gamma\gamma(\alpha)$ then by 1. there is a flat upper $\delta(\alpha)$-path $\gamma(b), b_1, \ldots, b_k, \gamma\gamma(\alpha)$ with $k > 0$. By the previous argument $b_k \leq^{\sim} a_0$ and hence $b <^{\sim} a_0$. Thus in either case a_0 is the $\sim$-largest element in $\delta(\alpha)$. For $a \in \delta(\alpha) - \{a_0\}$ we define the successor in $\delta(\alpha)$ as follows:

$$suc_\alpha(a) = \begin{cases} \inf_{\sim}(A) & \text{if } A = \{a' \in \delta^{\lambda}(\alpha) : \gamma(a) = \gamma(a') \text{ and } a' <^{\sim} a\} \neq \emptyset, \\ a'' & \text{such that } a'' \in \delta^{-\lambda}(\alpha),\ \gamma(a) \in \delta(a''), \text{ otherwise.} \end{cases}$$

The verification that it is a well defined successor is left for the reader.

Ad 5. Assume that $\gamma(\alpha) \in S^{-\lambda}$. First assume we have $a_0 \in \dot{\delta}^{-\lambda}(\alpha)$ such that $\gamma(a_0) = \gamma\gamma(\alpha)$. Then by 2. such a_0 is unique and by an argument similar to the one above a_0 is $\sim$-largest in $\dot{\delta}^{-\lambda}(\alpha)$ and all other elements in $\dot{\delta}^{-\lambda}(\alpha)$ have a successor there. Thus it remains to find a_0.

Note that to find a_0 it is enough to find $x \in \dot{\delta}\delta(\alpha)$ such that $x \neq \gamma\gamma(\alpha)$. Having such x, by 3., we have a flat upper $\dot{\delta}^{-\lambda}(\alpha)$-path $x, b_1, \ldots, b_k, \gamma\gamma(\alpha)$ with $k > 0$. We put $a_0 = b_k$.

To find x we consider two cases. If $\gamma \in S^{-\varepsilon}$ then $\gamma\gamma(\alpha) \not\in \delta\gamma(\alpha) \subseteq \dot{\delta}\delta(\alpha)$ and $\delta\gamma(\alpha) \neq \emptyset$. Then any element of $\delta\gamma(\alpha)$ can be taken as x.

If $\gamma \in S^{\varepsilon}$ then $1_{\gamma\gamma\gamma(\alpha)} = \delta\gamma(\alpha) \subseteq \delta\delta(\alpha)$. So there is $a \in \delta(\alpha)$ such that $\delta(\alpha) = 1_{\gamma\gamma\gamma(\alpha)}$. If $\gamma(a) = \gamma\gamma(\alpha)$ then we found $a_0 = a$ directly. Otherwise $\gamma\gamma(\alpha) \neq \gamma(a) \in \gamma\delta(\alpha)$. So $\gamma(a) \in \dot{\delta}\delta(a)$ and we put $x = \gamma(a)$.

Ad 6. Suppose $\gamma(\alpha) \in S^{-\lambda}$ and $x \in \dot{\delta}\gamma(\alpha)$. Then $\gamma(\alpha) \in S^{-\varepsilon}$ and we have $\gamma\gamma(\alpha) \not\in \delta\gamma(\alpha) \subseteq \dot{\delta}\delta(\alpha)$. By 3. there is a flat upper $\dot{\delta}^{-\lambda}(\alpha)$-path $x, a_1, \ldots, a_k, \gamma\gamma(\alpha)$. Then $x \in \delta(a_1)$ and $a_1 \in \dot{\delta}^{-\lambda}(\alpha)$, as required.

Ad 7. The essential case $a \lhd^+ b$ follows from 3. Then use induction. $\Box$

Notation. Having 4.2.4 we can introduce farther notation. If $\alpha \in S^{-\varepsilon}$ then the $\sim$-largest element in $\dot{\delta}(\alpha)$ will be denoted $\varrho(\alpha)$.

If $\gamma(\alpha) \in S^{-\lambda}$ then the $\sim$-largest element in $\dot{\delta}^{-\lambda}(\alpha)$ will be denoted $\varrho^{-\lambda}(\alpha)$.

Clearly, whenever the formulas make sense, we have

$$\gamma\gamma(\alpha) = \gamma(\varrho(\alpha)) = \gamma(\varrho^{-\lambda}(\alpha)).$$

Lemma 4.3 *Let S be an ordered face structure, $a, b, \alpha \in S$.*

1. *$\gamma(a) \in S^{\lambda}$ iff $\delta(a) \subseteq S^{\lambda}$ or $a \in S^{\varepsilon}$.*
2. *If $b \in S^{\lambda}$ and $a <^+ b$ then $a \in S^{\lambda}$.*
3. *If $a \in S^{\lambda}$ then there is $\alpha \in \mathcal{I}_{\gamma(a)}$ such that $\gamma(\alpha) \leq^+ a$ and $\delta(\alpha) = 1_{\gamma(a)}$.*
4. *If $a \in S^{\varepsilon}$ then there is $b \in \mathcal{I}$ such that $b \leq^+ a$. In that case $\delta(b) = \delta(a)$.*
5. *If $x \in S^{\lambda}$ then there is $b \in \mathcal{I}$ such that $\gamma(b) \leq^+ x$. In that case $\delta(b) = 1_{\gamma(x)}$.*

Proof. Ad 1. $\Leftarrow$. By Lemma 4.1.2 if $a \in S^{\varepsilon}$ then $\gamma(a)$ is a loop. So assume that $\delta(a) \subseteq S^{\lambda}$. Then $\dot{\delta}^{-\lambda}(a) = \emptyset(= \delta\dot{\delta}^{-\lambda}(a) = \gamma\dot{\delta}^{-\lambda}(a)$. As for $x \in \delta(a)$ we have $\gamma(x) = \delta(x)$, we also have $\gamma\delta(a) = \delta\delta(a)$. Thus by globularity, we have

$$\gamma\gamma(a) = \gamma\delta(a) - \delta\dot{\delta}^{-\lambda}(a) = \gamma\delta(a) = \delta\delta(a) = \delta\delta(a) - \gamma\dot{\delta}^{-\lambda}(a) = \delta\gamma(a)$$

i.e. $\gamma(a)$ is a loop, as required.

$\Rightarrow$. Suppose now that $\gamma(a) \in S^\lambda$. If $\delta(a) = 1_{\gamma\gamma(a)}$ then $a \in S^\varepsilon$. So assume that $\delta(a) \subseteq S$. By globularity, we have

$$\gamma\delta(a) - \delta\dot{\delta}^{-\lambda}(a) = \gamma\gamma(a) = \delta\gamma(a) = \delta\delta(a) - \gamma\dot{\delta}^{-\lambda}(a). \tag{6}$$

If $\dot{\delta}^{-\lambda}(a) = \emptyset$ then $\delta\dot{\delta}^{-\lambda}(a) = \gamma\dot{\delta}^{-\lambda}(a) = \emptyset$ and $\gamma\delta(a) = \delta\delta(a) = \gamma\gamma(a)$ i.e. $\delta(a) \subseteq S^\lambda$, as required.

So suppose now that $\dot{\delta}^{-\lambda}(a) \neq \emptyset$. Let $x \in \dot{\delta}^{-\lambda}(a)$ be the $\sim$-largest element in $\dot{\delta}^{-\lambda}(a)$. Such an x exists by Lemma 4.2. Then $\gamma(x) \in \gamma\dot{\delta}^{-\lambda}(a) \subseteq \gamma\delta(a)$. By (6) we have $\gamma(x) \in \delta\dot{\delta}^{-\lambda}(a)$, and hence we have $x' \in \dot{\delta}^{-\lambda}(a)$ such that $\gamma(x) \in \delta(x')$. As $x, x' \in \delta(a)$ we have $x \not<^+ x'$. Moreover if we were to have $\gamma(x') \in \delta(x)$ we would have $\gamma(x) <^+ \gamma(x)$ contradicting strictness. Thus, by pencil linearity, we have $x <^\sim x'$, i.e. x is not $\sim$-largest element in $\dot{\delta}^{-\lambda}(a)$ contrary to the supposition. Hence $\dot{\delta}^{-\lambda}(a) = \emptyset$ indeed, and $\delta(a) \subseteq S^\lambda$, as required.

Ad 2. Use 1. and then induction.

Ad 3. Use loop-filling, pencil linearity, strictness, and 1. to get a maximal upper $S - \gamma(S^{-\lambda})$-path $\alpha_1, \ldots, \alpha_k, a$ ending at a with $k > 0$. Then $\alpha_1 \in \mathcal{I}_{\gamma(a)}$ and $\gamma(\alpha_1) \leq^+ a$.

Ad 4. Suppose $\gamma(\alpha) \in S^\varepsilon$. By globularity we have $1_{\gamma\gamma\gamma(\alpha)} = \delta\gamma(\alpha) \subseteq \delta\delta(\alpha)$. Then there is $a \in \delta^\varepsilon(\alpha) \neq \emptyset$. The thesis follows form the above observation= , strictness, and inductions.

Ad 5. Fix $x \in S^\lambda$. By loop-filling and strictness there is $a \in S^\varepsilon$ such that $\gamma(a) \leq^+ x$. The rest follows from 4. $\Box$

Lemma 4.4 *Let S be an ordered face structure $\alpha, a, b \in S$.*

1. $\iota\delta(\alpha) = \iota\gamma(\alpha)$.
2. *If $a <^+ b$ then $\iota(a) \subseteq \iota(b)$.*

Proof. Ad 1. First we prove $\iota\gamma(\alpha) \subseteq \iota\delta(\alpha)$. Fix $u \in \iota\gamma(\alpha)$, i.e. we have $x, y \in \dot{\delta}^{-\lambda}\gamma(\alpha)$ such that $\gamma(x) = u \in \delta(y)$. Let $x, a_1, \ldots, a_k, x'$ be a maximal flat upper $\delta^{-\lambda}\gamma(\alpha)$-path starting from x such that $\gamma\gamma(a_i) = u$ for $i = 1, \ldots, k$. It might be empty in which case $x = x'$. As $x \in S^{-\lambda}$ by Lemma 4.3.2, $x' \in S^{-\lambda}$. Since $\gamma(x') = u \in \iota\gamma(\alpha)$ it follows that $\gamma(x') \neq \gamma\gamma\gamma(\alpha)$. Thus $x' \neq \gamma\gamma(\alpha)$. By Lemma 4.2.3 there is $a \in \dot{\delta}^{-\lambda}(\alpha)$ such that $x' \in \delta(a)$. By maximality of $x, a_1, \ldots, a_k, x'$ we have $\gamma\gamma(a) \neq \gamma(x')$. Again by Lemma 4.2.3 there is $y' \in \dot{\delta}^{-\lambda}(a)$ such that $\gamma(x') = u \in \delta(y')$. But then $u \in \iota(a) \subseteq \iota\delta(\alpha)$.

Now we shall show $\iota\delta(\alpha) \subseteq \iota\gamma(\alpha)$. Let $a \in \delta(\alpha)$ and $u \in \iota(a)$, i.e. there are $x', y' \in \delta^{-\lambda}(a)$ such that $\gamma(x') = u \in \delta(y')$. We shall construct $x, y \in \delta^{-\lambda}\gamma(\alpha)$ such that $\gamma(x) = u \in \delta(y)$, i.e. $u \in \iota(\alpha)$.

Construction of x. Let $a_l, \ldots, a_1, x'$ be the maximal flat $\delta^{-\lambda\varepsilon}(\alpha)$-path (possibly empty) ending at x' such that $\gamma\gamma(a_i) = \gamma(x')$ and $\gamma(a_i) \in S^{-\lambda}$, for $i = 1, \ldots, l$. Thus there is $x \in \delta^{-\lambda}(a_l)$ such that $\gamma(x) = \gamma\gamma(a_l)$. If $x \in \delta\gamma(\alpha)$ we have x with the required property (if the sequence is empty $x = x'$). So suppose contrary that $x \notin \delta\gamma(\alpha)$. Since $x \in \delta\delta(\alpha)$, by globularity, it follows that $x \in \gamma\dot{\delta}^{-\lambda}(\alpha)$. So there is $a_{l+1} \in \dot{\delta}^{-\lambda}(\alpha)$ such that $\gamma(a_{l+1}) = x$. Since $x \in S^{-\lambda}$, we have $a_{l+1} \in S^{-\varepsilon}$. But then the path $a_{l+1}, a_l, \ldots, a_1, x'$ is longer then the maximal one and we get a contradiction.

Construction of y. Let $b_k, \ldots, b_1, x'$ be the maximal flat $\delta^{-\lambda\varepsilon}(\alpha)$-path (possibly empty) ending at y' such that $u \in \delta\dot{\delta}^{-\lambda}(b_i) = \gamma(x')$ and $\gamma(b_i) \in S^{-\lambda}$, for $i = 1, \ldots, k$. By Lemma 4.2.6 there is $y \in \delta^{-\lambda}(b_k)$ such that $u \in \delta(y)$ (if the sequence is empty $y = y'$). If $y \in \delta\gamma(\alpha)$ we have y as required. So suppose that $y \notin \delta\gamma(\alpha)$. As

$y \in \delta\delta(\alpha)$, by globularity, we have that $y \in \gamma\dot{\delta}^{-\lambda}(\alpha)$. So there is $b_k + 1 \in \dot{\delta}^{-\lambda}(\alpha)$ such that $\gamma(b_{k+1}) = y$. Since $y \in S^{-\lambda}$, we have $b_{k+1} \in S^{-\varepsilon}$. But then the path $b_{k+1}, b_b, \ldots, b_1, y'$ is longer then the maximal one and we get a contradiction again.

Ad 2. Use 1. and induction. $\Box$

Lemma 4.5 *Let S be an ordered face structure $\alpha, a, b \in S$.*

1. *We have inclusions*

$$\begin{tikzcd} \dot{\delta}\gamma\gamma(\alpha) \arrow[r] \arrow[d] & \dot{\delta}\gamma\delta(\alpha) \arrow[d] \\ \dot{\delta}\delta\gamma(\alpha) \arrow[r] & \dot{\delta}\delta\delta(\alpha) \end{tikzcd}$$

2. $\dot{\theta}\theta(a) = \gamma\gamma(a) \cup \dot{\delta}\dot{\delta}^{-\lambda}(a)$, $\gamma\gamma(a) \cap \dot{\delta}\dot{\delta}^{-\lambda}(a) = \emptyset$.
3. $\dot{\theta}\theta(a) = \dot{\delta}\gamma(a) \cup \gamma\dot{\delta}^{-\lambda}(a)$, $\dot{\delta}\gamma(a) \cap \gamma\dot{\delta}^{-\lambda}(a) = \emptyset$.
4. $\dot{\theta}\theta(a) = \gamma\gamma(a) \cup \iota(a) \cup \delta\gamma^{-\lambda\varepsilon}(a)$ *(disjoint sum).*
5. $\dot{\theta}\theta(a) = \dot{\theta}\dot{\theta}(a) = \dot{\theta}\delta(a)$.
6. $\dot{\theta}\theta\theta(\alpha) = \dot{\theta}\theta\gamma(\alpha)$.
7. *If $a <^+ b$ then $\dot{\theta}\theta(a) \subseteq \dot{\theta}\theta(b)$.*
8. $\dot{\theta}\theta^{(k+1)}(a) = \dot{\delta}\dot{\delta}^{-\lambda}\gamma^{(k+2)}(a) \cup \gamma^{(k)}(a)$.
9. $\gamma\ddot{\theta}\theta^{(k+2)}(a) \subseteq \dot{\theta}^{(k)}(a)$. *(don't bother with 1_x's)*
10. $\gamma\dot{\delta}^{-\lambda}\delta(\alpha) = \gamma\dot{\delta}^{-\lambda}\gamma(\alpha)$.
11. $\gamma\dot{\delta}^{-\lambda}\theta^{(k+2)}(\alpha) = \gamma\dot{\delta}^{-\lambda}\gamma^{(k+2)}(\alpha)$.

Proof. Ad 1. This is an easy consequence of $\delta\gamma(a) \subseteq \delta\delta(a)$ and $\gamma\gamma(a) \subseteq \gamma\delta(a)$.

Ad 2. Let $A = \gamma\gamma(a) \cup \dot{\delta}\dot{\delta}^{-\lambda}(a)$. Clearly $A \subseteq \dot{\theta}\theta(a)$. We shall show the converse inclusion. From globularity we have $\gamma\gamma(a) \in \gamma\delta(a) \subseteq A$ and $\dot{\delta}\gamma \subseteq \dot{\delta}\delta(a) = \dot{\delta}\dot{\delta}^{-\lambda}(a) \cup \dot{\delta}\delta^{\lambda}(a) \cup \dot{\delta}\ddot{\delta}(a)$. Moreover $\dot{\delta}\delta^{\lambda}(a) = \gamma\delta^{\lambda}(a) \subseteq \gamma\delta(a)$. Finally, if $\ddot{\delta}(a) \neq \emptyset$ then $\ddot{\delta}(a) = 1_{\gamma\gamma(a)}$. So $\dot{\delta}\ddot{\delta}(a) = \delta(1_{\gamma\gamma(a)}) = \gamma\gamma(a) \in A$. Thus the other inclusion holds as well. The second part follows directly from γ-globularity.

Ad 3. Let $B = \dot{\delta}\gamma(a) \cup \gamma\dot{\delta}^{-\lambda}(a)$. Clearly $B \subseteq \dot{\theta}\theta(a)$. We shall show the converse inclusion. From globularity we have $\dot{\delta}\delta(a) \subseteq B$ and $\gamma\gamma \in \gamma\delta(a) = \gamma\dot{\delta}^{-\lambda}(a) \cup \gamma\delta^{\lambda}(a) \cup \gamma\ddot{\delta}(a)$ Moreover $\gamma\delta^{\lambda}(a) = \delta\delta^{\lambda}(a) \subseteq \delta\delta(a)$. Finally, if $\ddot{\delta}(a) \neq \emptyset$ then $\ddot{\delta}(a) = 1_{\gamma\gamma(a)}$. So $\gamma\ddot{\delta}(a) = \gamma(1_{\gamma\gamma(a)}) = \gamma\gamma(a) \in A$. Thus the other inclusion holds as well. The second part follows directly from δ-globularity.

Ad 4. Using 2. and 3. we have

$$\iota(a) \cap (\gamma\gamma(a) \cup \delta\gamma(a)) = (\gamma\dot{\delta}^{-\lambda}(a) \cap \delta\dot{\delta}^{-\lambda}(a)) \cap (\gamma\gamma(a) \cup \delta\gamma(a)) \subseteq$$

$$\subseteq (\gamma\dot{\delta}^{-\lambda}(a) \cap \delta\gamma(a)) \cup (\gamma\gamma(a) \cap \delta\dot{\delta}^{-\lambda}(a)) = \emptyset \cup \emptyset = \emptyset$$

$$\iota(a) \cup (\gamma\gamma(a) \cup \delta\gamma(a)) = (\gamma\dot{\delta}^{-\lambda}(a) \cap \delta\dot{\delta}^{-\lambda}(a)) \cup (\gamma\gamma(a) \cup \delta\gamma(a)) \supseteq$$

$$\supseteq (\gamma\dot{\delta}^{-\lambda}(a) \cup \delta\gamma(a)) \cap (\gamma\gamma(a) \cup \delta\dot{\delta}^{-\lambda}(a)) = \dot{\theta}\theta(a) \cap \dot{\theta}\theta(a) = \dot{\theta}\theta(a)$$

Note that $\dot{\delta}\gamma(a) = \delta\gamma^{-\varepsilon}(a)$ and if $\gamma(a) \in S^{\lambda}$ then $\dot{\delta}\gamma(a) = \gamma\gamma(a)$. From these observations the rest follows.

5. Easy application of 2. and 3.

6. We need to show that $\dot{\theta}\theta\delta(\alpha) \subseteq \dot{\theta}\theta\gamma(\alpha)$. By 4. it is enough to show the following three inclusions: $\iota\delta(\alpha) \subseteq \dot{\theta}\theta\gamma(\alpha)$, $\gamma\gamma\delta(\alpha) \subseteq \dot{\theta}\theta\gamma(\alpha)$, $\dot{\delta}\gamma\delta(\alpha) \subseteq \dot{\theta}\theta\gamma(\alpha)$. The first follows immediately from 4. If $\gamma(\alpha) \in S^\lambda$ then, as in this case $\gamma\delta(\alpha) = \gamma\gamma(\alpha$ the second and third inclusions hold as well. Thus we shall show the second and third inclusion in case $\gamma(\alpha) \in S^{-\lambda}$.

Assume $t \in \gamma\gamma\delta(\alpha)$. Pick $\sim$-minimal $a \in \delta(\alpha)$ such that $t = \gamma\gamma(a)$. Then either $a \in S^\varepsilon$ or $a \in S^{-\varepsilon}$. In the former case $\delta(a) = 1_{\gamma\gamma(a)} \in \delta\delta(\alpha)$, and by δ-globularity (see definition of $\equiv_1$) we have that $t = \gamma\gamma(a) \in \dot{\theta}\delta\gamma(\alpha)$. In the later case by Lemma 4.2.4 (see also notation after the proof) $x = \varrho^{-\lambda}(a) \in \delta^{-\lambda}(a)$ is well defined and we have $\gamma(x) = \gamma\gamma(a)$. By $\sim$-minimality of a, we get $x \notin \gamma\dot{\delta}^{-\lambda}(\alpha)$. As $x \in \delta\delta(\alpha)$ again by δ-globularity we have that $x \in \delta\gamma(\alpha)$. Thus $t = \gamma(x) \in \gamma\delta\gamma(\alpha)$. Thus in either case $t \in \dot{\theta}\theta\gamma(\alpha)$. This end the proof of the second inclusion.

Now assume $t \in \delta\gamma\delta(\alpha)$. Pick $\sim$-minimal $a \in \delta(\alpha)$ such that $t \in \delta\gamma(a)$. Then either $a \in S^\varepsilon$ or $a \in S^{-\varepsilon}$. In the former case $\gamma(a) \in S^\lambda$. Then using the second inclusion we get

$$t \in \delta\gamma(a) = \gamma\gamma(a) \in \gamma\gamma\delta(\alpha) \subseteq \dot{\theta}\theta\gamma(\alpha)$$

In the later case by Lemma 4.2.6 there is $x \in \delta(a)$ (i.e. $x \in \delta\delta(\alpha)$) such that $t \in \delta(x)$. By $\sim$-minimality of a, $x \notin \gamma\dot{\delta}^{-\lambda}$. So by δ-globularity we have $x \in \delta\gamma(\alpha)$. Thus $t \in \dot{\delta}(x) \subseteq \dot{\delta}\delta\gamma(\alpha)$. Thus in either case $t \in \dot{\theta}\theta\gamma(\alpha)$. This end the proof of the third inclusion and the whole statement 6.

For 7. and 8. Use 1., 5., 6., and induction.

9. Exercise.

Ad 10. $\supseteq$. As $\delta\gamma(\alpha) \subseteq \delta\delta(\alpha)$ we have $\dot{\delta}^{-\lambda}\gamma(\alpha) \subseteq \dot{\delta}^{-\lambda}\delta(\alpha)$. So $\gamma\dot{\delta}^{-\lambda}\gamma(\alpha) \subseteq \gamma\dot{\delta}^{-\lambda}\delta(\alpha)$.

$\subseteq$. Let $t \in \gamma\dot{\delta}^{-\lambda}\delta(\alpha)$. Pick $\sim$-minimal $a \in \delta(\alpha)$ such that there is $x \in \dot{\delta}^{-\lambda}(a)$ so that $t = \gamma(x)$. By $\sim$-minimality of a $x \notin \gamma\dot{\delta}^{-\lambda}(\alpha)$ and hence by δ-globularity $x \in \dot{\delta}^{-\lambda}\gamma(\alpha)$. Thus $t = \gamma(x) \in \gamma\dot{\delta}^{-\lambda}\gamma(\alpha)$, as required.

11. follows from 10. by induction. □

Lemma 4.6 ($\dot{\theta}\theta$ induction) *Let S be an ordered face structure $\alpha, a, b \in S$.*

1. *if $a \in \dot{\delta}^{-\lambda\varepsilon}(\alpha)$ then $\gamma\gamma(\alpha) \notin \delta(a)$;*
2. *if $a, b \in \dot{\delta}^{-\lambda\varepsilon}(\alpha)$ then $\delta(a) \cap \delta(b) = \emptyset$;*
3. *$\dot{\theta}\theta(\alpha) = \gamma\gamma(\alpha) \cup \delta\dot{\delta}^{-\lambda\varepsilon}(\alpha)$;*
4. *$\dot{\theta}\theta$-induction. Whenever*
   (a) *if $A \subset \dot{\theta}\theta(\alpha)$;*
   (b) *$\gamma\gamma(\alpha) \in A$;*
   (c) *for all $a \in \dot{\delta}^{-\lambda\varepsilon}(\alpha)$, if $\gamma(a) \in A$ then $\dot{\delta}(a) \subseteq A$*

   we have $A = \dot{\theta}\theta(\alpha)$.

Proof. 1. and 2. follows from pencil linearity. 3. follows from Lemma 4.5.2 and that $\delta\delta^{-\lambda\varepsilon} = \dot{\delta}\dot{\delta}^{-\lambda}$. The $\dot{\theta}\theta$-induction follows from 3. □

Lemma 4.7 *Let S be an ordered face structure $a, a' \in S$.*

1. *If $a, a' \in S^{-\lambda}$ and $\gamma(a) \in \delta(a')$ then $a <^\sim a'$.*
2. *If $a <^- a'$ and $\theta(a) \cap \theta(a') \neq \emptyset$ then $\gamma(a) \in \delta(a')$.*

Proof. Ad 1. Let $a, b \in S^{-\lambda}$ such that $\gamma(a) \in \gamma(b)$. By strictness we cannot have $b <^- a$. Thus by pencil linearity it is enough to to show that $a \not<^+ b$.

Suppose $a <^+ b$, i.e. there is a flat upper path $a, \alpha_1, \ldots, \alpha_r, b$. As $a \in S^{-\lambda}$, by Lemma 4.3, $\gamma(\alpha_i) \in S^{-\lambda}$, for $i = 1, \ldots, r$. Now either $\gamma(a) = \gamma\gamma(\alpha_r) = \gamma(b)$ or there is $1 \leq i \leq r$ such that $\gamma(a) \neq \gamma\gamma(\alpha_i)$. As $b \in S^{-\lambda}$ and $\gamma(a) \in \delta(b)$ the former is impossible. Fix minimal i_0 such that $\gamma(a) \neq \gamma\gamma(\alpha_{i_0})$. Then, by Lemma 4.5.2, $\gamma(a) \in \delta\dot{\delta}(\alpha_{i_0})$. As $\gamma(\alpha_{i_0-1}) \in S^{-\lambda}$ (or $a \in S^{-\lambda}$ if $i_0 = 1$) using Lemma 4.4 we get $\gamma(a) \in \iota(\alpha_{i_0}) \subseteq \iota(\alpha_r)$. On the other hand $\gamma(a) \in \delta(a) \subseteq \delta\gamma(\alpha_r)$ and $\delta\gamma(\alpha_r) \cap \iota(\alpha_r) = \emptyset$. But this contradicts Lemma 4.5.4. Thus $a <^+ b$ cannot hold.

Now suppose that $b <^+ a$, i.e. there is a flat upper path $b, \beta_1, \ldots, \beta_r, b$. As $a \in S^{-\lambda}$, by Lemma 4.3, $\gamma(\beta_i) \in S^{-\lambda}$, for $i = 1, \ldots, r$. Now either $\gamma(a) \in \delta\gamma(\beta_r)$ or there is $1 \leq i \leq r$ such that $\gamma(a) \in \delta\gamma(\beta_i)$. As $a \in S^{-\lambda}$ and $\gamma(\beta_r) = a$ the former is impossible. Fix minimal i_1 such that $\gamma(a) \not\in \delta\gamma(\beta_{i_1})$. By Lemma 4.5.3 we have $\gamma(a) \in \gamma\dot{\delta}(\beta_{i_1})$. As $b \in S^{-\lambda}$, if $i_1 > 1$ then $\gamma(\beta_{i_1-1}) \in S^{-\lambda}$, as well. Thus $\gamma(a) \in \iota(\beta_{i_1}) \subseteq \iota(\beta_r)$. But $\gamma(a) = \gamma\gamma(\beta_r)$. But this contradicts Lemma 4.5.4. again and hence $b <^+ a$ cannot hold either.

Therefore $a \not<^+ b$ and then $a <^\sim b$.

Ad 2. If $a <^- b$ then we have a lower path $a = a_0, a_1, \ldots, a_k, a_{k+1} = a'$, with $k \geq 0$, such that $a_1, \ldots, a_k$ is flat. If $k = 0$ then $\gamma(a) \in \delta(a')$ and we are done. We shall show, using $\theta(a) \cap \theta(a') \neq \emptyset$, that $k > 0$ is impossible.

If $\gamma(a) = \gamma(a')$ then $\gamma(a), a_1, \ldots, a_k, (a_{k+1}), \gamma(a')$ is a flat upper path, where the face (a_{k+1}) in parenthesis () is optional i.e. it is in the path iff it is not a loop. If $k > 0$ then $<^+$ is not strict.

If $x \in \delta(a) \cap \delta(a')$ then we have a flat upper $x, (a_0), a_1, \ldots, a_k, \gamma(a_k)$, with a_0 optional. If $k > 0$ then $x <^+ \gamma(a_k) \in \delta(a')$. If $x = \gamma(a_k)$ then $<^+$ is not strict and if $x \neq \gamma(a_k)$ then, as $x, \gamma(a_k) \in \delta(a')$, we get contradiction with local discreetness.

If $\gamma(a') = \delta(a)$ then $\gamma(a'), (a_0), a_1, \ldots, a_k, (a_{k+1}), \gamma(a')$ is a flat upper path, and again if $k > 0$, we get contradiction with strictness of $<^+$. $\Box$

Lemma 4.8 *Let S be an ordered face structure $\alpha, a, b, x \in S$.*

1. *If $\theta(a) \cap \iota(\alpha) \neq \emptyset$ then $a <^+ \gamma(\alpha)$.*

2. *If $\alpha \in S - \gamma(S^{-\lambda})$ and $x \in \theta(a) \cap \iota(\alpha)$ then there is $b \in \dot{\delta}(\alpha)$ such that $a \leq^+ b$ and $x \in \theta(b)$.*

Proof. 2. can be easily deduced from the proof of 1.

Ad 1. First we show that if $x \in \iota(\alpha)$ then there is $\alpha' \leq^+ \alpha$ such that $x \in \iota(\alpha')$ and $\alpha' \in S - \gamma(S^{-\lambda})$. Take as α' a +-minimal face such that $x \in \iota(\alpha')$ and $\alpha' \leq^+ \alpha$. If $\alpha' \in \gamma(S^{-\lambda})$ then there is $\xi \in S^{-\lambda}$ such that $\gamma(\xi) = \alpha'$. As $x \in \iota(\alpha') = \iota\gamma(\xi) = \iota\delta(\xi)$ there is $\alpha'' \in \delta(\xi)$ such that $x \in \iota(\alpha'')$. As $\alpha'' <^+ \alpha'$, α' was not minimal contrary to the supposition. Thus $\alpha' \in S - \gamma(S^{-\lambda})$.

Next we show that it is enough to show 1. in case $\alpha \in S - \gamma(S^{-\lambda})$. Suppose that $x \in \theta(a) \cap \iota(\alpha)$ for some $x \in S$. By the above there is $\alpha' \leq^+ \alpha$ such that $x \in \iota(\alpha')$ and $\alpha' \in S - \gamma(S^{-\lambda})$. So by Lemma 4.2.7 and the above we have $a <^+ \gamma(\alpha') \leq^+ \gamma(\alpha)$, as required.

So assume that $\alpha \in S - \gamma(S^{-\lambda})$ and $x \in \theta(a) \cap \iota(\alpha)$ for some $x \in S$. We consider three cases:

1. $a \in S^{-\lambda}$ and $\gamma(a) = x \in \iota(\alpha)$;

2. $a \in S^{-\lambda}$ and $x \in \delta(a) \cap \iota(\alpha)$;

3. $a \in S^{\lambda}$.

We fix $b, c \in \delta^{-\lambda}(\alpha)$ such that $\gamma(b) = x \in \delta(c)$ for the rest of the argument.

Case 1. By Lemma 4.2.1 we have $a \perp^+ b$ or $a = b$. If $a \leq^+ b$ then we are done. So we need to show that $b \not<^+ a$. Suppose contrary that $b <^+ a$ and $b, \beta_1, \ldots, \beta_r, a$ is a flat upper $(S - \gamma(S^{-\lambda}))$-path from b to a. As $b \in \delta(\beta_1) \cap \delta(\alpha)$ and $\beta_1, \alpha \in S - \gamma(S^{-\lambda})$ we have $\beta_1 = \alpha$. Thus $x = \gamma(a) \in \iota(\alpha) = \iota(\beta_1) \subseteq \iota(\beta_r)$ and $x = \gamma(a) = \gamma\gamma(\beta_r) \notin \iota(\beta_r)$ and we get a contradiction.

Case 2. Again by Lemma 4.2.1 we get that $a \perp^+ c$ or $a = c$. If $a \leq^+ c$ we are done. We shall show that $c \not<^+ a$. Suppose contrary that $c <^+ a$ and $c, \beta_1, \ldots, \beta_r, a$ is a flat upper $(S - \gamma(S^{-\lambda}))$-path from c to a. As $c \in \delta(\beta_1) \cap \delta(\alpha)$ and $\beta_1, \alpha \in S - \gamma(S^{-\lambda})$ we have $\beta_1 = \alpha$. Hence $x = \gamma(a) \in \iota(\alpha) = \iota(\beta_1) \subseteq \iota(\beta_r)$ and $x \in \delta(a) = \delta\gamma(\beta_r) \notin \iota(\beta_r)$ and we get a contradiction again.

Case 3. By loop-filling, pencil linearity, and strictness we have a flat $(S - \gamma(S^{-\lambda}))$-path $\alpha_0, \ldots, \alpha_k$ ending at a such that $\alpha_0 \in S^{\varepsilon}$. As $a \in S^{\lambda}$ we have $\gamma(\alpha_i) \in S^{\lambda}$,for $i = 0, \ldots, k$, and $\gamma\gamma(\alpha_0) = \gamma(a)$. Thus by pencil linearity we have either $\alpha_0 <^{\sim} \alpha$ or $\alpha_0 <^+ \alpha$. As $\alpha_0, \alpha \in S - \gamma(S^{-\lambda})$ the later is impossible. It remains to show that if $\alpha_0 <^{\sim} \alpha$ then $a <^+ \gamma(\alpha)$. Let $\alpha_0, \beta_1, \ldots, \beta_r = \alpha$ be a flat lower $S - \gamma(S^{-\lambda})$-path. Since $\alpha_i, \beta_j \in S - \gamma(S^{-\lambda})$ we have $\alpha_i = \beta_i$ for $i = 1, \ldots, \min(k, r)$. If $r \leq k$ then $\gamma(\alpha) \leq^+ a$. But $\gamma(\alpha) \notin S^{\lambda} \ni a$. This is a contradiction with Lemma 4.3.1. So $k < r$ and hence $a = \gamma(\alpha_k) \leq^+ \gamma(\beta_r) = \gamma(\alpha)$. Since $a \in S^{\lambda} \ni \gamma(\alpha)$, we have in fact that $a <^+ \gamma(a)$, as required. $\Box$

Lemma 4.9 *Let S be an ordered face structure $a, b, c, d \in S$.*

1. *If $a <^{\sim} b <^{\sim} c$ and $a, c <^+ d$ then $b <^+ d$.*

Proof. 1. is easy. $\Box$

Global properties

Let S be an ordered face structure, $n, i \in \omega$, $a, a_i \in S_n$, for $i = 1, \ldots, k$. The *weight* of a face a is the number

$$wt(a) = |\{b \in S^{-\lambda} \, : \, b <^+ a\}|$$

The *weight of a flat path* $\vec{a} = a_1, \ldots a_k$ is is the sum of weights of its faces $wt(\vec{a}) = \sum_{i=1}^{k} wt(a_i)$.

Lemma 4.10 *Let S be an ordered face structure $\alpha, a \in S$, and $a_1, \ldots, a_k$ flat lower $\dot{\delta}^{-\lambda}(\alpha)$-path with $k > 0$. Then*

$$wt(\gamma(\alpha)) > \sum_{i=1}^{k} wt(a_i).$$

Moreover, $wt(a) = 0$ iff $a \in S^{\lambda}$ or $a \notin \gamma(S^{-\lambda})$.

Proof. Let α, a_i be as in assumptions of Lemma. Then for $b \in S$, if $b <^+ a_i$ then $b <^+ \gamma(\alpha)$. As $a_i \not<^+ a_j$ for any $1 \leq i, j \leq k$, to prove the inequality we need to show that the faces on the right hand side are calculated at most once, i.e. for $b \in S^{-\lambda}$, if $b <^+ a_i$ then $b \not<^+ a_j$ for $j \neq i$. Suppose contrary, that $b <^+ a_i$ and $b <^+ a_j$, with $i < j$. Then, by Lemma 4.2.7, $\gamma(b) \leq^+ \gamma(a_i)$ and hence $b <^- a_j$. So by Lemma 4.7 $b <^{\sim} a_j$. But $b <^+ a_j$ and we get a contradiction with disjointness.

The last statement of Lemma is left for the reader. $\Box$

Lemma 4.11 *Let S be an ordered face structure, X convex subset of S, $x, y \in X$. If $x <^+ y$ then there is a unique flat upper $(X - \gamma(X^{-\lambda}))$-path from x to y.*

Proof. Let X, x, and y be as in assumptions of Lemma. As X is convex there is a flat upper X-path $x, a_1, \ldots, a_k, y$. Assume that it is a path with the smallest weight. Suppose that there is $i \leq k$ such that $a_i \in \gamma(X^{-\lambda})$. Hence there is $\alpha \in X^{-\lambda}$ such that $\gamma(\alpha) = a_i$. Let

$$x' = \begin{cases} x & \text{if } i = 1, \\ \gamma(a_{i-1}) & \text{otherwise.} \end{cases}$$

Then $x' \in \delta(a_i) \cap \delta\gamma(\alpha) \subseteq \delta\delta(a_i)$. By Lemma 4.2.3 there is a flat upper $\delta^{-\lambda}(\alpha)$-path $x', b_1, \ldots, b_r, \gamma(a_i)$. By Lemma 4.10, $wt(b_1, \ldots, b_r) < wt(a_i)$, and hence $wt(a_1, \ldots, a_{i-1}, b_1, \ldots, b_r, a_{i+1}, \ldots, a_k) < wt(a_1, \ldots, a_k)$, contrary to the supposition that the weight of the path $x, a_1, \ldots, a_k, y$ is minimal. Thus $x, a_1, \ldots, a_k, y$ is a $(X - \gamma(X^{-\lambda}))$-path, as required. The uniqueness of the path follows from Lemma 4.7 and pencil linearity. □

A lower flat path $a_0, \ldots, a_k$ is a *maximal path* if $\delta(a_1) \subseteq \delta(S) - \gamma(S^{-\lambda})$ and $\gamma(a_k) \in \gamma(S) - \delta(S^{-\lambda})$, i.e. if it can't be extended either way.

Lemma 4.12 (Path Lemma) *Let $k \geq 0$, $a_0, \ldots, a_k$ be a maximal lower flat path in an ordered face structure S, $b \in S$, $0 \leq s \leq k$, $a_s <^+ b$. Then there are $0 \leq l \leq s \leq p \leq k$ such that*

1. *$a_i <^+ b$ for $i = l, \ldots, p$;*
2. *$\gamma(a_p) = \gamma(b)$;*
3. *either $l > 0$ and $\gamma(a_{l-1}) \in \delta(b)$*
   or $l = 0$ and either $a_0 \in S^\varepsilon$ and $\gamma\gamma(a_0) \in \theta\delta(b)$ or $a_0 \in S^{-\varepsilon}$ and $\delta(a_0) \subseteq \delta(b)$;
4. *$a_i <^\sim b <^\sim a_j$, for $i = 1, \ldots, l-1$ and $j = p+1, \ldots, k$;*
5. *$\gamma(a_i) \in \iota(S)$, for $l \leq i < p$.*

Proof. We put

$$l = \min\{l' \leq s : \forall_{l' \leq i \leq s}\, a_i <^+ b\} \qquad p = \max\{p' \geq s : \forall_{s \leq i \leq p'}\, a_i <^+ b\}.$$

Then 1. holds by definition.

Ad 2. Suppose contrary that $\gamma(a_p) \neq \gamma(b)$. Let $a_p, \beta_0, \ldots \beta_r, b$ be a flat upper path from a_p to b. As $a \in S^{-\lambda}$ we have $\gamma(\beta_r) \in S^{-\lambda}$ for $i = 1, \ldots, r$. Let $i_0 = \min\{i : \gamma(a_p) \neq \gamma(\beta_i)\}$. Then $\gamma(a_p) \in \iota(\beta_{i_0})$ and hence, by maximality of the path, $p < k$ and $\delta(a_{p+1}) \cap \iota(\beta_{i_0}) \neq \emptyset$. Thus, by Lemma 4.8, $a_{p+1} <^+ \gamma(\beta_{i_0}) \leq \gamma(\beta_r) = b$ contrary to the definition of p.

Ad 3. Let $a_l, \beta_1, \ldots, \beta_r, b$ be a flat upper path. We consider two cases: $l > 0$ and $l = 0$.

Case $l > 0$. Suppose contrary that $\gamma(a_{l-1}) \not\in \delta\gamma(\beta_r)$. Let $i_1 = \min\{i : \gamma(a_{l-1}) \not\in \delta\gamma(\beta_i)\}$. Then $\gamma(a_{l-1}) \in \iota(\beta_{i_1})$ and hence, by Lemma 4.8, $a_{l-1} <^+ \gamma(\beta_{i_1}) \leq^+ \gamma(\beta_r) = b$ contrary to the definition of l.

Case $l = 0$. If $a_0 \in S^\varepsilon$ then, using Lemma 4.5, we have

$$\gamma\gamma(a_0) \in \theta\theta\delta(\beta_0) \subseteq \theta\theta\gamma(\beta_r) = \theta\theta(\alpha) = \theta\delta(\alpha),$$

as required in this case.

So now assume that $a_0 \in S^{-\varepsilon}$. As, by maximality of the path, there is no face $a \in S^{-\lambda}$ such that $\gamma(a) \in \delta(a_0)$, we have $\delta(a_0) \cap \gamma\dot{\delta}^{-\lambda}(\beta_i) = \emptyset$ for $i = 1, \ldots, r$. Clearly $\delta(a_0) \subseteq \delta\delta(\beta_0)$. Suppose that $\delta(a_0) \subseteq \delta\delta(\beta_i)$ with $i \leq r$. Then

$$\delta(a_0) \subseteq \delta\delta(\beta_i) - \gamma\dot{\delta}^{-\lambda}(\beta_i) = \delta\gamma(\beta_i) \subseteq \delta\delta(\beta_{i+1})$$

(last $\subseteq$ make sense only for $i < r$). Thus $\delta(a_0) \subseteq \delta\gamma(\beta_r) = \delta(b)$, as required.

4. follows easily from Lemma 4.3.7.

Ad 5. Fix $l \leq i \leq p$. Let $a_i, \beta_1, \ldots, \beta_r, b$ be a flat upper path. If we were to have $\gamma(a_i) \notin \bigcup_{i=0}^{r} \iota(\beta_r)$, by an argument similar as in 2., we would have $\gamma(a_i) = \gamma(b)$ contradicting strictness. $\Box$

Lemma 4.13 (Second Path Lemma) *Let $k \geq 0$, $a_0, \ldots, a_k$ be a flat lower path in an ordered face structure S, $x \in \delta(a_0) - \gamma(S^{-\lambda})$, $b \in S$, $a_k <^+ b$. Then either $x \in \delta(b)$ or there is $0 \leq i < k$, such that $\gamma(a_i) \in \delta(b)$, and hence $x \leq^+ y$ for some $y \in \delta(b)$, ($y = \gamma(a_i)$).*

Proof. This is an easy consequence of Path Lemma. $\Box$

Convex sets

Let S be an ordered face structure, $n, i \in \omega$, $a, a_i \in S_n$, for $i = 1, \ldots, k$. The *height* of a face a in S is the length of the longest flat upper $(S - \gamma(S^{-\lambda}))$-path ending at a. The height of a is denoted by $ht_S(a)$ or if it does not lead to confusions by $ht(a)$. The *height of a flat path* $\vec{a} = a_1, \ldots a_k$ is is the sum of heights of its faces $ht(\vec{a}) = \sum_{i=1}^{k} ht(a_i)$.

The *depth* of a face a in S is the length of the longest flat upper $(S - \gamma(S^{-\lambda}))$-path starting from a. The depth of a is denoted by $dh_S(a)$ or if it does not lead to confusions by $dh(a)$. The *depth of a flat path* $\vec{a} = a_1, \ldots a_k$ is is the sum of depths of its faces $dh(\vec{a}) = \sum_{i=1}^{k} dh(a_i)$.

Let X be a subhypergraph of S. We say that X is a *convex subset* in S if it is non-empty and the relation $<^{X,+}$ is the restriction of $<^{S,+}$ to X.

Let X be a convex subset of S, $a \in X$. The X-*depth* of a face a is the length of the longest flat upper $(X - \gamma(X^{-\lambda}))$-path starting from a. The X-depth of a is denoted by $dh_X(a)$. If $X = S$ and it does not lead to confusions we write $dh(a)$. The X-*depth of a flat path* $\vec{a} = a_1, \ldots a_k$ is is the sum of depths of its faces $dh_X(\vec{a}) = \sum_{i=1}^{k} dh_X(a_i)$.

Lemma 4.14 *Let X be a convex subset of an ordered face structure S. Then X satisfy all the axioms of ordered face structures but loop-filling, where as $<^{X,\sim}$ we take $<^{S,\sim}$ restricted to X.*

Proof. The only fact that needs a comment is that if $a <^{X,\sim} b$ then $a <^{X,-} b$. But this follows from the observation that $a = a_0, a_1, \ldots, a_{k-1}, a_k = b$, ($k > 0$) is a lower path iff $\gamma(a_0), a_1, \ldots, a_{k-1}, \gamma(a_{k-1})$ is a (possibly empty) upper path. $\Box$

Lemma 4.15 *Let X be a convex subset of an ordered face structure S, and $\alpha, a, b \in X$.*

1. *$dh_X(a) = 0$ iff $a \notin \dot{\delta}^{-\lambda}(X)$.*
2. *If $\alpha \in X^{-\lambda} - \gamma(X^{-\lambda})$ and $b \in \dot{\delta}(\alpha)$ then $dh_X(b) = dh_X(\gamma(\alpha)) + 1$.*
3. *$a <^+ b$ then $dh_X(a) > dh_X(b)$).*

Proof. Easy. $\Box$

Lemma 4.16 *Let S be an ordered face structure, X convex subset of S, $x, y \in X - \iota(X)$ and $x <^+ y$. Then there is a flat upper $(X - \delta(X^{-\lambda}))$-path from x to y.*

Proof. Let $x, y \in X - \iota(X)$ and $x <^+ y$. Let $x, a_1, \ldots, a_m, y$ be a flat upper X-path of least X-depth. Suppose that it is not $(X - \delta(X^{-\lambda}))$-path. Thus by Lemma 4.15 $dh_X(a_1, \ldots, a_m) > 0$. Pick a_s of maximal X-depth in $a_1, \ldots, a_m$. Let $\alpha \in X - \gamma(X^{-\lambda})$ such that $a_s \in \delta(\alpha)$. Let

$$l = \min\{l' \leq s : \forall_{l' \leq i \leq s}\ a_i \in \delta(\alpha)\} \qquad p = \max\{p' \geq s : \forall_{s \leq i \leq p'}\ a_i \in \delta(\alpha)\}.$$

Since $x, y \in X - \iota(X)$, by an argument similar to the one given in Path Lemma, we get that $\gamma(a_p) = \gamma\gamma(\alpha)$ and with

$$x' = \begin{cases} x & \text{if } l = 1, \\ \gamma(a_{l-1}) & \text{otherwise.} \end{cases}$$

$x' \in \delta\gamma(\alpha)$. Thus $x, a_1, \ldots, a_{l-1}, \gamma(\alpha), a_{p+1} \ldots, a_m, y$ is a path of a smaller X-depth then $x, a_1, \ldots, a_m, y$, contrary to the assumption. Therefore $x, a_1, \ldots, a_m, y$ is in fact a $(X - \delta(X^{-\lambda}))$-path. $\Box$

Order

Lemma 4.17 *Let S be an ordered face structure, $a \in S$. Then the set*

$$\{b \in S \,:\, a \leq^+ b\}$$

is linearly ordered by $\leq^+$.

Proof. Let $a, \alpha_1, \ldots, \alpha_k$ be a maximal flat upper $S - \gamma(S^{-\lambda})$-path starting from a. Then the set

$$\{a\} \cup \{\gamma(\alpha_i) : i = 1, \ldots, k\} = \{b \in S \,:\, a \leq^+ b\}$$

is obviously linearly ordered. $\Box$

Lemma 4.18 *Let S be an ordered face structure $a, b, c \in S$.*

1. *If $a <^+ b$ and $b <^\sim c$ then $a <^\sim c$.*
2. *If $a <^\sim b$ and $b <^+ c$ then either $a <^\sim c$ or $a <^+ c$.*

Proof. Ad 1. Assume $a <^+ b$ and $b <^\sim c$. Let $a, \alpha_1, \ldots, \alpha_k, b$ be a flat upper path from a to b and $b = b_0, b_1, \ldots, b_l = c$ a lower path from from b to c. Using Lemma 4.2 we get a flat upper $\bigcup_i \delta(\alpha_i)$-path $\gamma(a), a_1, \ldots, a_r\gamma(b)$. Thus we have a lower path $a, a_1, \ldots, a_r, b_1, \ldots, b_l = c$, i.e. $a <^- c$. If $\theta(a) \cap \theta(c) = \emptyset$ then clearly $a <^\sim c$.

If $\theta(a) \cap \theta(c) \neq \emptyset$ then by pencil linearity we have $a \perp^+ c$ or $a \perp^\sim c$. We shall show that the only condition that does not lead to a contradiction is $a <^\sim c$.

If $a <^+ c$, then, as $a <^+ b$, by Lemma 4.17 we have $b \perp^+ c$. Contradiction.

If $c <^+ a$, then, as $a <^+ b$, we have $c \perp^+ b$. Contradiction.

If $c <^\sim a$, then, as $b <^\sim c$, we have $b \perp^\sim a$. Contradiction.

Ad 2. Assume $a <^\sim b$ and $b <^+ c$. First note that by an argument as above we can show that if a and c are comparable at all then either $a <^\sim c$ or $a <^+ c$. Thus it is enough to show that a and c are comparable. Let $a, a_1, \ldots, a_k, b$ be a lower path with $k \geq 0$. We can assume that $a_1, \ldots, a_k$ is a flat path. As, $b <^+ c$ by Path Lemma either $a_0 <^\sim b$ or $a_0 <^+ b$. In the former case we have $\theta(a) \cap \theta(c) = \emptyset$ and that $a <^- c$. Thus $a <^\sim c$. In the later either $\gamma(a) \in \delta(c)$ and we are done or there is a flat upper path $a, \alpha_1, \ldots, \alpha_r, c$ and $i \leq r$ such that $\gamma(a) \in \iota(\alpha_i)$. Then by Lemma 4.8 we have $a <^+ \gamma(\alpha_i) \leq^+ \gamma(\alpha_r) = c$, as required. $\Box$

Lemma 4.19 *Let S be an ordered face structure, $a, b \in S$. Then we have*

1. *If $a <^+ b$ then $\gamma(a) \leq^+ \gamma(b)$;*
2. *If $a <^\sim b$ then $\gamma(a) \leq^+ \gamma(b)$;*
3. *If $\gamma(a) = \gamma(b)$ then either $a = b$ or $a \perp^+ b$ or $a \perp^\sim b$;*
4. *If $\gamma(a) <^+ \gamma(b)$ then either $a <^+ b$ or $a <^\sim b$;*
5. *If $\gamma(a) \perp^\sim \gamma(b)$ then $a \not<^\sim b$ and $a \not<^+ b$.*

Proof. 1. is repeated from Lemma 4.2.7.

Ad 2. If $a <^\sim b$ then there is a lower path $a = a_0, a_1, \ldots, a_m = b$. Hence $\gamma(a), a_1, \ldots, a_m, \gamma(b)$ is an upper path. So either $\gamma(a) = \gamma(b)$ or after dropping loops from the sequence $a_1, \ldots, a_m$ we get a flat upper path from $\gamma(a)$ to $\gamma(b)$, as required.

Ad 3. This is an immediate consequence of pencil linearity.

Ad 4. Suppose that $\gamma(a) <^+ \gamma(b)$. If $\theta(a) \cap \theta(b) \neq \emptyset$ then the thesis is obvious. So assume that $\theta(a) \cap \theta(b) = \emptyset$. Thus, by disjointness, it is enough to show that either $a <^+ b$ or $a <^- b$. There is a flat upper path $\gamma(a), a_1, \ldots, a_m, \gamma(b)$, with $m \geq 1$.

Now we argue by cases. If b is a loop the clearly $a <^- b$. Similarly, if $b = a_m$ then $m > 1$ and hence $a <^- b$. Finally, assume that $a_m <^+ b$. If $a_1 <^\sim b$ then $a <^b$. If $a_1 <^+ b$ then we have a flat path $a_1, \alpha_1, \ldots, \alpha_r, b$. Using our assumptions we find $i \leq r$ such that $\gamma(a) \in \iota(\alpha_i)$. Then by Lemma 4.8.1 we get that $a <^+ \gamma(\alpha_i) \leq^+ b$, as required.

Ad 5. It is an immediate consequence of 1., 2. and disjointness. $\Box$

Proposition 4.20 *Let S be an ordered face structure, $a, b \in S$. Let $\{a_i\}_{0 \leq i \leq n}$, $\{b_i\}_{0 \leq i \leq n}$ be two sequences of codomains of a and b, respectively, so that*

$$a_i = \gamma^{(i)}(a), \qquad\qquad b_i = \gamma^{(i)}(b)$$

(i.e. $dim(a_i) = i$), for $i = 0, \ldots, n$. Then, there are two numbers l and k such that $0 \leq l \leq k \leq n$, $1 \leq k$ and either

1. *$a_i = b_i$ for $i < l$,*
2. *$a_i <^+ b_i$ for $l \leq i < k$,*
3. *$a_i <^\sim b_i$ for $k = i \leq n$,*
4. *$a_i \not< b_i$ for $k < i \leq n$,*

or 1.-4. holds with the roles of a and b interchanged.

Proof. The above conditions we can present more visually as:

$$a_0 = b_0, \ldots, a_{l-1} = b_{l-1}, a_l <^+ b_l, \ldots a_k <^+ b_k,$$

$$a_{k+1} <^- b_{k+1}, a_{k+2} \not< b_{k+2}, \ldots, a_n \not< b_n.$$

These conditions we will verify from the bottom up. Note that by strictness $<^{S_0,+}$ is a linear order. So either $a_0 = b_0$ or $a_0 \perp^+ b_0$. In the later case $l = 0$. As $a \neq b$ then there is $i \leq n$ such that $a_i \neq b_i$. Let l be minimal such, i.e. $l = \min\{i : a_i \neq b_i\}$. By Lemma 4.19.3, $a_l \perp^+ b_l$ or $a_l \perp^\sim b_l$. We put $k = \max\{i \leq n : a_i \perp^+ b_i \text{ or } i = l\}$. If $k = n$ we are done. If $k < n$ then by Lemma 4.19.4, we have $a_{k+1} \perp^\sim b_{k+1}$. Then if $k+1 < n$, by Lemma 4.19.5, $a_i \not< b_i$ for $k+2 \leq i \leq n$. Finally, by Lemma 4.19.1 and .2 all the inequalities head in the same direction. This ends the proof. $\Box$

For $a, b \in S$ we define $a <^\sim_l b$ iff $\gamma^{(l)}(a) <^\sim \gamma^{(l)}(b)$.

Corollary 4.21 *Let S be an ordered face structure, $a, b \in S_n$, $a \neq b$. Then either $a \perp^+ b$ or there is a unique $0 \leq l \leq n$ such that $a \perp_l^\sim b$, but not both.*

The above Corollary allows us to define an order $<^S$ (also denoted $<$) on all cells of S as follows. For $a, b \in S_n$,

$$a <^S b \quad \text{iff} \quad a <^+ b \text{ or } \exists_l \; a <_l^\sim b.$$

Corollary 4.22 *For any ordered face structure S, and $k \in \omega$, the relation $<^S$ restricted to S_k is a linear order.*

Proof. In the proof we use Lemma 4.18 without mention. We need to verify that $<^S$ is transitive. Let $a, b, c \in S_n$, $l, k \leq n \in \omega$. We argue by cases.

Case $a <^+ b <^+ c$. Then by transitivity of $<^+$ we have $a <^+ c$.

Case $a <^+ b <_l^\sim c$. Then $\gamma^{(l)}(a) <^+ \gamma^{(l)}(b) <^\sim \gamma^{(l)}(c)$. Therefore $\gamma^{(l)}(a) <^\sim \gamma^{(l)}(c)$ and hence $a <_l^\sim c$.

Case $a <_l^\sim b <^+ c$. Then $\gamma^{(l)}(a) <^\sim \gamma^{(l)}(b) <^+ \gamma^{(l)}(c)$. Thus either $\gamma^{(l)}(a) <^\sim \gamma^{(l)}(c)$ and hence $a <_l^\sim c$ or $\gamma^{(l)}(a) <^+ \gamma^{(l)}(c)$. In the later case by Proposition 4.20 we have either $a <^+ c$ or there is l' such that $l \leq l' \leq n$ and $a <_{l'}^\sim c$.

Case $a <_k^\sim b <_l^\sim c$. If $k = l$ then by transitivity of $<^\sim$ we have $a <_k^\sim c$.

If $k > l$ then $\gamma^{(l)}(a) \leq^+ \gamma^{(l)}(b) <^\sim \gamma^{(l)}(c)$. Therefore $\gamma^{(l)}(a) <^\sim \gamma^{(l)}(c)$ and hence $a <_l^\sim c$.

If $k < l$ then $\gamma^{(k)}(a) \leq^\sim \gamma^{(k)}(b) <^+ \gamma^{(k)}(c)$. Therefore either $\gamma^{(k)}(a) \leq^\sim \gamma^{(k)}(c)$ and hence $a <_k^\sim b$ or $\gamma^{(k)}(a) \leq^+ \gamma^{(k)}(c)$. In the later case again by Proposition 4.20 we have either $a <^+ c$ or there is k' such that $k \leq k' \leq n$ and $a <_{l'}^\sim c$. $\Box$

Monotone morphisms

From Corollary 4.22 we also get

Corollary 4.23 *Let $f : S \to T$ be a monotone morphism of ordered face structures, and $l, k \in \omega$, $l \leq k$, $x, y \in S_k$. Then*

1. *$x <_l^\sim y$ iff $f(x) <_l^\sim f(y)$;*
2. *$x \leq^+ y$ iff $f(x) \leq^+ f(y)$.*

Proof. Obvious. $\Box$

Remak. Note however that monotone morphisms do not preserve the relation $<^+$ in general.

Corollary 4.24 *Any monotone morphism of ordered face structures which is a bijection is an isomorphism.*

Proof. If f is a monotone bijection of ordered face structures it is clearly a local isomorphism. But by Lemma 4.23 it reflects $<^\sim$ as well, i.e. it is a monotone isomorphism. $\Box$

Lemma 4.25 *Let $f : S \to T$ be a monotone morphism of ordered face structures. If $a \in T$ then if $f^{-1}(a) \neq \emptyset$ there are $b, c \in f^{-1}(a)$ and a flat upper $S^u - \gamma(S^{-\lambda})$-path $b, \alpha_1, \ldots, \alpha_r, c$, with $r \geq 0$, such that $f^{-1}(a) = \{b\} \cup \{\gamma(\alpha_i)\}_{1 \leq i \leq r}$. In particular all faces in $f^{-1}(a)$ are parallel to each other and the whole set is linearly ordered by $<^+$.*

Proof. Suppose $b, c \in S$ such that $f(b) = f(c) = a \in T$. Then for any l, $f(b) \not<_l^\sim f(c)$. Therefore for any l, $b \not<_l^\sim c$. Thus, by Lemma 4.20, $b <^+ c$. Hence There is a flat upper $S - \gamma(S^{-\lambda})$-path $b, \alpha_1, \ldots, \alpha_r, c$. As $f(b) = f(c)$, we have $f(\alpha_i) \in T^\lambda$, for $i = 1, \ldots, r$. In particular, $\alpha_i \in S^u$, for $i = 1, \ldots, r$. So we have shown that between any two different elements of $f^{-1}(a)$ there is a flat upper $S^u - \gamma(S^{-\lambda})$-path. This clearly imply all the remaining parts of Lemma. □

Corollary 4.26 *Any endomorphism of an ordered face structures is an identity.*

Proof. Let $f : S \to S$ be a monotone morphism. First note that as S_0 is linearly ordered by $<^+$, if $x, y \in S_0$ then $ht(x) + dh(x) = ht(y) + dh(y)$, and $x = y$ iff $ht(x) = ht(y)$. As f preserves $<^\sim$, using Lemma 4.7, we get that $f_0 : S_0 \to S_0$ is an identity.

In order to get a contradiction we suppose that f is not identity. Let k be the minimal such that $f_k \neq 1_{S_k}$ and let $a \in S_k$ be $<^S$-minimal such that $f(a) \neq a$. By minimality of k we have $f(a) \| a$. We shall show that $f(a) \perp^+ a$. By previous observation and pencil linearity we have that either $f(a) \perp^+ a$ or $f(a) \perp^\sim a$. If $a <^\sim f(a)$ then we get an infinite sequence $a <^\sim f(a) <^\sim ff(a) <^\sim \ldots$ contradicting strictness of $<^\sim$. The condition $f(a) <^\sim a$ cannot hold for the similar reasons. Suppose that $a <^+ f(a)$ (the argument for the case $f(a) <^+ a$ is similar). Let $a, \alpha_1, \ldots, \alpha_r, f(a)$ be a flat upper path with $r > 0$. Then, by Lemma 4.7, we have $\alpha_i <^\sim \alpha_{i+1}$ for $i = 1, \ldots, r-1$. Hence we get an infinite sequence

$$\alpha_1 <^\sim f(\alpha_1) <^\sim ff(\alpha_1) <^\sim \ldots$$

again contradicting strictness of $<^\sim$. Thus f must be an identity indeed. □

Proposition 4.27 *Let $f : S \to T$ be a local morphism of ordered face structures that preserves $<^\sim$ on sets $S_k - \delta(S_{k+1}^{-\lambda})$, for $k \in \omega$. Then f is a monotone morphism. In particular if S is n-normal then f is a monotone morphism iff $f_n : S_n \longrightarrow T_n$ preserves $<^\sim$.*

Proof. Let $\prec$ denote the relation $<^{S,\sim}$ restricted to such pairs of elements a, b that either $a, b \in \delta(\alpha)$ for some $\alpha \in S$ or $a, b \in S - \delta(S^{-\lambda})$. Thus we must show that if a hypergraph morphism $f : S \to T$ between ordered face structures preserves $\prec$, i.e. is such that for any $a, b \in S$ if $a \prec b$ then $f(a) <^{T,\sim} b$ then it preserves $<^\sim$, i.e. for any $a, b \in S$ if $a <^{S,\sim} b$ then $f(a) <^{T,\sim} b$.

So fix $f : S \to T$ preserving $\prec$ and $a, b \in S$ such that $a <^\sim b$. Then by disjointness and few other facts, there is a lower path $a = a_0, \ldots, a_k = b$ such that $a_i <^\sim a_{i+1}$, for $i = 0, \ldots, k-1$. By transitivity of $<^\sim$ it is enough to show that $f(a) <^\sim f(b)$ only in case $\gamma(a) \in \delta(b)$. We shall prove by induction of the sum of depth of a and b, $s = dh(a) + dh(b)$ that if $\gamma(a) \in \delta(b)$ and $a <^\sim b$ then $f(a) <^\sim f(b)$.

If $s = 0$ then $a, b \in S - \delta(S^{-\lambda})$ and hence, by assumption on f, $f(a) <^\sim f(b)$.

So assume that $s > 0$ and that for s' smaller than s the inductive hypothesis holds. We consider two cases.

Case $dh(a) \geq dh(b)$. So we have $\alpha \in S^{-\lambda} - \gamma(S^{-\lambda})$ such that $a \in \delta(\alpha)$. Hence $\delta(b) \cup \theta\delta(\alpha) \neq \emptyset$. If $\delta(b) \cap \iota(\alpha) \neq \emptyset$ then, by Lemma 4.8, $b <^+ \gamma(\alpha)$. If $\delta(b) \cap \theta\gamma(\alpha) \neq \emptyset$ then either $b <^+ \gamma(\alpha)$ or $\gamma(\alpha) <^\sim b$, as two other cases easily lead to a contradiction. If $b <^\sim \gamma(\alpha)$ then $a <^\sim \gamma(\alpha)$, and if $\gamma(\alpha) \leq^+ b$ then $a <^+ b$. So in both cases we get a contradiction. Now if $b <^+ \gamma(\alpha)$, as $\alpha \in S^{-\lambda} - \gamma(S^{-\lambda})$ and $dh(a) \geq dh(b)$ we have $b \in \delta(\alpha)$. Hence $a \prec b$ and then by assumption on f we get $f(a) <^\sim f(b)$, as required. If $\gamma(\alpha) <^\sim b$ then, as $\gamma\gamma(\alpha) \in \delta(b)$ by strictness and induction hypothesis, we have $f(a) \leq^+ f(\gamma(\alpha)) <^\sim f(b)$. So by Lemma 4.18 $f(a) <^\sim f(b)$, as well.

Case $dh(a) < dh(b)$. So we have $\beta \in S^{-\lambda} - \gamma(S^{-\lambda})$ such that $b \in \delta(\beta)$. Then $a \not\in \delta(\alpha)$ and $a \not<^+ \gamma(\beta)$. We shall show that $\gamma(a) \in \delta\gamma(\beta)$. Clearly we have $\gamma(a) \in \delta\delta(\beta)$. If we were to have $\gamma(a) \in \iota(\beta)$ then $a <^+ \gamma(\beta)$ and hence $dh(a) \geq dh(\gamma(\beta)) + 1 \geq dh(b)$ contradicting our assumption. Therefore $\gamma(a) \in \delta\gamma(\beta)$. Now, to get a contradiction, we assume that $\gamma(a) = \gamma\gamma(\beta) \not\in \delta\gamma(\beta)$. By δ-globularity we have $c \in \dot{\delta}^{-\lambda}(\beta)$ such that $\gamma(c) = \gamma(a)$. Thus, by pencil linearity, either $c \perp^{\sim} a$ or $c \perp^+ a$. As if $a <^+ c$ then $a <^+ \gamma(\beta)$, if $c <^+ a$ then $b <^+ a$, and if $a <^{\sim} c$ then $\gamma(a) \neq \gamma(c)$ the only non-trivial case, we have to consider, is $c <^{\sim} a$. Thus by Lemma 4.9 we have $a <^+ \gamma(\beta)$, and we get a contradiction again. Therefore $\gamma(a) \in \delta\gamma(\beta)$ as claimed.

As $b <^+ \gamma(\beta) <^{\sim} a$ would lead to $b <^{\sim} a$ and contradiction, we must have $a <^{\sim} \gamma(\beta)$. Thus by induction hypothesis we have $f(a) <^{\sim} f(\gamma(\beta))$. Clearly, we also have $f(b) \leq^+ f(\gamma(\beta))$. As $\gamma(a) \in \delta(b)$ we have $\gamma(f(a)) \in \delta(f(b))$ and hence, by pencil linearity, either $f(a) \perp^{\sim} f(b)$ or $f(a) \perp^+ f(b)$. We shall show that the only condition that does not lead to a contradiction is $f(a) <^{\sim} f(b)$. If $f(a) <^+ f(b)$ then $f(a) <^+ f(\gamma(\beta))$ and contradiction. If $f(b) <^{\sim} f(a)$ then, by Lemma 4.17, $f(a) \perp^+ f(\gamma(\beta))$ and contradiction. If $f(b) <^{\sim} f(a)$ then $f(b) <^{\sim} f(\gamma(\beta))$ and again we get a contradiction. Thus $f(a) <^{\sim} f(b)$, as required. $\Box$

Corollary 4.28 *The ordered face structure S is uniquely determined by its local face structure part $|S|$ and the order $<^{\sim}$ restricted to the sets $S_k - \delta(S^{-\lambda}_{k+1})$, for $k \in \omega$.*

Proof. Let S and S' be ordered face structures such that their local parts are equal, i.e. $|S| = |S'|$ and that the relation $<^{\sim,S}$ agree with $<^{\sim,S'}$ on the set $S - \delta(S^{-\lambda})$. Then the identity morphism is preserving $<^{\sim}$ on the set $S - \delta(S^{-\lambda})$. Thus by Proposition 4.27 it is a monotone morphism, considered as a map either way, i.e. $S = S'$. $\Box$

In general, there are more local than monotone morphisms between ordered face structures. However if we restrict our attention to the principal ordered face structures those two notions agree. We have

Corollary 4.29 *The embedding $| - | :$ **pFs** $\longrightarrow$ **lFs** is full and faithful.*

Proof. Fix a local morphism $f : S \longrightarrow T$ between ordered face structures, with S being principal. Then for $k \in \omega$ the sets $S_k - \delta(S^{-\lambda}_{k+1})$ has at most one element. So $<^{\sim}$ is obviously preserved on these sets, i.e. by Lemma 4.27 f is monotone. $\Box$

The limits and colimits in **oFs** are rather rare and in **lFs** also do not always exist. For example if we take a local face structure S such that $S_0 = \{u\}$, $S_1 = \{x, y, z\}$, $S_2 = \{a\}$, and with $\gamma(x) = \gamma(y) = \gamma(z) = \delta(x) = \delta(y) = \delta(z) = u$, $\delta(a) = \{x, y\}$, $\gamma(a) = z$, and $<_a$ being empty relation, then we have two local isomorphisms from S to itself identity 1_S and a map σ switching x and y. Clearly in the coequalizer Q of 1_S and σ the faces x and y should be identified but then the map $q : S \to Q$ in to it cannot be local as there cannot be a bijection from $\delta(a)$ to $\delta(q(a))$. Thus the coequalizer 1_S and σ does not exists in **lFs**. However we have

Proposition 4.30 *The colimits and connected limits of diagrams from* **oFs** *exists in* **lFs** *and are calculated in Set in each dimension separately.*

Proof. The main property of the monotone morphisms of order face structures that allow calculations of the above limits and colimits is the following. If $f, g : S \to T$ are monotone morphisms and $a \in S_{>0}$ such that $f(a) = g(a) = b$ then the functions $f_a, g_a : \delta(a) \to \delta(b)$ are equal. This is an immediate consequence of Corollary 4.23. $\Box$

5 Stretching the convex subhypergraphs

From Corollary 4.23 follows immediately that the (hypergraph) image of a monotone morphism is a convex subset of the codomain. In this section we shall show that the converse is also true and it is true in an essentially unique way, i.e. if X is a convex subset of T then there is a monotone morphism $\nu_X : [X] \to T$ such that image of ν_X is X, i.e. we can cover a convex set by an ordered face structure. Moreover, if $f_i : S_i \to T$, $i = 0, 1$, are monotone morphisms such that their images are equal, $im(f_0) = im(f_1)$, then there is a monotone isomorphism $g : S_0 \to S_1$ making the triangle

$$\begin{tikzcd} S_0 \arrow[rr, "g"] \arrow[dr, "f_0"'] & & S_1 \arrow[dl, "f_1"] \\ & T & \end{tikzcd}$$

commutes, i.e. the covering is essentially unique. As the title of the section suggests, the construction of $[X]$ is done by stretching X. The stretching means in this case the splitting all the empty loops in the convex set X.

Let T be an ordered face structure, and $X \subseteq T$ a subhypergraph. Recall that X is *convex* in T if it is non-empty and the relation $<^{X,+}$ is the restriction of $<^{T,+}$ to X. For the rest of the section assume that X is a convex subhypergraph of T. We shall define an ordered face structure $[X]$ and a monotone morphism $\nu_X : [X] \longrightarrow T$. But first we need to explain what are cuts of empty loops.

We define the set of empty loops in X as

$$\mathcal{E}^X = X^\lambda - \gamma(X^{-\lambda})$$

and the set of empty loops in X over $x \in X$ as

$$\mathcal{E}^X_x = \{a \in \mathcal{E} : \gamma(a) = x\}.$$

As X is convex, the relation $<^{T,\sim}$ restricted to $\mathcal{E}^X_x$ is a linear order. We say that a triple (x, L, U) is an x-*cut* i.e. a cut of $\mathcal{E}^X_x$ iff $L \cup U = \mathcal{E}^X_x$ and for $l \in L$ and $l' \in U$, $l <^\sim l'$. Note that both L and U might be empty and hence, there is an x-cut for any $x \in X$ (e.g. $(x, \emptyset, \mathcal{E}^X_x)$). Let $\mathcal{C}(\mathcal{E}^X_x)$ be the set of x-cuts.

We need some notation for cuts in X. If (x, L, U) is a x-cut then L determines U and vice versa ($L = \mathcal{E}^X_x - U$ and $U = \mathcal{E}^X_x - L$).

Therefore we sometimes denote this cut by describing only the lower cut $(x, L, -)$ or only the upper cut $(x, -, U)$, whichever is easier to define. Let a be an arbitrary face in X, $y \in \dot{\delta}(a)$. We define two sets

$$\uparrow a = \{b \in \mathcal{E}^X_{\gamma(a)} : a <^\sim b\}, \qquad \downarrow_y a = \{b \in \mathcal{E}^X_y : b <^\sim a\}$$

that determine the cuts $(x, -, \uparrow a)$ and $(y, \downarrow_y a, -)$. We often omit subscript y inside the cuts, i.e. we usually write $(y, \downarrow a, -)$ when we mean $(y, \downarrow_y a, -)$. If $\gamma(a) =$ then we sometimes write $\uparrow_x a$ instead of $\uparrow a$ to emphasis that the cut is over$x$.

Now we are ready to define $[X]$. We put for $k \in \omega$

$$[X]_k = \bigcup_{x \in X_k} \mathcal{C}(\mathcal{E}^X_x)$$

We put, for $(x, L, U) \in [X]_l$

$$\gamma(x, L, U) = (\gamma(x), -, \uparrow x)$$

and

$$\delta(x, L, U) = \begin{cases} 1_{(\gamma\gamma(x), -, \uparrow\gamma(x))} & \text{if } \delta(x) = 1_{\gamma\gamma(x)}, \\ \{(t, \downarrow x, -) : t \in \dot{\delta}(x)\} & \text{otherwise.} \end{cases}$$

We have a hypergraph map

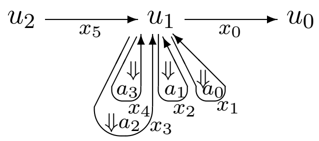
$$\nu_X : [X] \longrightarrow T$$

such that $\nu_X(x, L, U) = x$, for $(x, L, U) \in [X]$. The order $<^{[X],\sim}$ is inherited from T via ν_X, i.e. $(x, L, U) <^{[X],\sim} (x', L', U')$ iff $x <^{T,\sim} x'$, for $(x, L, U), (x', L', U') \in [X]$.

Let $Z \subseteq T$. By $< Z >$ we mean the least subhypergraph of T containing Z. We call Z *convex set* if $< Z >$ is a convex subhypergraph. If Z is a convex set we write $\mathcal{E}^Z$, $[Z]$ and ν_Z instead of $\mathcal{E}^{<Z>}$, $[< Z >]$ and $\nu_{<Z>}$. Moreover, if $Z = \{\alpha\}$ we write $\mathcal{E}^\alpha$, $[\alpha]$ and ν_α instead of $\mathcal{E}^{\{\alpha\}}$, $[\{\alpha\}]$ and $\nu_{\{\alpha\}}$.

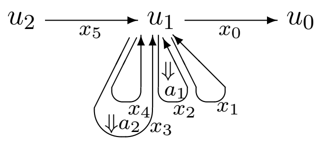
Example. An ordered face structure T

T



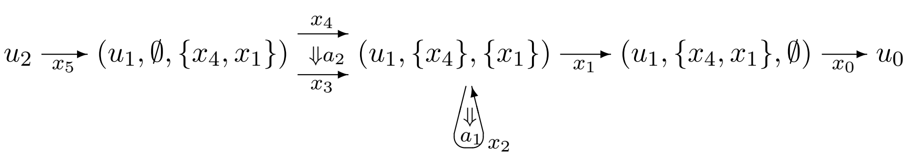
has a convex subset X

X



with $\mathcal{E}^X = \{x_4, x_1\}$ whose stretching is the following ordered face structure $[X]$:

$[X]$



We adopt the convention that the empty cut in $[X]$, say $(x_5, \emptyset, \emptyset)$, is identified with the corresponding face in X, x_5 in this case.

Lemma 5.1 *Let T be an ordered face structure, X be a convex subset of T, $(a, L, U), (a', L', U') \in [X]$. Then*

1. *(a, L, U) is an empty domain face in $[X]$ iff a is an empty domain face in T;*

2. *(a, L, U) is a loop in $[X]$ iff a is a loop in T and there is no empty loop $l \in \mathcal{E}^X$ such that $l \leq^{T,+} a$;*

3. *$(a, L, U) <^+ (a', L', U')$ iff $a <^+ a'$ or ($a = a'$ and $L \subsetneq L'$);*

4. *$(a, L, U) <^- (a', L', U')$ iff $a <^- a'$ and (if $\gamma(a) \in \delta(a')$ then $\mathcal{E}_{\gamma(a)} = \uparrow_{\gamma(a)} a \cup \downarrow_{\gamma(a)} a'$);*

5. *If $(a, L, U) \in [X]^{-\varepsilon}$ then '$\delta(a, L, U)$ is $\delta(a)$ with colors': if $x, y \in \delta(a)$ such that y is $<^\sim$-successor of x in $\delta(a)$ then $(y, \downarrow a, -)$ is $<^\sim$-successor of $(x, \downarrow a, -)$ in $\delta(a, L, U)$; in particular $\gamma(x, \downarrow a, -) \in \delta(y, \downarrow a, -)$.*

6. *If $(a, L, U) \in [X]^\varepsilon$ then $\delta(a, L, U) = 1_{(\gamma\gamma(a), \downarrow\gamma(a), -)}$.*

Proof. 1., 5. and 6. are obvious.

Ad 2. If (a,L,U) is a loop in $[X]$ so must be a in T. So fix a cut (a,L,U) in $[X]$ such that a is a loop. Let us denote $\gamma(a,L,U)=(\gamma(a),-,\uparrow a)$ and $\delta(a,L,U)=(\gamma(a),\downarrow a,-)$.

If there is $l\in\mathcal{E}^X_{\gamma(a)}$ such that $l\leq^+ a$, then $a\not<^\sim l$. Hence $l\in L'$ and $l\in U'$. So $\gamma(a,L,U)\neq\delta(a,L,U)$ and (a,L,U) is not a loop.

If there is no $l\in\mathcal{E}^X_{\gamma(a)}$ such that $l\leq^+ a$ then any empty loop $l\in\mathcal{E}^X_{\gamma(a)}$ is $<^\sim$-comparable with a. Thus $\gamma(a,L,U)=(\gamma(a),-,\uparrow a)=(\gamma(a),\downarrow a,-)=\delta(a,L,U)$.

Ad 3. Fix (a,L,U), (a',L',U') in $[X]$.

First we shall show that the condition is necessary. Suppose that $(a,L,U),(\alpha_1,L_1,U_1),\ldots,(\alpha_k,L_k,U_k),(a',L',U')$ is a flat upper path in $[X]$.

Suppose that $a\neq a'$. Clearly $a,\alpha_1,\ldots,\alpha_k,a'$ is an upper path in X. So after deleting loops we get a flat upper X-path from a to a', i.e. $a<^+ a'$.

Suppose now that $a=a'$ and $L\neq L'$. As $(a,L,U)\in\delta(\alpha_1,L_1,U_1)\neq\emptyset$ we have $L=\downarrow_a\alpha_1$. Moreover, $\gamma(\alpha_k,L_k,U_k)=(a',L',U')$ implies that $\uparrow_a\alpha_k=U'$. Since

$$(a,-,\uparrow_a\alpha_i)=\gamma(\alpha_i,L_i,U_i)\in\delta(\alpha_{i+1},L_{i+1},U_{i+1})=(a,\downarrow_a\alpha_{i+1},-)$$

we have that $(a,\downarrow_a\alpha_{i+1},\uparrow_a\alpha_i)$ is a cut, for $i=1,\ldots,k-1$. As $\uparrow_a\alpha_i\cap\downarrow_a\alpha_i=\emptyset$, we have that $\downarrow_a\alpha_i\subseteq\downarrow_a\alpha_{i+1}$, for $i=1,\ldots,k-1$. Thus

$$L=\downarrow_a\alpha_1\subseteq\downarrow_a\alpha_k\subseteq\mathcal{E}^X_a-\uparrow_a\alpha_k=\mathcal{E}^X_a-U'=L'$$

i.e. the condition is necessary.

Now we shall show that the condition is sufficient. First note that if $l\in\mathcal{E}^X_a$ then

$$\delta(l,\emptyset,-)=(a,\downarrow l,-)=(a,\downarrow l,\{l\}\cup\uparrow l),$$

$$\gamma(l,\emptyset,-)=(a,-,\uparrow l)=(a,\downarrow l\cup\{l\},\uparrow l).$$

Thus if $a=a'$ we have that $(a,L,U)<^+(a',L',U')$ iff $L\subsetneq L'$.

Assume now that $a<^+ a'$. Let $a,\alpha_1,\ldots,\alpha_k,a'$ be a flat upper X-path of minimal weight. We claim that it is $X-\gamma(X^{-\lambda})$-path. Suppose contrary that there is $A\in X^{-\lambda}$ such that $\gamma(A)=\alpha_i$, for some i. Then we will have a flat upper $\delta(A)$-path $\beta_1,\ldots,\beta_r$ from $\gamma(\alpha_{i-1})$ (or a if $i=0$) to $\gamma(\alpha_i)$. Replacing α_i by $\beta_1,\ldots,\beta_r$ we get a flat upper X-path of smaller weight than $\alpha_1,\ldots,\alpha_k$ contrary to the choice of this path. This $\alpha_1,\ldots,\alpha_k$ is a flat upper $X-\gamma(X^{-\lambda})$-path indeed.

As $\alpha_i\in X-\gamma(X^{-\lambda})$ we have

$$\delta(\alpha_i,\emptyset,-)=\{(b,\downarrow\alpha_i,\emptyset):b\in\delta(\alpha_i)\},\qquad \gamma(\alpha_i,\emptyset,-)=(\gamma(\alpha_i),\emptyset,\uparrow\alpha_i).$$

From this and previous we get that

$$(a,L,U)\leq^+(a,\downarrow\alpha_1,\emptyset)\in\delta(\alpha_{i+1},\emptyset,-),$$

$$\gamma(\alpha_i,\emptyset,-)\leq^+(\gamma(\alpha_i),\downarrow\alpha_{i+1},\emptyset)\in\delta(\alpha_1,\emptyset,-)$$

$$\gamma(\alpha_k,\emptyset,-)\leq^+(a',L',U')$$

and this shows that $(a,L,U)<^+(a',L',U')$, as required.

Ad 4. Using 3. we have the following equivalent statement

$$\begin{array}{c}
(a,L,U)<^-(a',L',U')\\ \hline
\exists_{x\in\delta(a')}\ (\gamma(a),-,\uparrow a)\leq^+(x,\downarrow a',-)\\ \hline
\exists_{x\in\delta(a')}\ \gamma(a)<^+x \text{ or } \ (\gamma(a)=x \text{ and } \ \mathcal{E}_{\gamma(a)}-\uparrow_{\gamma(a)}a\subseteq\downarrow_{\gamma(a)}a')\\ \hline
a<^-a' \text{ and (if } \ \gamma(a)\in\delta(a') \text{ then } \mathcal{E}_{\gamma(a)}=\uparrow_{\gamma(a)}a\cup\downarrow_{\gamma(a)}a')
\end{array}$$

□

Lemma 5.2 *Let S be an ordered face structure, X a convex subset of S, $a \in X$, $u \in \theta\delta(a)$, $l \in \mathcal{E}^X_u$. Then*

1. *If $a \in S^{\varepsilon}$ then $\gamma(a) \perp^{\sim} l$.*
2. *If $a \in S^{-\varepsilon}$ then*
   (a) *if $u = \gamma\gamma(a)$ then $\gamma(a) <^{\sim} l$ iff $\varrho(a) <^{\sim} l$;*
   (b) *if $u \in \iota(a)$ then if $x, y \in \delta(a)$ and $x <^{\sim} l <^{\sim} y$ there is $z \in \delta(a)$ such that $l \leq^{+} z$ and $x <^{\sim} z <^{\sim} y$;*
   (c) *if $u \in \delta\gamma(a)$ then $l <^{\sim} \gamma(a)$ iff $l <^{\sim} x_0$, where x_0 is the $\sim$-minimal element in $\delta(a)$ such that $u \in \delta(x)$.*

Proof. Ad 1. By pencil linearity we have that either $\gamma(a) \perp^{\sim} l$ or $\gamma(a) \perp^{+} l$. As l is an empty loop we cannot have $\gamma(a) \leq^{+} l$. We shall show that $l <^{+} \gamma(a)$ is impossible, as well.

Suppose not, and that we have a flat upper $X - \gamma(X^{-\lambda})$-path $l, a_1, \ldots, a_k, \gamma(a)$. As $a_k \in X^{-\varepsilon}$ we have that $a_k <^{+} a$. But $a \in X^{\varepsilon}$, so by Path Lemma, we have $a_i <^{+} a$, for $i = 1, \ldots, n$. Let $a_1, \alpha_1, \ldots, \alpha_r, a$ be a flat upper X-path. As $l \in \delta(a)$ and $\delta\gamma(\alpha_i) \cup \gamma\dot{\delta}^{-\lambda}(\alpha_i) = \theta\delta(\alpha_i)$, either for all i, $l \in \delta\gamma(\alpha_i)$, or there is i_0 such that $l \in \gamma\dot{\delta}^{-\lambda}(\alpha_{i_0})$. In the former case $l \in \delta\gamma(\alpha_r) = \delta(a)$ and we get a contradiction, as $a \in T^{\varepsilon}$. In the later case l is not an empty loop in X contrary to the assumption.

Ad 2(a). First, assume $\gamma(a) <^{\sim} l$. As $\gamma\gamma(a) = \gamma(l)$, by pencil linearity we have either $\varrho(a) \perp^{\sim} l$ or $\varrho(a) \perp^{+} l$. We shall show that the other cases then $\varrho(a) <^{\sim} l$ lead quickly to a contradiction. If $l <^{\sim} \varrho(a)$ then $\gamma(a) <^{\sim} \varrho(a)$ and this is a contradiction. If $l <^{+} \varrho(a)$ then $l <^{+} \gamma(a)$ and this is a contradiction. If $\varrho(a) <^{+} l$ then $l \perp^{+} \gamma(a)$ and this is again a contradiction. Thus we must have $\varrho(a) <^{\sim} l$[7].

Next we assume $\varrho(a) <^{\sim} l$. By pencil linearity we have either $\gamma(a) \perp^{\sim} l$ or $\gamma(a) \perp^{+} l$. We need to show that $\gamma(a) <^{\sim} l$. We shall show that the condition $l <^{+} \gamma(a)$ leads to a contradiction. The other two are easily excluded. Clearly $l \notin \delta(a)$.

So suppose that $l <^{+} \gamma(a)$. Let $l, a_1, \ldots, a_k, \gamma(a)$ be a flat upper $X - \gamma(X^{-\lambda})$-path. By Path Lemma, either there is $i < k$ such that $\gamma(a_i) \in \delta(a)$ or $a_i <^{+} a$ for $i = 1, \ldots, k$. In the former case we have $l <^{+} \gamma(a_i) <^{\sim} \varrho(a)$. Thus, by Lemma 4.18, $l <^{\sim} \varrho(a)$ and this is a contradiction. In the later case, there is a flat upper X-path $a_1, \alpha_1, \ldots, \alpha_r, a$. As $l \in \delta(a_1)$ and $l \notin \delta(a)$ there is i such that $l \in \iota(\alpha_i)$. In particular l is not an empty loop in X and we get a contradiction again.

Ad 2(b). Fix $x, y \in \delta(a)$ such that $\gamma(x) = u \in \delta(a)$, $l \in \mathcal{E}^X_u$ such that $x <^{\sim} l <^{\sim} y$. If $l \in \delta(a)$ 2(ii) obviously holds, so assume that $l \notin \delta(a)$. We have $\gamma(l) \leq^{+} \gamma(y) \leq^{+} \gamma\gamma(a)$. Thus by Proposition 4.19, either $l <^{\sim} \gamma(a)$ or $l <^{+} \gamma(a)$. The former case gives immediately $x <^{\sim} \gamma(a)$ and a contradiction.

Thus we have $l <^{+} \gamma(a)$. Fix a flat X-path $l, a_1, \ldots, a_k, \gamma(a)$. By Path Lemma, either there is $i < k$ such that $\gamma(a_i) \in \delta(a)$ or $a_i <^{+} a$ for all $1 \leq i \leq k$. In the former case $\gamma(a_i)$ is the z we are looking for. We shall show that the later case leads to a contradiction. Take a flat upper X-path $a_1, \alpha_1, \ldots, \alpha_r, a$. As $l \in \delta(a_1)$ and $l \notin \delta(a)$, there is $1 \leq i \leq r$ such that $l \in \gamma\dot{\delta}^{-\lambda}(\alpha_i)$. In particular, $l \notin \mathcal{E}_u$, contrary to the assumption.

Ad 2(c). Suppose $l <^{\sim} \gamma(a)$. Then, as other cases are easily excluded, we have $l < x_0$ indeed.

On the other hand, if $l <^{\sim} x_0$ the only case not easily excluded is $l <^{+} \gamma(a)$. Let $l, a_1, \ldots, a_k, \gamma(a)$ be a flat upper $X - \gamma(X^{-\lambda})$-path. By Path Lemma either there is

[7] In the following, the similar simple arguments we will describe in a shorter form as follows: as other cases are easily excluded, we have $\varrho(a) <^{\sim} l$.

$i_0 < k$ such that $\gamma(a_{i_0}) \in \delta(a)$ or $a_i <^+ a$ for $i = 1, \ldots, k$. In the later case, there is a flat upper X-path $a_1, \alpha_1, \ldots, \alpha_k, a$. As $l \not\in \delta(a)$ and $l \in \delta(a_1)$, there is $1 \leq i \leq k$ such that $l \in \iota(\alpha_i)$. In particular $l \not\in \mathcal{E}^X_u$, contrary to the assumption.

In the former case we shall show that $u \in \delta\gamma(a_i)$, for $i = 1, \ldots, i_0$. We have $u = \gamma(l) \in \delta\delta(a_1)$. Note that if for some $i \leq i_0$, we would have that $u \in \iota(a_i)$, then, as $u \in \delta(x_0)$, by Lemma 4.8, we would have $x_0 <^+ \gamma(a_{i_0})$, contradicting local discreteness. Now suppose contrary, that for some $i_1 \leq i_0$, we have $u \not\in \delta\gamma(a_{i_1})$. Then, by the previous observation, we have $u = \gamma\gamma(a_{i_1}) \in \gamma\dot{\delta}^{-\lambda}(a_{i_1})$. In particular, $\dot{\delta}^{-\lambda}(a_{i_1}) \neq \emptyset$ and $\gamma(a_i)$ is not a loop, for $i_1 \leq i \leq i_0$. As $u \not\in \iota(a_i)$ for $i \leq i_0$, we have $u = \gamma\gamma(a_i)$ for $i \leq i_0$. In particular $u = \gamma\gamma(a_{i_0}) \in \gamma\dot{\delta}^{-\lambda}(a)$. But $u \in \delta\gamma(a)$ and $\delta\gamma(a) \cap \gamma\dot{\delta}^{-\lambda}(a) = \emptyset$ so we get a contradiction. $\Box$

Proposition 5.3 *Let T be an ordered face structure, X be a convex subset of T. Then*

1. *$[X]$ is an ordered face structure, and $\nu_X : [X] \to T$ is a monotone morphism;*

2. *if $f_i : S_i \to T$, $i = 0, 1$, are monotone morphisms such $im(f_0) = im(f_1)$, then there is a monotone isomorphism $g : S_0 \to S_1$ making the triangle*

$$\begin{tikzcd} S_0 \arrow[rr, "g"] \arrow[dr, "f_0"'] & & S_1 \arrow[dl, "f_1"] \\ & T & \end{tikzcd}$$

commutes;

3. *$\mathcal{E}^X$ is empty iff ν_X is an embedding;*

4. *if X is a proper subset of T then $size(T) > size([X])$.*

Proof. Ad 1. The fact that ν_X is a monotone morphism is immediate from the definition of $[X]$. We need to check that $[X]$ satisfies the axioms of ordered face structures. *Local discreteness* and *Strictness*, are easy using Lemma 5.1.

Disjointness. We shall check that if $\theta(a, L, U) \cap \theta(a', L', U') = \emptyset$ and $(a, L, U) <^- (a', L', U')$ then $(a, L, U) <^\sim (a', L', U')$. The remaining parts of the condition are easy.

Suppose that $\theta(a, L, U) \cap \theta(a', L', U') = \emptyset$ and $(a, L, U) <^- (a', L', U')$. By Lemma 5.1 we have that $a <^- a'$. If $\theta(a) \cap \theta(a') = \emptyset$, we get by disjointness in T that $a <^\sim a'$, and we are done. Assume that $\theta(a) \cap \theta(a') \neq \emptyset$. Thus by Lemma 4.7.2 $\gamma(a) \in \delta(a')$. By characterization of $<^-$ in $[X]$ we have $\uparrow_{\gamma(a)} a \cup \downarrow_{\gamma(a)} a' = \mathcal{E}_{\gamma(a)}$. As $(\gamma(a), -, \uparrow a) \in \theta(a, L, U)$ and $(\gamma(a), \downarrow a', -) \in \theta(a', L', U')$ we get that $(\gamma(a), -, \uparrow a) \neq (\gamma(a), \downarrow a', -)$. So $\mathcal{E}_{\gamma(a)} - \uparrow_{\gamma(a)} a \neq \downarrow_{\gamma(a)} a'$. But then there is $l \in \uparrow_{\gamma(a)} a \cap \downarrow_{\gamma(a)} a'$. Hence $a <^\sim l <^\sim a'$, i.e. $a <^\sim a'$ as required.

Loop filling. Suppose (a, L, U) is a loop in $[X]$. If $L \neq \emptyset$ then let $l = \max_\sim(L)$. By Lemma 5.1 $(l, \emptyset, -)$ is not a loop in $[X]$. We have $\gamma(l, \emptyset, -) = (a, -, \uparrow l) = (a, L, U)$.

Now consider the case $L = \emptyset$. a is not an empty loop since otherwise (a, L, U) wouldn't be a loop in $[X]$. Thus there is $\alpha \in X^{-\lambda}$ such that $\gamma(\alpha) = a$. Clearly, we can choose such α in $X^{-\lambda} - \gamma(X^{-\lambda})$. Then $(\alpha, \emptyset, -)$ is not a loop and

$$\gamma(\alpha, \emptyset, -) = (\gamma(\alpha), -, \uparrow \alpha) = (a, \emptyset, \uparrow \alpha) = (a, L, U).$$

Pencil linearity. Let $(a, L, U) \neq (a', L', U')$ be some faces in $[X]$ such that $\dot{\theta}(a, L, U) \cap \dot{\theta}(a', L', U') \neq \emptyset$. Then either $a \perp^\sim b$ or $a \perp^+ b$ or ($a = b$ and $L \neq L'$). In the first case we have $(a, L, U) \perp^\sim (a', L', U')$ and in the remaining cases we have $(a, L, U) \perp^+ (a', L', U')$.

To see the second part of the pencil linearity assume that $\check{a} = (a, L, U) \in [X]^{\varepsilon}$, $\check{b} = (b, L', U') \in [X]$, $x \in \delta(b)$, such that $(x, \downarrow b, -), (y, \downarrow b, -) \in \delta^{-\lambda}(b, L', U')$, and

$$\gamma\gamma(\check{a}) = \gamma(x, \downarrow b, -) \in \delta(y, \downarrow b, -)$$

i.e. for some $t \in \delta(y)$

$$(\gamma\gamma(a), -, \uparrow \gamma(a)) = (\gamma(x), -, \uparrow x) = (t, \downarrow y, -) \tag{7}$$

We need to show that either $\check{a} <^{\sim} \check{b}$ or $\check{a} <^{+} \check{b}$. From the characterization of $<^{\sim}$ and $<^{+}$ in $[X]$ it is enough to show that either $a <^{\sim} b$ or $a <^{+} b$. And for that, by Lemma 4.19.4, it is enough to show that $\gamma(a) <^{+} \gamma(b)$. We shall consider four cases separately:

1. $x, y \in T^{-\lambda}$;
2. $x \in T^{-\lambda}$ and $y \in T^{\lambda}$ and there is $l_y \in \mathcal{E}_{\gamma\gamma(a)}$ such that $l_y \leq^{+} y$;
3. $y \in T^{-\lambda}$ and $x \in T^{\lambda}$ and there is $l_x \in \mathcal{E}_{\gamma\gamma(a)}$ such that $l_x \leq^{+} x$;
4. $x, y \in T^{\lambda}$ and there are $l_x, l_y \in \mathcal{E}_{\gamma\gamma(a)}$ such that $l_x \leq^{+} x$ and $l_y \leq^{+} y$.

Case 1. If $x, y \in T^{-\lambda}$ then $\gamma\gamma(a) \in \iota(b)$. So, by pencil linearity in T, we get that either $a <^{\sim} b$ or $a <^{+} b$.

Case 2. In this case we have $\gamma(a) <^{\sim} l_y$. As $\gamma(x) = \gamma\gamma(a)$ we have either $x \perp^{\sim} \gamma(a)$ or $x \perp^{+} \gamma(a)$. We have $\gamma(a) \not<^{-} x$. Moreover, by Lemma 4.3, $x \not<^{+} \gamma(a)$, as $x \in T^{-\lambda}$ and $\gamma(a) \in T^{\lambda}$. So we have either either $\gamma(a) <^{+} x$ or $x <^{\sim} \gamma(a)$. In the former case we get immediately $\gamma(a) <^{+} \gamma(b)$. In the later case, we have

$$x <^{\sim} \gamma(a) <^{\sim} l_y, \qquad x <^{+} \gamma(b) >^{+} l_y. \tag{8}$$

So we have $\gamma\gamma(a) \leq^{+} \gamma(l_y) \leq^{+} \gamma\gamma(b)$. Thus, by Lemma 4.19, we have $\gamma(a) \perp^{+} \gamma(b)$ or $\gamma(a) \perp^{\sim} \gamma(b)$. Using (8) we see that of four conditions only the $\gamma(a) <^{+} \gamma(b)$ does not lead to a contradiction.

Case 3. First note that $\gamma(l_x) = \gamma(x) = \gamma\gamma(a)$ and hence $\gamma(a) \perp^{+} l_x$ or $\gamma(a) \perp^{\sim} l_x$. The inequality $\gamma(a) <^{\sim} l_x$ is impossible as $l_x \not\in \uparrow x = \uparrow \gamma(a)$. $\gamma(a) \leq^{+} l_x$ is impossible since l_x is an empty loop and $\gamma(a)$ is not. Finally, $l_x <^{+} \gamma(a)$ is impossible as $(\gamma(a), -, -)$ is a loop in $[X]$, and cannot contain any empty loops. So we have shown that $l_x <^{\sim} \gamma(a)$. As $\gamma\gamma(a) \in \delta(y)$ and $y \in T^{-\lambda}$, we have $\gamma(a) <^{-} y$ and $y \not<^{-} \gamma(a)$. As $y \in T^{-\lambda}$ and $\gamma(a) \in T^{\lambda}$, by Lemma 4.3, we cannot have $y <^{\sim} \gamma(a)$. So we must have either $\gamma(a) <^{+} y$ or $\gamma(a) <^{\sim} y$. If $\gamma(a) <^{+} y$ then clearly $\gamma(a) <^{+} \gamma(b)$. If $l_x <^{\sim} \gamma(a) <^{\sim} y$ then having $l_x <^{+} \gamma(b) >^{+} y$ we can easily verify, as before in (8), that we must have $\gamma(a) <^{+} \gamma(b)$.

Case 4. As $l_y <^{+} y$ and $\downarrow y = \mathcal{E}_{\gamma\gamma(a)} - \uparrow \gamma\gamma(a)$, we have $\gamma(a) <^{\sim} l_y$. As $\gamma\gamma(a) = \gamma(l_x)$, we also have $\gamma(a) \perp^{+} l_x$ or $\gamma(a) \perp^{\sim} l_x$. It is easy to see that the only inequality that does not lead to a contradiction in $l_x <^{\sim} \gamma(a)$. So we have

$$l_x <^{\sim} \gamma(a) <^{\sim} l_y, \qquad l_x <^{+} \gamma(b) >^{+} l_y. \tag{9}$$

From (9) we get, as before from (8), that $\gamma(a) <^{+} \gamma(b)$.

Globularity. Let us fix a face $\check{a} = (a, L, U) \in [X]_{\geq 2}$. As different a-cuts are parallel, i.e. they have the same domains and codomains, to verify Globularity condition in $[X]$ we don't need know in fact the very cut over a for which we check the condition. It is enough to know that it is a cut over a. So in the following $\check{a}$ will be treated as a cut over a, for which we don't bother to specify exactly which one it is. In the following γ and δ when applied to cuts are meant in $[X]$ and when applied to faces are meant in T.

γ-globularity. If $\check{a} \in [X]^{\varepsilon}$ then we have

$$\gamma\gamma(\check{a}) = (\gamma\gamma(a), -, \uparrow \gamma(a)) = \gamma(1_{(\gamma\gamma(a),-,\uparrow\gamma(a))}) = \gamma\delta(\check{a})$$

Now assume that $\check{a} \in [X]^{-\varepsilon}$. We need to verify the following three conditions:

(i) $\gamma\gamma(\check{a}) \in \gamma\delta(\check{a})$;

(ii) $\gamma\gamma(\check{a}) \notin \delta\dot{\delta}^{-\lambda}(\check{a})$;

(iii) $\gamma\delta(\check{a}) \subseteq \gamma\gamma(\check{a}) \cup \delta\dot{\delta}^{-\lambda}(\check{a})$.

Ad *(i)*. We have $(\varrho(a), \downarrow a, -) \in \delta(\check{a})$ and then using Lemma 5.2.2.(a) we have

$$\gamma\gamma(\check{a}) = (\gamma\gamma(a), -, \uparrow \gamma(a)) = (\gamma\varrho(a), -, \uparrow \gamma(a)) = \gamma(\varrho(a), \downarrow a, -)$$

Ad *(ii)*. Suppose that $x \in \delta(a)$ and $u \in \delta(x)$ so that $(x, \downarrow a, -) \in \dot{\delta}^{-\lambda}(\check{a})$ and $(u, \downarrow x, -) \in \delta\dot{\delta}^{-\lambda}(\check{a})$. If $u \neq \gamma\gamma(a)$ then clearly $(u, \downarrow x, -) \neq (\gamma\gamma(a), -, \uparrow \gamma(a))$. If $u = \gamma\gamma(a)$ then $x \in \delta^{\lambda}(a)$ and by characterization of loops in $[X]$, Lemma 5.1, there is $l \in \mathcal{E}^X_{\gamma\gamma(a)}$ such that $l \leq^+ x$. Then $l \leq^+ \gamma(a)$. So $\downarrow_{\gamma\gamma(a)} x \not\ni l \notin \uparrow \gamma(a)$ and hence

$$\gamma\gamma(\check{(a)}) = (\gamma\gamma(a), -, \uparrow \gamma(a)) \neq (u, \downarrow x, -) \in \delta(x, \downarrow a, -).$$

Ad *(iii)*. Let $x \in \delta(a)$ so that

$$\gamma(x, \downarrow a, -) = (\gamma(x), -, \uparrow x) \neq (\gamma\gamma(a), -, \uparrow \gamma(a)) = \gamma\gamma(\check{a}).$$

Then either $\gamma(x) \neq \gamma\gamma(a)$ or $\gamma(x) = \gamma\gamma(a)$ and there is $l \in \mathcal{E}^X_{\gamma\gamma(a)}$ so that $x <^{\sim} l$ and $\gamma(a) \not<^{\sim} l$. This implies that the face

$$y_0 = \min_{\sim}\{y \in \delta(a) : \gamma(x) \in \delta(y) \text{ and either } y \in T^{-\lambda} \text{ or } \exists_{l \in \mathcal{E}^X_{\gamma(x)}}\ l \leq^+ y\}$$

is well defined, i.e. the set over which the minimum is taken is not empty. Then we have $(y_0, \downarrow a, -) \in \dot{\delta}^{-\lambda}(\check{a})$ and $(\gamma(x), -, \uparrow x) = (\gamma(x), \downarrow y_0, -) \in \delta\dot{\delta}^{-\lambda}(\check{a})$.

δ-globularity. We consider separately two cases $\gamma(\check{a}) \in [X]^{\varepsilon}$ and $\gamma(\check{a}) \in [X]^{-\varepsilon}$. In the former case we need to verify two conditions

(i) $\dot{\delta}\delta(\check{a}) \subseteq \gamma\dot{\delta}^{-\lambda}(\check{a})$;

(ii) $\gamma\gamma\gamma(\check{a}) = \gamma\gamma\delta^{\varepsilon}(\check{a})$.

Ad *(i)*. Let $x \in \dot{\delta}(a)$ and $u \in \dot{\delta}(x)$ so that $(u, \downarrow x, -) \in \dot{\delta}\delta(\check{a})$. As T is an ordered face structure, $\dot{\delta}\delta(a) \subseteq \gamma\dot{\delta}^{-\lambda}(a)$ and hence $u \in \gamma\dot{\delta}^{-\lambda}(a)$. Thus there is a $y \in \dot{\delta}^{-\lambda}(a)$ such that $\gamma(y) = u$. From this follows that the face

$$y_1 = \max_{\sim}\{y \in \delta(a) : \gamma(y) = u,\ y <^{\sim} x \text{ and either } y \in T^{-\lambda} \text{ or } \exists_{l \in \mathcal{E}^X_{\gamma(x)}}\ l \leq^+ y\}$$

is well defined. Then we have $(y_1, \downarrow a, -) \in \dot{\delta}^{-\lambda}(\check{a})$ and, using Lemma 5.2.2.(b), we get

$$\gamma(y_1, \downarrow a, -) = (\gamma(y_1), -, \uparrow y_1) = (u, \downarrow x, -).$$

This shows *(i)*.

Ad *(ii)*. First note that if $\gamma(\check{a}) \in [X]^{\varepsilon}$ then $\gamma(a) \in X^{\varepsilon}$ and hence $\delta^{\varepsilon}(a) \neq \emptyset$. So $\delta^{\varepsilon}(\check{a}) \neq \emptyset$, as well. Thus we need to show that $\gamma\gamma\delta^{\varepsilon}(\check{a}) \subseteq \gamma\gamma\gamma(\check{a})$.

Fix $(x, \downarrow a, -) \in \delta^{\varepsilon}(\check{a})$ and $l \in \mathcal{E}^X_{\gamma\gamma\gamma(a)}$. It is enough to show that

$$\gamma(x) <^{\sim} l \quad \text{iff} \quad \gamma\gamma(a) <^{\sim} l. \tag{10}$$

Clearly $x \in \delta^{\varepsilon}(a)$. By Lemma 5.2.1 $\gamma\gamma(a) \perp^{\sim} l$. Since $l \in \mathcal{E}^X$ and $\gamma(x) \leq \gamma\gamma(a)$ we have $l \not\perp^{\sim} \gamma(x)$.

Thus ($\gamma\gamma(a) <^{\sim} l$ and $l <^{\sim} \gamma(x)$) or ($\gamma(x) <^{\sim} l$ and $l <^{\sim} \gamma\gamma(a)$) then $\gamma(x) \perp^{\sim} \gamma\gamma(a)$ and this is a contradiction. Therefore (10) holds. This shows *(ii)* and end up the case $\gamma(\check{a}) \in [X]^{\varepsilon}$.

In case $\gamma(\check{a}) \in [X]^{-\varepsilon}$ we need to verify the following four conditions

(i) $\dot{\delta}\delta(\check{a}) \subseteq \delta\gamma(\check{a}) \cup \gamma\dot{\delta}^{-\lambda}(\check{a})$;

(ii) $\delta\gamma(\check{a}) \subseteq \dot{\delta}\delta(\check{a})$;

(iii) $\delta\gamma(\check{a}) \cap \gamma\dot{\delta}^{-\lambda}(\check{a}) = \emptyset$;

(iv) $\gamma\gamma\delta^{\varepsilon}(\check{a}) \subseteq \theta\delta\gamma(\check{a})$.

Ad *(i)*. Fix $x \in \delta(a)$ and $u \in \delta(x)$ so that $(u, \downarrow x, -) \in \dot{\delta}\delta(\check{a})$. Assume that $(u, \downarrow x, -) \notin \delta\gamma(\check{a})$. Then either $u \notin \delta\gamma(a)$ or $u \in \delta\gamma(a)$ and there is $l \in \mathcal{E}^X_u$ such that $l <^{\sim} x$ and $l \not<^{\sim} \gamma(a)$. Then, by Lemma 5.2.2.(c), the face

$$y_2 = \underset{\sim}{\max}\{y \in \delta(a) : \gamma(y) = u, \text{ and either } y \in T^{-\lambda} \text{ or } \exists_{l \in \mathcal{E}^X_u}\, l \leq^{+} y\}$$

is well defined. Clearly, $(y_2, \downarrow a, -) \in \dot{\delta}^{-\lambda}(\check{a})$. By Lemma 5.2.2.(b), we have

$$(u, \downarrow x, -) = (u, -, \uparrow y_2) = \gamma(y_2, \downarrow a, -) \in \gamma\dot{\delta}(\check{a})$$

Ad *(ii)*. Fix $u \in \delta\gamma(a)$ so that $(u, \downarrow \gamma(x), -) \in \delta\gamma(\check{a})$. If $a \in X^{\varepsilon}$ then $\delta(\check{a}) = 1_{(u, \downarrow\gamma(x), -)}$ and hence $(u, \downarrow \gamma(x), -) \in \delta\delta(\check{a})$. So suppose that $a \in X^{-\varepsilon}$. Then the face

$$y_3 = \underset{\sim}{\min}\{y \in \dot{\delta}(a) : u \in \delta(y)\}$$

is well defined. By Lemma 5.2.2.(c), we have

$$(u, \downarrow \gamma(a), -) = (u, \downarrow y_3, -) \in \delta\delta(\check{a})$$

Ad *(iii)*. Let $u \in \delta\gamma(a)$ so that $(u, \downarrow \gamma(a), -) \in \delta\gamma(\check{a})$. We shall show that if $(u, \downarrow \gamma(a), -) \in \gamma\delta(\check{a})$ then $(u, \downarrow \gamma(a), -) \in [X]^{\lambda}$. So fix $z \in \delta(a)$ such that $(u, \downarrow a, -) = \gamma(z, \downarrow a, -) = (\gamma(z), -, \uparrow z)$. As $\gamma(z) = u \in \delta\gamma(a)$, z is a loop. If we were to have $l \in \mathcal{E}_u$ such that $l \leq^{+} z$ then $l \leq^{+} \gamma(a)$ and hence $l \notin \downarrow_u \gamma(a)$ and $l \notin \uparrow z$. Thus

$$(u, \downarrow \gamma(a), -) \neq (\gamma(z), -, \uparrow z)$$

contrary to the assumption.

Ad *(iv)*. Let $x \in \delta^{\varepsilon}(a)$ so that $(x, \downarrow a, -) \in \delta^{\varepsilon}(\check{a})$. We shall show that

$$\gamma\gamma(x, \uparrow a, -) = (\gamma\gamma(x), -, \uparrow \gamma(x)) \in \theta\delta\gamma(\check{a}). \tag{11}$$

Note that as T is an ordered face structure, we have $\gamma\gamma(x) \in \gamma\gamma\delta^{\varepsilon}(a) \subseteq \theta\delta\gamma(a)$.

First we claim that for any $t \in \delta\gamma(a)$ we have $t \not\perp^{+} \gamma(x)$. Fix $t \in \delta\gamma(a)$. As $\gamma(x) \in \gamma\dot{\delta}^{-\lambda}(a)$, using $\gamma\dot{\delta}^{-\lambda} \cap \delta\gamma(a) = \emptyset$ we get that $\gamma(x) \not\leq^{+} t$. Now suppose contrary, that $t <^{+} \gamma(x)$. Thus there is a flat upper path $t, x_1, \ldots, x_n, \gamma(x)$, with $r \geq 1$. As $\gamma(x)$ is a loop and x is an empty domain face, by Path Lemma, we have $x_i <^{+} x$ for $i = 1, \ldots, n$. In particular, there is a flat upper path in T, $x_1, a_1, \ldots, a_m, x$. As $x \in T^{\varepsilon}$ and $x_1 \in T^{-\varepsilon}$, for some $1 \leq j_0 \leq m$, we have $t \in \iota(a_{j_0})$.

Thus $t \in \delta\gamma(a) \cap \iota(a_{j_0})$. Hence by Lemma 4.8, we have $\gamma(a) <^+ \gamma(a_{j_0}) \leq^+ \gamma(a_s) = x$. But $x \in \delta(a)$ and we get a contradiction with strictness. This ends the proof of the claim.

Now let $u = \gamma\gamma(x)$. Using the claim it is easy to see that one of the following conditions holds:

(a) $u = \gamma\gamma\gamma(a)$ and $\varrho\gamma(a) <^\sim \gamma(x)$;

(b) $u \in \delta\gamma\gamma(a)$ and with $s_1 = \min_\sim\{s \in \delta\gamma(a) : u \in \delta(s)\}$ we have $\gamma(x) <^\sim s_1$;

(c) there are $s_0, s_1 \in \delta\gamma(a)$ such that $\gamma(s_0) = u \in \delta(s_1)$ and $s_0 <^\sim \gamma(x) <^\sim s_1$.

In each case we shall show (11).

Ad *(a)*. Using Lemma 5.2.2.(a), we have

$$(u, -, \uparrow \varrho\gamma(a)) \leq^+ (u, -, \uparrow \gamma(x)) \leq^+ (u, -, \uparrow \gamma\gamma(a)) \leq^+ (u, -, \uparrow \varrho\gamma(a))$$

Thus $\gamma\gamma(x, \downarrow a, -) = (u, -, \uparrow \varrho\gamma(a)) \in \gamma\delta\gamma(\breve{a})$.

Ad *(b)*. Using Lemma 5.2.1 and 5.2.2.(c), we have

$$(u, \downarrow \gamma\gamma(a), -) \leq^+ (u, \downarrow s_1, -) \leq^+ (u, \downarrow \gamma(x), -) \leq^+ (u, \downarrow \gamma\gamma(a), -)$$

Thus $\gamma\gamma(x, \downarrow a, -) = (u, \downarrow s_1, -) \in \delta\delta\gamma(\breve{a})$.

Ad *(c)*. Let s_1 be as above in (c) and s_0 maximal such as in (c), i.e.

$$s_0 = \max_\sim\{s \in \delta\gamma(a) : \gamma(s) = u,\ s <^\sim \gamma(x)\}.$$

Suppose $\uparrow s_0 \neq \uparrow \gamma(x)$. Then there is $l \in \mathcal{E}_u$ such that $s_0 <^\sim l <^\sim \gamma(x)$. Then by Lemma 5.2.2.(b) there is $t \in \delta\gamma(a)$ such that $l <^+ t$ and $s_0 <^\sim t <^\sim s_1$. We shall show that the existence of such a t leads to a contradiction. Note that t is a loop and that $\gamma(t) = u$. By the claim proven above it follows that $t \not\perp^+ \gamma(x)$. So, by pencil linearity, we should have $t \perp^\sim \gamma(x)$. But if $\gamma(x) <^\sim t$ then by transitivity we get $l <^\sim t$ and we get a contradiction with disjointness. On the other hand, it $t <^\sim \gamma(x)$ then, as other cases are easily excluded, we have that $s_0 <^\sim t$. But this contradicts the choice of s_0. Thus $\uparrow s_0 = \uparrow \gamma(x)$ holds and we have $\gamma\gamma(x, \downarrow a, -) = (u, -, \uparrow s_0) \in \gamma\delta\gamma(\breve{a})$. This ends *(iv)* and 1.

Ad 2. By 1. it is enough to show that if $f : S \to T$ is a monotone morphism such that $f(S) = X$ then there is a monotone isomorphism $g : S \to [X]$ making the triangle

$$\begin{tikzcd} S \arrow[rr, "g"] \arrow[dr, "f"'] & & {[X]} \arrow[dl, "\nu_X"] \\ & T & \end{tikzcd}$$

commutes. We put, for $a \in S$

$$g(a) = \begin{cases} (f(a), -, \uparrow f(a)) & \text{if } \alpha \in S^{-\lambda} - \gamma(S^{-\lambda}) \text{ such that } a = \gamma(\alpha), \\ (f(a), \emptyset, -) & \text{such that } a \not\in \gamma(S^{-\lambda}). \end{cases}$$

Note that if $a \in \gamma(S^{-\lambda})$ then there is a unique $\alpha \in S^{-\lambda} - \gamma(S^{-\lambda})$ such that $a = \gamma(\alpha)$. This shows that g is a well defined function. As monotone morphisms preserves and reflects $<^\sim$ (in particular f and ν_X does) it follows that g preserves $<^\sim$.

Before we verify the other properties of g let us make one observation. Fix $x \in X$ and let

$$y_{\min} = \min_{<^+}\{y' \in S : f(y) = x\}, \qquad y_{\max} = \max_{<^+}\{y' \in S : f(y) = x\}$$

and $y_{\min}, l_1, \ldots, l_k, y_{\max}$ be a flat upper $S - \gamma(S^{-\lambda})$-path from $y_{\min}$ to $y_{\max}$. Then $\mathcal{E}^X_x = \{f(l_i)\}_{1\leq i\leq k}$.

With this description it is easy to see that g preserves both γ and δ. Fix $a \in S_{\geq 1}$. Then if $b \in S^{-\lambda} - \gamma(S^{-\lambda})$ and $\gamma(b) = \gamma(a)$ we have

$$g(\gamma(a) = (f(\gamma(a)), -, \uparrow f(b)) = (\gamma(f(a)), -, \uparrow f(a)) = \gamma(f(a)).$$

If $\gamma(a) \notin \gamma(S^{-\lambda})$ we can show that $g(\gamma(a) = (\gamma(f(a))$ in a similar way.

If $a \in S^{-\varepsilon}$ then we have

$$g(\delta(a) = \{(f(x), -, \uparrow f(b)) : \gamma(b) = x \in \delta(a),\ b \in S^{-\lambda} - \gamma(S^{-\lambda})\}\cup$$

$$\cup\{(f(x), \emptyset, -) : x \in \delta(a),\ x \notin \gamma(S^{-\lambda})\} =$$

$$= \{(f(x), \downarrow f(a), -) : x \in \delta(a)\} = \delta(g(a)).$$

If $a \in S^{\varepsilon}$ we clearly have $g(\delta(a) = \delta(g(a))$, as well.

It remains to show that g is a bijection. Suppose g is not one-to-one. Let $a, b \in S$ such that $g(a) = g(b)$ and $a \neq b$. In particular $f(a) = f(b)$. By Lemma 4.25 we can assume that there is $\alpha \in S^u - \gamma(S^{-\lambda})$ such that $a = \delta(\alpha)$ and $\gamma(\alpha) = b$. Then

$$g(a) = (f(a), L_a, U_a) = (f(b), L_b, U_b) = g(b).$$

But $U_a \ni f(\alpha) \notin U_b$, and we get a contradiction, i.e. g is one-to-one.

To see that g is onto fix $(a, L, U) \in [X]$. First assume that $L \neq \emptyset$. Then let $\alpha = \max_{<^{\sim}}(L) \in X$ and let $\alpha' = \min_{<^{+}}\{\alpha'' \in S : f(\alpha'') = \alpha\}$. Clearly $\alpha' \in S^{-\lambda} - \gamma(S^{-\lambda})$. Then

$$g(\gamma(\alpha')) = (f(\gamma(\alpha')), -, \uparrow \alpha') = (a, L, U).$$

If $L = \emptyset$ then with $b = \min_{<^{+}}\{b' : f(b') = a\}$ we have

$$g(b) = (f(b), \emptyset, -) = (a, L, U)$$

in this case (a, L, U) is in the image of g, as well. Thus g is onto and hence a bijection.

Ad 3. If $\mathcal{E}^X$ is empty there is exactly one cut $(x, \emptyset, \emptyset)$ over any face $x \in X$. So ν_X is an embedding in that case. If there is $l \in \mathcal{E}^X$ then $(\gamma(l), \downarrow l, -) \neq (\gamma(l), -, \uparrow l)$ and $\nu_X(\gamma(l), \downarrow l, -) = \nu_X(\gamma(l), -, \uparrow l)$, i.e. ν_X is not an embedding.

Ad 4. First note that for any $k \in \omega$, $[X]_k - \delta([X]^{-\lambda}_{k+1}) = \{(a, -, \emptyset) : a \in X - \delta(X^{-\lambda}_{k+1})\}$. In particular, we have $size([X])_k = |X - \delta(X^{-\lambda}_{k+1}|$. Now fix a and k so that $a \in T_k - X_k$, a is a face of the maximal dimension not in X. Then $T_{k+1} = X_{k+1}$ and hence $a \notin \delta(T^{-\lambda}_{k+1})$ so we have

$$a \in T_k - \delta(T^{-\lambda}_{k+1}) = T_k - \delta(X^{-\lambda}_{k+1}) \supset X_k - \delta(X^{-\lambda}_{k+1}) \not\ni a$$

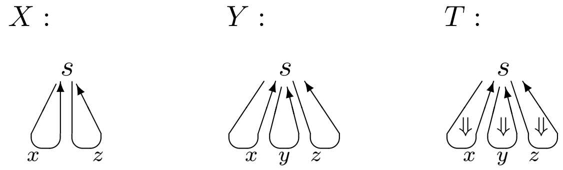
Thus

$$size(T)_k = |T_k - \delta(T^{-\lambda}_{k+1})| > |X_k - \delta(X^{-\lambda}_{k+1})| = size([X]_k.$$

As $size(T)_l = size([X]_l$, for $l > k$, we have $size(T) > size([X])$. $\Box$

Even if the equivalence classes of objects of the comma category $\mathbf{oFs} \downarrow T$ corresponds to the elements of the poset $Convex((T)$ of convex subsets of T we are not saying that $\mathbf{oFs} \downarrow T$ and $Convex((T)$ are equivalent as categories. In fact, if $X \subset Y$ are convex subsets of T, it does not mean that there is a morphism from $[X]$ to $[Y]$ over T as following example shows.

Example 1. Let $X \subset Y$ be convex subsets of an ordered face structure T as shown on the diagram below.



Clearly $X \subseteq Y$. And the stretching of X and Y gives

$$[X]: \quad (s, \emptyset, -) \xrightarrow{x} (s, \{x\}, -) \xrightarrow{z} (s, \{x, z\}, -)$$

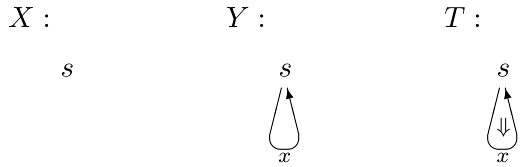
and

$$[Y]: \quad (s, \emptyset, -) \xrightarrow{x} (s, \{x\}, -) \xrightarrow{y} (s, \{x, y\}, -) \xrightarrow{z} (s, \{x, y, z\}, -)$$

respectively. Clearly there is no map from $[X]$ to $[Y]$ over T.

Moreover is such a comparison map exists it does not need to be unique, as the following example shows.

Example 2.



Clearly $X \subseteq Y$. The stretching of Y gives

$$[Y]: \qquad (s, \emptyset, -) \xrightarrow{x} (s, \{x\}, -)$$

Thus from $[X] = X$ to $[Y]$ there are two monotone morphisms and both of them commutes over T.

6 Quotients of positive face structures

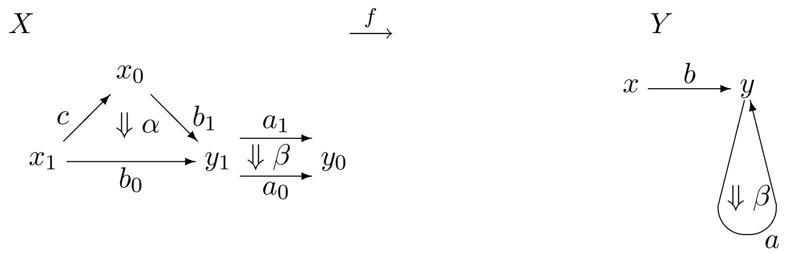
Positive face structures can be thought of as ordered face structures without empty-domain faces. If we collapse to 'identity' some domains of some unary faces which are not codomains of any other face in a positive face structure we obtain an ordered face structures which is not necessarily positive. In this section we shall describe this construction of a quotient of a positive face structure and prove its properties. In the next section we shall show that, we can obtain any ordered face structure in this way.

Let T, S be ordered (or positive) face structures. We say that $f : T \to S$ is a *collapsing morphism* if $f = \{f_k : T_k \to S_k \cup 1_{S_{k-1}}\}_k \in \omega$, f preserves $<^{\sim}$, γ and δ i.e. for $k > 0$, $a, b \in S_k$, we have

1. if $a <^{\sim} b$ and $f(a), f(b) \in T_k$ then $f(a) <^{\sim} f(b)$
2. $f(\gamma(a)) = \gamma(f(a))$ and $f(\delta(a)) \equiv_1 \delta(f(a))$.

The *kernel* of the morphism f is the set of faces $ker(f) = f^{-1}(1_S)$.

Example.



where f is given by: $x_i \mapsto x$, $y_i \mapsto y$, $a_0 \mapsto a$, $a_1 \mapsto 1_y$, $b_i \mapsto b$, $c \mapsto 1_x$, $\alpha \mapsto 1_b$, $\beta \mapsto \beta$. We have $ker(f)\{c, \alpha, a_1\}$.

Remark. The collapsing morphisms do not compose. If a map $f : X \to Y$ sends α to 1_a and a map $g : Y \to Z$ sends a to 1_x then $g \circ f$ should send α to 1_{1_x} but we don't consider such faces in ordered face structures.

Let T be a positive face structure, T^u unary faces in T. A set $\mathcal{J} \subseteq T^u$ is an *ideal* iff

1. $\mathcal{J} \cap \gamma(T) = \emptyset$;
2. $\mathcal{J} \cap \delta(\mathcal{J}) = \emptyset$.

$\sim_{\mathcal{J}_{k+1}}$ is the least equivalence relation on T_k containing $\sim'_{\mathcal{J}_{k+1}}$; for $x, x' \in T_k$ we have $x \sim'_{\mathcal{J}_{k+1}} x'$ iff there is $a \in \mathcal{J}_{k+1}$ such that $x = \delta(a)$ and $\gamma(a) = x'$. The kernel of any collapsing morphism is an ideal.

We define an ordered hypergraph $T_{/\mathcal{J}}$ the quotient of T by the ideal $\mathcal{J}$:

1. $T_{/\mathcal{J},k} = (T_k - \mathcal{J}_k)_{/\sim_{\mathcal{J}_{k+1}}}$,
2. $\gamma_{/\mathcal{J}} : T_{/\mathcal{J},k+1} \longrightarrow T_{/\mathcal{J},k}$, $\quad \delta_{/\mathcal{J}} : T_{/\mathcal{J},k+1} \longrightarrow T_{/\mathcal{J},k} \sqcup \mathbf{1}_{T_{/\mathcal{J},k-1}}$,
$$\gamma_{/\mathcal{J}}([a]) = [\gamma^T(a)], \qquad \delta_{/\mathcal{J}}([a]) = \begin{cases} 1_{[\gamma^T\gamma^T(a)]} & \text{if } \delta(a) \subseteq \mathcal{J}, \\ \{[x] : x \in \delta^T(a) - \mathcal{J}\} & \text{otherwise.} \end{cases}$$
for $[a] \in T_{/\mathcal{J},k+1}$,
3. $[x] <^{T_{/\mathcal{J}},k,\sim} [x']$ iff $x <^{T_k,-} x'$, for $[x], [x'] \in T_{/\mathcal{J},k}$.

We define $q_{\mathcal{J}} : T \longrightarrow T_{/\mathcal{J}}$ by
$$q_{\mathcal{J}}(a) = \begin{cases} 1_{[\gamma(a)]} & \text{if } a \in \mathcal{J}, \\ [a] & \text{otherwise.} \end{cases}$$

In the remaining part of the section we are going to prove

Theorem 6.1 *Let T be a positive face structure, $\mathcal{J} \subseteq T^u$ is an ideal. Then $T_{/\mathcal{J}}$ is an ordered face structure and $q_{\mathcal{J}}$ is a collapsing morphism with kernel $\mathcal{J}$.*

Before we prove this theorem we need some Lemmas.

The class $\mathcal{L}_{\mathcal{J}}$ of *$\mathcal{J}$-loops* in T, is defined as the least set $X \subseteq T$ such that
$$\text{if } \alpha \in T \text{ and } \delta(\alpha) \subseteq \mathcal{J} \cup X \text{ then } \gamma(\alpha) \in X.$$

Note that $\mathcal{L}_{\mathcal{J}} \cap \mathcal{J} = \emptyset$.

The following three lemmas concerns positive face structures and their quotients.

Lemma 6.2 *Let T be a positive face structure, $\mathcal{J} \subseteq T^u$ is an ideal. Then*

1. *If $a \in T$ then the following are equivalent:*
   (a) *$\gamma(a) \in T^u$;*

(b) $\delta(a) \subseteq T^u$;

(c) there is $v \in \delta\gamma(a)$ and a $\delta^u(a)$-path from v to $\gamma\gamma(a)$.

2. $\mathcal{L}_{\mathcal{J}} \subseteq T^u$.

3. If $a <^+ b$ and $b \in T^u$ then $a \in T^u$.

Proof. 2. and 3. follows from 1. We shall prove 1.
Fix $a \in T$. Let $x_0 = \max_{<^\sim}(\delta(a))$. From globularity we have

$$\delta\gamma(a) = \delta\delta(a) - (\gamma\delta(a) - \gamma\gamma(a)) \quad \text{and} \quad (\gamma\delta(a) - \gamma\gamma(a)) \subseteq \delta\delta(a)$$

Recall that if $x, y \in \delta(a)$ then $\delta(x) \cap \delta(y) = \emptyset$. Using these observations, we get

$$|\delta\gamma(a)| = |\delta\delta(a)| - |\gamma\delta(a) - \gamma\gamma(a)| = \bigcup_{x \in \delta(a)} |\delta(x)| - |\gamma(\delta(a) - x_0)| =$$

$$= \bigcup_{x \in \delta(a)} |\delta(x)| - (|\delta(a)| - 1) = 1 + \bigcup_{x \in \delta(a)} (|\delta(x)| - 1)$$

This shows that the set $\delta\gamma(a)$ is a singleton if and only if for $x \in \delta(a)$ the sets $\delta(x)$ are singletons. This shows that *(a)* is equivalent to *(b)*.

Clearly, *(b)* implies *(c)*. We shall show the converse. Let $v \in \delta\gamma(a)$ and $v, x_1, \ldots, x_k, \gamma\gamma(a)$ be an upper $\delta^u(a)$-path from v to $\gamma\gamma(a)$. We shall show that $\delta(a) = \{x_1, \ldots, x_k\}$. Suppose contrary, that there $y \in \delta(a) - \{x_1, \ldots, x_k\}$. Let $y = y_0, y_1, \ldots, y_r, \gamma\gamma(a)$ be an upper $\delta(a)$-path to $\gamma\gamma(a)$. Hence $\gamma(y_r) = \gamma(x_k)$ and then $y_r = x_k$. Let $r' = \min\{i : y_i \not\in \{x_1, \ldots, x_k\}\}$. Then $r' < r$ and $y_{r'+1} = x_j$ for some j. If $j = 1$ then $\gamma(y_{r'}) = v \in \delta(x_1) \subseteq \delta\gamma(a)$. But then $v \in \delta\gamma(a) \cap \gamma\delta(a)$, which contradicts globularity. If $j > 1$ then, as $x_j \in T^u$, we have $\gamma(y_{r'}) = \gamma(x_{j-1})$. But, as $y_{r'}, x_{j-1} \in \delta(a)$ we must have $y_{r'} = x_{j-1}$ contrary to the choice of r'. Thus $\delta(a) = \{x_1, \ldots, x_k\}$ and *(c)* implies *(b)* as well. □

The following lemma describes some basic properties $T_{/\mathcal{J}}$.

Lemma 6.3 *Let T be a positive face structure, $\mathcal{J} \subseteq T^u$ is an ideal, $a, x, y \in T - \mathcal{J}$. Then*

1. $[x]_{\sim_{\mathcal{J}}} = [y]_{\sim_{\mathcal{J}}}$ *iff $x = y$ or there is an upper $\mathcal{J}$-path from x to y or from y to x.*

2. *The functions $\gamma_{/\mathcal{J}}$ and $\delta_{/\mathcal{J}}$ are well defined.*

3. $[a] \in T^{\varepsilon}_{/\mathcal{J}}$ *if and only if* $\delta(a) \subseteq \mathcal{J}$.

4. $[a] \in T^{\lambda}_{/\mathcal{J}}$ *if and only if* $a \in \mathcal{L}_{\mathcal{J}}$.

Proof. Ad 1. It is enough to note that it $a, b \in \mathcal{J}$ then $a, b \in T^u - \gamma(T)$ and therefore if $\gamma(a) = \gamma(b)$ or $\delta(a) \cap \delta(b) \neq \emptyset$ then $a = b$.

Ad 2. Since for $a \in T$, the value $\gamma^{T/\mathcal{J}}[a]$ and $\delta_{/\mathcal{J}}[a]$ depend only on $\gamma(a)$ and $\delta(a)$ (and not on a itself) it is enough to show that if $a \sim'_{\mathcal{J}} b$ then $\gamma(a) = \gamma(b)$ and $\delta(a) = \delta(b)$.

So assume that there is $\alpha \in T$ such that $a = \delta(\alpha)$ and $\gamma(\alpha) = b$. Since T is a positive face structures $\gamma(a) \cap \delta(a) = \emptyset$. Thus using globularity (in positive face structures), we have

$$\gamma(b) = \gamma\gamma(\alpha) = \gamma\delta(\alpha) - \delta\delta(\alpha) = \gamma(a) - \delta(a) = \gamma(a)$$

and

$$\delta(b) = \delta\gamma(\alpha) = \delta\delta(\alpha) - \gamma\delta(\alpha) = \delta(a) - \gamma(a) = \delta(a)$$

as required.

Ad 3. This follows immediately from the definition of $\delta_{/\mathcal{J}}([a])$.

Ad 4. We argue by induction on the height $ht(a)$. The inductive assumption is:

Ind_n: for $a \in T - \mathcal{J}$ such that $ht(a) = n$ we have: $a \in \mathcal{L}_{\mathcal{J}}$ iff $[a] \in T^{\lambda}_{/\mathcal{J}}$.

We can assume that $a \in T^u$, as each of the conditions $a \in \mathcal{L}_{\mathcal{J}}$ and $[a] \in T^{\lambda}_{/\mathcal{J}}$ implies it.

If $ht(a) = 0$ then neither $a \in \mathcal{L}_{\mathcal{J}}$ nor $[a] \in T^{\lambda}_{/\mathcal{J}}$. Hence Ind_0 holds.

Assume that $ht(a) = 1$. Let $\alpha \in T - \gamma(T)$ such that $\gamma(\alpha) = a$.

Suppose that $a \in \mathcal{L}_{\mathcal{J}}$. Then $\delta(\alpha) \subseteq \mathcal{J}$. Hence $\delta(a) \sim_{\mathcal{J}} \gamma(a)$ and $[a] \in T^{\lambda}_{/\mathcal{J}}$.

On the other hand, if $[a] \in T^{\lambda}_{/\mathcal{J}}$, then $[\gamma(a)] = \gamma_{/\mathcal{J}}([a]) = \delta_{/\mathcal{J}}([a]) = [\delta(a)]$. So there is a $\mathcal{J}$-path from $\delta(a)$ to $\gamma(a)$. As $ht(a) = 1$ this must be a $\delta(\alpha)$-path. Since it is a T^u-path, we have $\delta(\alpha) \subseteq \mathcal{J}$. Thus $a = \gamma(\alpha) \in \mathcal{L}_{\mathcal{J}}$.

Finally, assume that $ht(a) = n > 1$. Let $\alpha \in T - \gamma(T)$ such that $\gamma(\alpha) = a$. As $a \in T^u$, so $\delta(\alpha) \subseteq T^u$. Let $a_1, \ldots, a_k$ be the lower path containing all elements of $\delta(\alpha)$.

First suppose that $a \in \mathcal{L}_{\mathcal{J}}$. Then $\delta(\alpha) \subseteq \mathcal{L}_{\mathcal{J}} \cup \mathcal{J}$. If $a_i \in \mathcal{J}$ then, by def $\delta(a_i) \sim_{\mathcal{J}} \gamma(a_i)$. If $a_i \in \mathcal{L}_{\mathcal{J}}$ then, as $ht(a_i) < n$, by induction hypothesis $[a_i]$ is a loop. But this means that $\delta(a_i) = [\gamma(a_i)]$ and in this case again we have that $\delta(a_i) \sim_{\mathcal{J}} \gamma(a_i)$. By transitivity of $\sim_{\mathcal{J}}$ we have $\delta(a) = \delta(a_1) \sim_{\mathcal{J}} \gamma(a_k) = \gamma(a)$, i.e. $[a]$ is a loop in $T_{/\mathcal{J}}$, as required.

Now suppose that $[a] \in T^{\lambda}_{/\mathcal{J}}$. Thus there is an upper $\mathcal{J}$-path $\delta(a), b_1, \ldots, b_m, \gamma(a)$. We claim that there are numbers $0 = m_0 < m_1 < m_2 < \ldots < m_k = m$ such that

(i) either $m_i = m_{i-1} + 1$ and $a_i = b_{m_i} \in \mathcal{J}$

(ii) or $b_{m_{i-1}+1}, \ldots, b_{m_i}$ is a path from $\delta(a_i)$ to $\gamma(a_i)$ (i.e. $\delta(a_i) \in \delta(b_{m_{i-1}+1})$ and $\gamma(a_i) = \gamma(b_{m_i})$).

Having the above claim it follows that either $a_i \in \mathcal{J}$ or $[a_i] \in T^{\lambda}_{\mathcal{J}}$, for $i = 1, \ldots, k$. As $ht(a_i) < n$, by inductive hypothesis this means that $a_i \in \mathcal{L}_{\mathcal{J}} \cup \mathcal{J}$, for $i = 1, \ldots, k$, i.e. $\delta(\alpha) \subseteq \mathcal{L}_{\mathcal{J}} \cup \mathcal{J}$. So by definition of $\mathcal{L}_{\mathcal{J}}$, $a = \gamma(\alpha) \in \mathcal{L}_{\mathcal{J}}$, as required.

It remains to prove the claim. Suppose contrary, that the claim is not true. Let $1 \leq i_0 \leq k$ be the least number for which it does not hold, i.e. we have $m_0, m_1, \ldots, m_{i_0-1}$ satisfying (**i**) or (**ii**). In particular, either $i_0 = 1$ or $\gamma(b_{m_{i_0-1}}) = \gamma(a_{i_0-1}) = \delta(a_{i_0})$. So $\delta(b_{m_{i_0}}) = \delta(a_{i_0})$. As (**i**) does not hold $a_{i_0} \neq b_{m_{i_0}}$. Since $b_{m_{i_0}} \in T - \gamma(T)$, $b_{m_{i_0}} <^+ a_{i_0}$. As (**ii**) does not hold, by Path lemma (for positive face structures), $b_i <^+ a_{i_0}$, for $i = m_{i_0-1} + 1, \ldots, m$ and $\gamma(a_k) = \gamma(b_m) \neq \gamma(a_{i_0})$. Again by Path lemma, the upper path $\delta(a_{i_0}), b_{m_{i_0-1}+1}, \ldots, b_m, \gamma(a_k)$ can be extended to an upper path reaching $\gamma(a_{i_0})$:

$$\delta(a_{i_0}), b_{m_{i_0-1}+1}, \ldots, b_m, \gamma(a_k), c_1, \ldots, c_r, \gamma(a_{i_0})$$

But this means that we have both $a_{i_0} \leq^- a_k$ and $a_{i_0} <^+ a_k$. This contradicts the disjointness and ends the proof of the claim and the Lemma. $\Box$

The following lemma describes some relations between pathes in T and $T_{/\mathcal{J}}$.

Lemma 6.4 *Let T be a positive face structure, $\mathcal{J} \subseteq T^u$ is an ideal. $x, y, a \in T - \mathcal{J}$. Then*

1. *If $a \not\in \gamma(T)$ then $[x]_{\sim_{\mathcal{J}}} \in \delta([a]_{\sim_{\mathcal{J}}})$ iff there are $y \in \delta(a) - \mathcal{J}$ and a $\mathcal{J}$-path (possibly empty) from x to y.*

2. *If $a_1, \ldots, a_k$ is a flat path in $T - \gamma(T)$ then $< [a_i]_{\sim_{\mathcal{J}}} : 1 \leq i \leq k,\, a_i \not\in \mathcal{J} \cup \mathcal{L}_{\mathcal{J}} >$ is a flat path in $T_{/\mathcal{J}} - \gamma_{/\mathcal{J}}(T^{-\lambda}_{/\mathcal{J}})$.*

3. *Assume $a_1, \ldots, a_k \in T - (\mathcal{J} \cup \mathcal{L}_{\mathcal{J}})$. Then $[a_1]_{\sim_{\mathcal{J}}}, \ldots, [a_k]_{\sim_{\mathcal{J}}}$ is a flat path in $T_{/\mathcal{J}} - \gamma_{/\mathcal{J}}(T^{-\lambda}_{/\mathcal{J}})$ iff for $1 \leq j < k$ there there is a $\mathcal{J}$-path $b_{j,1}, \ldots, b_{j,l_j}$ so that*

$$a_1, b_{1,1}, \ldots, b_{1,l_1}, a_2, b_{2,1}, \ldots, b_{2,l_2}, a_3, \ldots, a_{k-1}, b_{k-1,1}, \ldots, b_{k-1,l_{k-1}} a_k$$

   is a path in T.

4. *$[a]_{\sim_{\mathcal{J}}} \in T_{/\mathcal{J}} - \gamma(T^{-\lambda}_{/\mathcal{J}})$ iff there is $a' \in T - \gamma(T)$ such that $a' \sim_{\mathcal{J}} a$.*

5. *$[x]_{\sim_{\mathcal{J}}} <^{T_{/\mathcal{J}},+} [y]_{\sim_{\mathcal{J}}}$ iff $x <^{T,+} y$ and the upper $(T - \gamma(T))$-path from x to y is not a $\mathcal{J}$-path.*

Proof. 1. follows easily from pencil linearity of positive face structures, 2. is easy and 3. is a consequence of 1.

Ad 4. $\Rightarrow$: Let $a' = \min_{<^{T,+}}([a])$, i.e. a' is the least element of $[a]$. Suppose that $a' \in \gamma(T)$, i.e. there is $\alpha \in T$ such that $\gamma(\alpha) = a'$. Clearly, we can assume that $\alpha \not\in \gamma(T)$. If $\alpha \in \mathcal{J} \subseteq T^u$ then $\delta(\alpha) \sim_{\mathcal{J}} a'$ and $\delta(\alpha) <^+ a'$, contrary to the choice of a'. If $\alpha \not\in \mathcal{J}$ then $\gamma_{/\mathcal{J}}([\alpha]) = [a]$ and by assumption $[\alpha] \in T^{\lambda}_{/\mathcal{J}}$. But then by description of the loops is $T_{/\mathcal{J}}$, $\alpha \in \gamma(T)$, contrary to the choice of α. The contradiction shows that $a' \not\in \gamma(T)$, as required.

$\Leftarrow$: Suppose $a \in T - \gamma(T)$. We need to show that $[a] \in T_{/\mathcal{J}} - \gamma(T^{-\lambda}_{/\mathcal{J}})$.

Suppose contrary, that there is $\alpha \in T - (\mathcal{J} \cup \mathcal{L}_{\mathcal{J}})$ such that $\gamma_{/\mathcal{J}}([\alpha]) = a$. Since $a \not\in \gamma(T)$ there is an upper $\mathcal{J}$-path $a, \alpha_1, \ldots, \alpha_k, \gamma(\alpha)$. Since $\alpha_k \in \mathcal{J} \subseteq T - \gamma(T)$ and $\alpha \not\in \mathcal{J}$, by pencil linearity in positive face structures, we have that $\alpha_k <^k \alpha$. By Path Lemma, (since $a \not\in \gamma(T)$) either $a \in \delta(\alpha)$ or there is $i < k$ such that $\gamma(\alpha_i) \in \delta(\alpha)$. Using the characterization of loops in $T_{/\mathcal{J}}$ we get in either case that $[\alpha]$ is a loop in $T_{/\mathcal{J}}$, i.e. $\alpha \in \mathcal{L}_{\mathcal{J}}$ contrary to the assumption.

Ad 5. The 'if' part is obvious. We shall show the 'only if' part. The essential argument consists of showing that in case the path from $[x]$ to $[y]$ has length 1 the conclusion hold. Then use induction.

So assume that $[x], [a], [y]$ is an upper path in $T_{/\mathcal{J}}$ with $[a] \in T^{-\lambda}_{/\mathcal{J}}$. Thus we have $x' \in \delta(a) - \mathcal{J}$ so that one of the following four cases holds. There are $\mathcal{J}$-pathes

1. from x to x' and from $\gamma(a)$ to y;

2. from x to x' and from y to $\gamma(a)$;

3. from x' to x and from $\gamma(a)$ to y;

4. from x' to x and from y to $\gamma(a)$.

In case 1. the conclusion follows immediately. The case 4. is most involved of 2., 3., and 4. and we will deal with this case only.

Let $x', b_1, \ldots, b_k, x$ and $y, c_1, \ldots, c_r, \gamma(a)$ be (non-empty) $\mathcal{J}$-pathes. We have $x' \in \delta(b_1) \cap \delta(a)$. As $b_1 \in \mathcal{J}$ and $a \not\in \mathcal{J}$ we have $b_1 <^+ a$. As $[a] \in T^{-\lambda}_{/\mathcal{J}}$, and $b_1, \ldots, b_k$ is a $\mathcal{J}$-path we have $\gamma(b_i) \neq \gamma(a)$, for $i = 1, \ldots, k$. By Path Lemma, we have a (non-empty) upper $T - \gamma(T)$-path $x, b'_1, \ldots, b'_l, \gamma(a)$. As $[a]$ is not a loop $b'_1, \ldots, b'_l$ is not a $\mathcal{J}$-path. Since $\gamma(c_r) = \gamma(b'_l)$ and both $c_1, \ldots, c_r$ and $b'_1, \ldots, b'_l$ are $T - \gamma(T)$-pathes, it follows that one is the end-part of the other. As the former is a

$\mathcal{J}$-path and the latter is not, $c_1, \ldots, c_r$ is the end $b'_1, \ldots, b'_l$. Thus we have an upper path $x, b'_1, \ldots, b'_{l-r}, y$ which is not a $\mathcal{J}$-path, as required. $\Box$

Proof of Theorem 6.1. We shall check that $T_{/\mathcal{J}}$ satisfies all the conditions of the definition of an ordered face structure.

Local discreteness, *Strictness*, and *Loop-filling* are obvious from the Lemmas above.

Globularity. First we shall spell the definitions of the sets involved. For $a \in T_{\geq 2} - \mathcal{J}$, we have

$$\gamma_{/\mathcal{J}}\gamma_{/\mathcal{J}}([a]) = [\gamma\gamma(a)]$$

$$\delta_{/\mathcal{J}}\gamma_{/\mathcal{J}}([a]) = \begin{cases} [\gamma\gamma(a)] & \text{if } \delta(a) \subseteq \mathcal{J}, \\ 1_{[\gamma\gamma\gamma(a)]} & \text{if } \delta\gamma(a) \subseteq \mathcal{J}, \\ \{[x] : x \in \delta\gamma(a) - \mathcal{J}\} & \text{otherwise.} \end{cases}$$

$$\gamma_{/\mathcal{J}}\delta_{/\mathcal{J}}([a]) = \begin{cases} [\gamma\gamma(a)] & \text{if } \delta(a) \subseteq \mathcal{J}, \\ \{[\gamma(x)] : x \in \delta(a) - \mathcal{J}\} & \text{otherwise.} \end{cases}$$

$$\delta_{/\mathcal{J}}\delta_{/\mathcal{J}}([a]) = \begin{cases} [\gamma\gamma(a)] & \text{if } \delta(a) \subseteq \mathcal{J}, \\ \{[u] : \exists_x u \in \delta(x) - \mathcal{J},\ x \in \delta(a) - \mathcal{J}\} \cup & \\ \{1_{[\gamma\gamma(x)]} : \delta(x) \subseteq \mathcal{J},\ x \in \delta(a) - \mathcal{J}\} & \text{otherwise.} \end{cases}$$

$$\dot{\delta}^{-\lambda}_{/\mathcal{J}}([a]) = \begin{cases} \emptyset & \text{if } \delta(a) \subseteq \mathcal{J}, \\ \{[x] : x \in \delta(a) - (\mathcal{L}_{\mathcal{J}} \cup \mathcal{J})\} & \text{otherwise.} \end{cases}$$

$$\gamma_{/\mathcal{J}}\dot{\delta}^{-\lambda}_{/\mathcal{J}}([a]) = \begin{cases} \emptyset & \text{if } \delta(a) \subseteq \mathcal{J}, \\ \{[\gamma(x)] : x \in \delta(a) - (\mathcal{L}_{\mathcal{J}} \cup \mathcal{J})\} & \text{otherwise.} \end{cases}$$

$$\delta_{/\mathcal{J}}\dot{\delta}^{-\lambda}_{/\mathcal{J}}([a]) = \begin{cases} \emptyset & \text{if } \delta(a) \subseteq \mathcal{J}, \\ \{[u] : \exists_x u \in \delta(x) - \mathcal{J},\ x \in \delta(a) - (\mathcal{L}_{\mathcal{J}} \cup \mathcal{J})\} \cup & \\ \{1_{[\gamma\gamma(x)]} : \delta(x) \subseteq \mathcal{J},\ x \in \delta(a) - \mathcal{J}\} & \text{otherwise.} \end{cases}$$

Thus if $\delta(a) \subseteq \mathcal{J}$ it is easy to see that globularity holds. So we assume that $\delta(a) \not\subseteq \mathcal{J}$.

γ-globularity. We shall show:

1. $[\gamma\gamma(a)] \in \gamma_{/\mathcal{J}}\delta_{/\mathcal{J}}([a])$
2. $[\gamma\gamma(a)] \not\in \delta_{/\mathcal{J}}\dot{\delta}^{-\lambda}_{/\mathcal{J}}([a])$
3. $\gamma_{/\mathcal{J}}\delta_{/\mathcal{J}}([a]) - [\gamma\gamma(a)] \subseteq \delta_{/\mathcal{J}}\dot{\delta}^{-\lambda}_{/\mathcal{J}}([a])$

Ad 1. Let $x_0, x_1, \ldots, x_k$ be a lower $\delta(a)$-path such that $\gamma(x_k) = \gamma\gamma(a)$, $x_0 \not\in \mathcal{J}$ and $x_i \in \mathcal{J}$, for $i > 0$ (k is possibly equal 0). Such a sequence exists since $\delta(a) \not\subseteq \mathcal{J}$ and is unique since $x_i \in T^u$, for $i > 0$. Then

$$[\gamma\gamma(a)] = [\gamma(x_0)] \in \gamma_{/\mathcal{J}}\delta_{/\mathcal{J}}([a]),$$

as required.

Ad 2. Suppose contrary, that there is $x \in \delta(a) - (\mathcal{L}_{\mathcal{J}} \cup \mathcal{J})$ (i.e. $[x]$ is not a loop) and $u \in \delta(x) - \mathcal{J}$ such that $[u] = [\gamma\gamma(a)]$. Let $u, x = x_0, x_1, \ldots, x_k, \gamma\gamma(a)$ be the upper $\delta(a)$-path from u to $\gamma\gamma(a)$. Since $u \sim_{\mathcal{J}} \gamma\gamma(a)$ and $u <^+ \gamma\gamma(a)$ there is an upper $\mathcal{J}$-path $u, y_0, \ldots, y_l, \gamma\gamma(a)$. As $x_0 \notin \mathcal{J}$ and $u \in \delta(x_0) \cap \delta(y_0)$, by pencil linearity, we have $y_0 <^+ x_0$. By Path Lemma, there is $0 \leq i \leq l$ such, that $\gamma(y_i) = \gamma(x_0)$. Hence using the characterization of the loops in $T_{/\mathcal{J}}$, $[x]$ is a loop contrary to our assumption. From this contradiction if follows that indeed $[\gamma\gamma(a)] \notin \delta_{/\mathcal{J}}\dot{\delta}^{-\lambda}_{/\mathcal{J}}([a])$.

Ad 3. Fix $x \in \delta(a) - \mathcal{J}$. Then $[\gamma(x)] \in \gamma_{/\mathcal{J}}\delta_{/\mathcal{J}}([a])$. Let $x_0, \ldots, x_k$ be the $\delta(a)$-path such that $x_0 \notin \mathcal{L}_{\mathcal{J}} \cup \mathcal{J}$ and $x_i \in \mathcal{L}_{\mathcal{J}} \cup \mathcal{J}$ for $i > 0$. Clearly $x_i \in T^u$ if $i > 0$ and possibly $k = 0$. Then $\gamma_{/\mathcal{J}}([x_0]) = \gamma_{/\mathcal{J}}\gamma_{/\mathcal{J}}([a])$. So an arbitrary element of $\gamma_{/\mathcal{J}}\delta_{/\mathcal{J}}([a]) - [\gamma\gamma(a)]$ is of form $[\gamma(x)]$ for $x \in \delta(a) - (\mathcal{J} \cup \{x_0\})$. Then we have a lower $\delta(a)$-path $x = y_1, \ldots, y_l = x_0$ with $l > 1$. Put

$$l' = \max(\{l'' : \{y_2, \ldots, y_{l''}\} \subseteq (\mathcal{L}_{\mathcal{J}} \cup \mathcal{J})\} \cup \{1\})$$

As $x_0 \notin \mathcal{L}_{\mathcal{J}} \cup \mathcal{J}$, we have $1 \leq l' < l$, and $y_{l'+1} \notin \mathcal{L}_{\mathcal{J}} \cup \mathcal{J}$, i.e. $[y_{l'+1}] \in \dot{\delta}^{-\lambda}_{/\mathcal{J}}([a])$. Clearly, $\gamma(y_{l'}) \in \delta(y_{l'+1}) - \mathcal{J}$ and hence

$$[\gamma(x)] = [\gamma(y_{l'})] \in \delta_{/\mathcal{J}}\dot{\delta}^{-\lambda}_{/\mathcal{J}}([a]),$$

which ends the proof of γ-globularity.

δ-globularity. We have there different cases:

I $\delta(a) \subseteq \mathcal{J}$,

II $\delta(a) \not\subseteq \mathcal{J}$ and $\delta\gamma(a) \subseteq \mathcal{J}$,

III $\delta(a) \not\subseteq \mathcal{J}$ and $\delta\gamma(a) \not\subseteq \mathcal{J}$.

Case **I**, as we already mentioned, is obvious.

Case **II**: $\delta(a) \not\subseteq \mathcal{J}$ and $\delta\gamma(a) \subseteq \mathcal{J}$.

In this case we have:

$$\delta_{/\mathcal{J}}\gamma_{/\mathcal{J}}([a]) = 1_{[\gamma\gamma\gamma(a)]}.$$

$$\delta_{/\mathcal{J}}\delta_{/\mathcal{J}}([a]) = \{[u] : \exists_{x\in\delta(a)-\mathcal{J}}\ u \in \delta(x) - \mathcal{J}\} \cup \{1_{[\gamma\gamma(x)]} : \delta(x) \subseteq \mathcal{J},\ x \in \delta(a) - \mathcal{J}\}$$

and

$$\gamma_{/\mathcal{J}}\dot{\delta}^{-\lambda}_{/\mathcal{J}}([a]) = \{[\gamma(x)] : x \in \delta(a) - (\mathcal{L}_{\mathcal{J}} \cup \mathcal{J})\}.$$

Let $u \in \delta(x) - \mathcal{J}$ and $x \in \delta(a) - \mathcal{J}$. As $\delta\gamma(a) \subseteq \mathcal{J}$ and $u \notin \mathcal{J}$, by globularity (of positive face structures), $u \in \gamma\delta(a)$. Thus there is $y_0 \in \delta(a)$ such, that $\gamma(y_0) = u$. Since $\delta(\mathcal{J}) \cap \mathcal{J} = \emptyset$ and $\delta\gamma(a) \subseteq \mathcal{J}$ there is a $\delta(a)$-path $y_k, \ldots, y_0$ such that $y_k \notin (\mathcal{J} \cup \mathcal{L}_{\mathcal{J}})$ and $y_{k-1}, \ldots, y_0 \subseteq (\mathcal{J} \cup \mathcal{L}_{\mathcal{J}})$, $k \geq 0$. Then $[y_k] \in \dot{\delta}^{-\lambda}([a])$ and $\gamma_{/\mathcal{J}}([y_k]) = [u]$. Thus $\{[u] : \exists_{x\in\delta(a)-\mathcal{J}} u \in \delta(x) - \mathcal{J}\} \subseteq \gamma_{/\mathcal{J}}\dot{\delta}^{-\lambda}_{/\mathcal{J}}([a])$.

It remains to show that

1. there is $x \in \delta(a) - \mathcal{J}$, $\delta(x) \subseteq \mathcal{J}$,
2. for any such x, we have $1_{[\gamma\gamma(x)]} = 1_{[\gamma\gamma\gamma(a)]}$.

Ad 1. The existence of such x follows easily from Path Lemma.

Ad 2. Suppose $x \in \delta(a) - \mathcal{J}$, $\delta(x) \subseteq \mathcal{J}$. As $\mathcal{J} \cap \gamma(T) = \emptyset$, by globularity, we have that $\delta(x) \subseteq \delta\gamma(a)$. Hence $\gamma\gamma(x) \in \gamma\delta(x) \subseteq \gamma\delta\gamma(a)$, and then there is an upper $\delta\gamma(a)$-path (possibly empty) from $\gamma\gamma(x)$ to $\gamma\gamma\gamma(a)$. But $\delta\gamma(a) \subseteq \mathcal{J}$, so this is a $\mathcal{J}$-path and this means that $\gamma\gamma(x) \sim_{\mathcal{J}} \gamma\gamma\gamma(a)$, i.e. $[\gamma\gamma(x)] = [\gamma\gamma\gamma(a)]$, as required.

Case **III**: $\delta(a) \not\subseteq \mathcal{J}$ and $\delta\gamma(a) \not\subseteq \mathcal{J}$. This is the only case, where we do not have equality but $\equiv_1$ only. We need to verify:

1. $\delta_{/\mathcal{J}}\gamma_{/\mathcal{J}}([a]) \subseteq \delta_{/\mathcal{J}}\delta_{/\mathcal{J}}([a])$;

2. $\delta_{/\mathcal{J}}\gamma_{/\mathcal{J}}([a]) \cap \gamma_{/\mathcal{J}}\dot{\delta}^{-\lambda}_{/\mathcal{J}}([a]) = \emptyset$;

3. $\delta_{/\mathcal{J}}\delta_{/\mathcal{J}}([a]) - \delta_{/\mathcal{J}}\gamma_{/\mathcal{J}}([a]) \subseteq \gamma_{/\mathcal{J}}\dot{\delta}^{-\lambda}_{/\mathcal{J}}([a])$;

4. $\gamma_{/\mathcal{J}}\delta_{/\mathcal{J}}\delta^{\varepsilon}_{/\mathcal{J}}([a]) \subseteq \theta_{/\mathcal{J}}\delta_{/\mathcal{J}}\gamma_{/\mathcal{J}}([a])$.

Ad 1. We have $\delta_{/\mathcal{J}}\gamma_{/\mathcal{J}}([a]) = \{[u] : u \in \delta\gamma(a) - \mathcal{J}\}$.

So, let $u \in \delta\gamma(a) - \mathcal{J}$ and $u, x_1, \ldots, x_k, \gamma\gamma(a)$ be a $\delta(a)$-path, $k \geq 1$. There is $1 \leq l \leq k$ such that $x_i \in \mathcal{J}$, for $i < l$, and $x_l \not\in \mathcal{J}$. Such l exists, since $\delta(a) \not\subseteq \mathcal{J}$. Let

$$v = \begin{cases} u & \text{if } l = 1, \\ \gamma(x_{l-1}) & \text{otherwise.} \end{cases}$$

Then $[u] = [v]$. Moreover, $v \in \delta(x_{l+1}) - \mathcal{J}$, $x_{l+1} \in \delta(a) - \mathcal{J}$, i.e. $[v] \in \delta_{/\mathcal{J}}\delta_{/\mathcal{J}}([a])$, as required.

Ad 2. Suppose contrary that there is $u \in \delta\gamma(a) - \mathcal{J}$ and $x \in \delta(a) - (\mathcal{L}_{\mathcal{J}} \cup \mathcal{J})$ so that $[u] = [\gamma(x)] \in \gamma_{/\mathcal{J}}\delta_{/\mathcal{J}}([a])$.

Thus we have a $\mathcal{J}$-path $u, x_1, \ldots, x_k, \gamma(x)$. As $\delta\gamma(a) \cap \gamma\delta(a) = \emptyset$, $u \neq \gamma(x)$ and $k \geq 1$. Since $\gamma(x) = \gamma(x_k)$, $x_k \in \mathcal{J}$ and $x \not\in \mathcal{J}$, by pencil linearity $x_k <^+ x$. We have that $\delta(x_l) \cap \delta(x) = \emptyset$ for $1 \leq l \leq k$, since otherwise $[x]$ would be a loop. Let $y_1, \ldots, y_r, x_1, \ldots, x_k$ be a continuation of the path $y_1, \ldots, y_r, x_1, \ldots, x_k$ through u (i.e. $\gamma(y_r) = u$) such that there is $v \in \delta(y_1) \cap \delta(x)$. Since $x \in \delta(a)$, $v \in \delta\delta(a)$. So there is $v' \in \delta\gamma(a)$ such that $v' \leq^+ v$. But then $v' <^+ u$ and $v', u \in \delta\gamma(a)$, which is impossible. Thus 2. holds, as well.

Ad 3. Let $u \in \delta(x) - \mathcal{J}$ and $x \in \delta(a) - \mathcal{J}$, i.e. $[u] \in \delta_{/\mathcal{J}}\delta_{/\mathcal{J}}([a])$ and suppose that $[u] \not\in \delta_{/\mathcal{J}}\gamma_{/\mathcal{J}}([a])$. Let $x_0, \ldots, x_l, u$ be a $\delta(a)$-path, $l \geq 0$, such that $x_1, \ldots, x_l \subseteq (\mathcal{L}_{\mathcal{J}} \cup \mathcal{J})$, and $x_0 \not\in (\mathcal{L}_{\mathcal{J}} \cup \mathcal{J})$. Such a path exists since $[u] \not\in \delta_{/\mathcal{J}}\gamma_{/\mathcal{J}}([a])$. Then $[x_0] \in \dot{\delta}^{-\lambda}_{/\mathcal{J}}([a])$ and hence $[u] = [\gamma(x_0)] \in \gamma_{/\mathcal{J}}\dot{\delta}^{-\lambda}_{/\mathcal{J}}([a])$, as required.

Ad 4. We have

$$\delta_{/\mathcal{J}}\delta^{\varepsilon}_{/\mathcal{J}}([a]) = \{1_{[\gamma\gamma(x)]} : x \in \delta(a) - \mathcal{J},\, \delta(x) \subseteq \mathcal{J}\}$$

Fix $x \in \delta(a) - \mathcal{J}$ such that $\delta(x) \subseteq \mathcal{J}$. We need to show that $\gamma_{/\mathcal{J}}\gamma_{/\mathcal{J}}([x]) = [\gamma\gamma(x)] \in \theta_{/\mathcal{J}}\delta_{/\mathcal{J}}\gamma_{/\mathcal{J}}([a])$.

We have $\gamma\gamma(x) \in \gamma\gamma\delta(a) \subseteq \gamma\gamma\gamma(a) \cup \delta\delta\gamma(a)$. If $\gamma\gamma(x) = \gamma\gamma\gamma(a)$ then, using γ-globularity of positive face structures, we have

$$[\gamma\gamma(x)] = [\gamma\gamma\gamma(a)] = \gamma_{/\mathcal{J}}\gamma_{/\mathcal{J}}\gamma_{/\mathcal{J}}([a]) \subseteq \gamma_{/\mathcal{J}}\delta_{/\mathcal{J}}\gamma_{/\mathcal{J}}([a]) \subseteq \theta_{/\mathcal{J}}\delta_{/\mathcal{J}}\gamma_{/\mathcal{J}}([a])$$

and 4. holds.

So now assume that $\gamma\gamma(x) \in \delta\delta\gamma(a)$. Thus there is an upper $\delta\gamma(a)$-path $\gamma\gamma(x), u_1, \ldots, u_k, \gamma\gamma\gamma(a)$. If it is a $\mathcal{J}$-path then $[\gamma\gamma(x)] = [\gamma\gamma\gamma(a)] \in \theta_{/\mathcal{J}}\delta_{/\mathcal{J}}\gamma_{/\mathcal{J}}([a])$. If it is not a $\mathcal{J}$-path, then let $i_0 = \min\{i' : u_{i'} \not\in \mathcal{J}\}$ and

$$t = \begin{cases} \gamma\gamma(x) & \text{if } i_0 = 1, \\ \gamma(u_{i_0-1}) & \text{otherwise.} \end{cases}$$

Then $u_1, \ldots, u_{i_0-1}$ is a $\mathcal{J}$-path and $\gamma\gamma(x) \sim_{\mathcal{J}} t \in \delta(u_i) - \mathcal{J}$. Thus $[\gamma\gamma(x)] \in \delta_{/\mathcal{J}}([u_{i_0}])$. But $u_{i_0} \in \delta\gamma(a) - \mathcal{J}$, so $[u_{i_0}] \in \delta_{/\mathcal{J}}\gamma_{/\mathcal{J}}([a])$ and then

$$\gamma\gamma(x)] \in \delta_{/\mathcal{J}}\delta_{/\mathcal{J}}\gamma_{/\mathcal{J}}([a]) \subseteq \theta_{/\mathcal{J}}\delta_{/\mathcal{J}}\gamma_{/\mathcal{J}}([a]),$$

as required. This ends the proof of globularity of $T_{/\mathcal{J}}$.

Disjointness. From the description of the orders we get immediately $\perp^{T_{/\mathcal{J}},+} \cap \perp^{T_{/\mathcal{J}},\sim} = \emptyset$. If $a, b \in T - \mathcal{J}$ and $[a] <^{\sim} [b]$ the by definition $a <^{T,-} b$. So we have a lower $T - \gamma(T)$-path $a = a_0, \ldots, a_k = b$. Let $b_1, \ldots, b_l$ be the path (possibly empty) obtained from $a_1, \ldots, a_{k-1}$ by dropping elements that belong to $\mathcal{L}_{\mathcal{J}} \cup \mathcal{J}$. Then $[a], [b_1], \ldots, [b_l], [b]$ is a lower flat path in $T_{/\mathcal{J}}$. Hence $[a] <^{\sim} [b]$ implies $[a] <^{-} [b]$.

Now assume that $\theta_{/\mathcal{J}}([a]) \cap \theta_{/\mathcal{J}}([b]) = \emptyset$ and $[a] <^{-} [b]$. We need to show that $[a] <^{\sim} [b]$.

So we have a lower flat path $[a] = [a_0], \ldots, [a_k] = [b]$ in $T_{/\mathcal{J}}$, with $k > 1$. There are some cases to be considered. We will deal with the one which is most involved: $k = 2$, there are $x \in \delta(a_1) - \mathcal{J}$, $y \in \delta(a_2) - \mathcal{J}$ there are upper $\mathcal{J}$-pathes $x, b_1, \ldots, b_l, \gamma(a_0)$, and $y, c_1, \ldots, c_r, \gamma(a_1)$.

As $[a_1]$ is not a loop $b_i <^{+} a_1$, for $i \leq l$ and $c_i <^{+} a_1$, for $i \leq r$. Moreover they are $T - \gamma(T)$-pathes. It is easy to see that if we continue the path $b_1, \ldots, b_l$ as $T - \gamma(T)$-path we shall get to $c_1, \ldots, c_r$ (note that $\mathcal{J}$-faces are unary). Thus there is a path $b_1, \ldots, b_l, d_1, \ldots, d_s, c_1, \ldots, c_r$, with $s \geq 0$. Then $a_0, d_1, \ldots, d_s, a_2$ is a lower path showing that $a = a_0 <^{-} a_2 =$, i.e. $[a] <^{\sim} [b]$, as required. This ends the proof of disjointness.

Pencil linearity. Let $[a], [b] \in T_{/\mathcal{J}}$, and $[a] \neq [b]$. First assume that $\theta_{/\mathcal{J}}([a]) \cap \theta_{/\mathcal{J}}([b]) \neq \emptyset$. Thus we have three cases to consider:

I $\gamma_{/\mathcal{J}}([a]) = \gamma_{/\mathcal{J}}([b])$,

II $\gamma_{/\mathcal{J}}([a]) \in \delta_{/\mathcal{J}}([b])$,

III $\delta_{/\mathcal{J}}([a]) \cap \delta_{/\mathcal{J}}([b]) \neq \emptyset$.

Case **I**. Possibly changing the roles of a and b there is a $\mathcal{J}$-path $\gamma(a), a_1, \ldots, a_k, \gamma(b)$. If $\gamma(a) \in \delta(b)$ or there is $i < k$ such that $\gamma(a_i) \in \delta(b)$ then $a <^{T,-} b$ and hence $[a] <^{T_{/\mathcal{J}},-} [b]$. If it is not the case, that is $\gamma(a) \notin \delta(b)$ and for all $i < k$ $\gamma(a_i) \notin \delta(b)$ then by Path Lemma there is $y \in \delta(b)$ and an upper path $y, b_1, \ldots, b_l, a, a_1, \ldots, a_k, \gamma(b)$ and $a <^{+} b$. Therefore $[a] <^{+} [b]$.

Case **II**. In this case we have $x \in \delta(b) - \mathcal{J}$ and a $\mathcal{J}$-path $\gamma(a), a_1, \ldots, a_k, x$ or $x, a_1, \ldots, a_k, \gamma(a)$. In the former case we have $a <^{T,-} b$ and hence $[a] <^{T_{/\mathcal{J}},-} [b]$. In the later case either there is $1 \leq i \leq k$ such that $\gamma(b) = \gamma(a_i)$ and then $\gamma_{/\mathcal{J}}([b]) = \gamma_{/\mathcal{J}}([a])$, i.e. this case is reduced to **I** or, by Path Lemma, we have that $a_i <^{+} b$, for $1 \leq i \leq k$. Let $a_k, \alpha_1, \ldots, \alpha_l, b$ be an upper path in T. As $\gamma(a_k) \neq \gamma(b)$ there is $1 \leq j \leq l$ such that $\gamma(a_k) \in \iota(\alpha_j)$. Since $\gamma(a_k) = \gamma(a)$ we have that $\gamma(a) \in \iota(\alpha_j)$ and then $a <^{+} \gamma(\alpha_j) \leq^{+} \gamma(\alpha_l) = b$, i.e. $[a] <^{+} [b]$.

Case **III**. Possibly changing the roles of a and b there are $x \in \delta(a) - \mathcal{J}$ and $y \in \delta(b) - \mathcal{J}$ and $\mathcal{J}$-path $x, a_1, \ldots, a_k, y$. If there is $1 \leq i \leq k$ such that $\gamma(a_i) = \gamma(a)$ then $a <^{-} b$ and hence $[a] <^{\sim} [b]$. If for all $i \leq n$ we have $\gamma(a_i) \neq \gamma(a)$ then, by Path Lemma, $b <^{+} a$ and hence $[b] <^{+} [a]$.

Next let assume that $[a] \in T^{\varepsilon}_{/\mathcal{J}}$, $[b] \in T_{/\mathcal{J}}$ and $\gamma_{/\mathcal{J}}\gamma_{/\mathcal{J}}([a]) \in \iota_{/\mathcal{J}}([b])$. Thus there are $x, y \in \delta(b) - (\mathcal{J} \cup \mathcal{L}_{\mathcal{J}})$ such that $\gamma_{/\mathcal{J}}\gamma_{/\mathcal{J}}([a]) = \gamma_{/\mathcal{J}}([x]) \in \delta_{/\mathcal{J}}([y])$. Hence there is $u \in \delta(y) - \mathcal{J}$ such that $\gamma(x) \sim_{\mathcal{J}} u$. If we were to have a $\mathcal{J}$-path $u, x_1 \ldots, x_k, \gamma(x)$ then, as $u \in \delta(b)$ by Path Lemma, either there is $1 \leq i \leq k$ such that $\gamma(x_i) = \gamma(y)$ or $\gamma(x_k) \neq \gamma(y)$ and $x_k <^{+} y$. In the former case $[y]$ would be a loop in the later, we would have $x <^{+} y$ and $x, y \in \delta(b)$. As none of the above is possible, it follows that we cannot have a $\mathcal{J}$-path from u to $\gamma(x)$. Hence we have a $\mathcal{J}$-path $\gamma(x), x_1 \ldots, x_k, u$.

Claim. Exactly one of the following conditions holds:

(i) there is a $\mathcal{J}$-path (possibly empty) from $\gamma\gamma(a)$ to $\gamma(x)$;

(ii) there is a $\mathcal{J}$-path (possibly empty) from u to $\delta\gamma(a)$;

(iii) $\delta(a) \subseteq \{x_1, \ldots, x_k\}$.

Clearly no two of the above three conditions can hold simultaneously. We assume that **(i)** and **(ii)** does not hold and we shall prove **(iii)**. We can assume that $k \geq 1$. As $\delta(a) \subseteq \mathcal{J}$ and $\{x_1, \ldots, x_k\} \subseteq \mathcal{J}$, it is enough to show:

(a) there is $1 \leq i \leq k$ such that $\gamma(x_i) = \gamma\gamma(a)$;

(b) either $\gamma(x) \in \delta\gamma(a)$ or there is $1 \leq j < i$ such that $\gamma(x_j) = \delta\gamma(a)$;

Ad **(a)**. Suppose that **(a)** does not hold. Then, as **(i)** does not hold, we have an upper $\mathcal{J}$-path $u, x_1, \ldots, x_l, \gamma\gamma(a)$, with $l > k$. As $x_l \in \mathcal{J}$ and $\gamma(a) \notin \mathcal{J}$, we have $x_l <^+ \gamma(a)$. So by Path Lemma, either $\gamma(x_{i_0}) \in \delta\gamma(a)$, for some $k \leq i_0 < l$ or $u \in \iota(a)$. In the former case we get **(ii)** contrary to the supposition. In the later case, on one hand, as $u \in \delta(y) \cap \iota(a)$, we have that $y <^+ \gamma(a)$. Thus $\gamma(y) \leq^+ \gamma\gamma(a)$. On the other hand, if we were to have $k < i_1 \leq l$ such that $\gamma(x_{i_1}) = \gamma(y)$ then, as $x_i \in \mathcal{J}$, we would have that $[y]$ is a loop. Hence, by Path Lemma $x_i <^+ y$, for $i = k+1, \ldots, l$, and $\gamma\gamma(a) = \gamma(x_l) \neq \gamma(y)$. But then, again by Path Lemma, $\gamma\gamma(a) <^+ \gamma(y)$. Therefore we get a contradiction once more. This ends the proof of **(a)**.

Ad **(b)**. As we have established **(a)**, let us fix $1 \leq i_1 \leq k$ such that $\gamma(x_{i_1}) = \gamma\gamma(a)$. Suppose that **(b)** does not hold. Then $\{x_1, \ldots, x_{i_1}\} \subseteq \delta(a)$ and $\gamma(x) \in \iota(a)$. Thus $x <^+ \gamma(a)$. Let $y_1, \ldots, y_r, x_1, \ldots, x_{i_1}$ be the lower $\mathcal{J}$-path consisting of all the faces in $\delta(a)$. Clearly, $\delta(y_1) = \delta\gamma(a)$, $\gamma(y_r) = \gamma(x)$ and $y_r <^+ x$. If we were to have $1 \leq i \leq r$ such that $\delta(y_i) \cap \delta(x) \neq \emptyset$ then the face $[x]$ would be a loop contrary to the supposition. Thus, by Path Lemma, $y_i <^+ x$, for $i = 1, \ldots, r$ and there is $v \in \delta(x)$ such that $v <^+ \delta(y_1) = \delta\gamma(a)$ (both $\delta(y_1)$ and $\delta\gamma(a)$ are singletons). On the other hand, as $x <^+ \gamma(a)$, we have, by Path Lemma, a lower path $z_1, \ldots, z_s = x$, with $s \geq 1$, and $w \in \delta(z_1) \cap \delta\gamma(a)$. Then, for $w' = \gamma(z_{s-1}) \in \delta(x)$ (or $w' = w$ if $s = 1$) we have $\delta\gamma(a) \leq^+ w'$. Thus $v, w' \in \delta(x)$ and $v <^+ \delta\gamma(a) \leq^+ w'$. But this is impossible by Proposition 5.1 of [Z]. This ends the proof of **(b)** and of the Claim.

Having the Claim, it is easy to see, that in case **(i)** $\gamma(a) \leq x$ and in case **(ii)** $\gamma(a) \leq y$. Thus in both cases we have $[a] <^+ [b]$.

Finally assume that **(iii)** holds. Thus $x <^- \gamma(a)$ and $\gamma\gamma(a) <^+ \gamma\gamma(b)$. From the later and [Z], we have that either $\gamma(a) <^- \gamma(b)$ or $\gamma(a) <^+ \gamma(b)$. If $\gamma(a) <^- \gamma(b)$ then using the former and transitivity of $<^-$ we would have $x <^- \gamma(b)$. But $x <^+ \gamma(b)$ and this contradicts disjointness. Thus $\gamma(a) <^+ \gamma(b)$. Then, again by [Z], we have that either $a <^- b$ or $a <^+ b$, as required.

The fact that $q_{\mathcal{J}}$ is a collapsing morphism with the kernel $\mathcal{J}$ is left for the reader. This ends the proof of the theorem. $\Box$

Proposition 6.5 *Let T be a positive face structure, and $\mathcal{I}$ be an ideal in T. Then $size(T_{/\mathcal{I}}) = size(T)$.*

Proof. We have a quotient morphism $q : T \to T_{/\mathcal{J}}$ such that, for $a \in T - \mathcal{J}$, we have $q(a) = [a]_{\mathcal{J}}$. We shall show that, for any $k \in \omega$, the restriction of this function

$$\tilde{q}_k : T_k - \delta(T_{k+1}) \longrightarrow (T_{/\mathcal{J}})_k - \delta_{/\mathcal{J}}((T^{-\lambda}_{/\mathcal{J}})_{k+1})$$

is a bijection. This is clearly sufficient to establish 2. To see that $\tilde{q}_k$ is one-to-one, note that for $a, a' \in T_k - \delta(T_{k+1})$, by Lemma 6.3.1, we have $a \not\sim_{\mathcal{J}} a'$.

We shall verify that $\tilde{q}_k$ is onto. Fix $[a]_{\mathcal{J}} \in (T_{/\mathcal{J}})_k - \delta_{/\mathcal{J}}((T^{-\lambda}_{/\mathcal{J}})_{k+1})$ such that $a \in T - \mathcal{J}$ is $<^+$-maximal in its class $[a]_{\mathcal{J}}$. Suppose that $a \in \delta(T_{k+1})$, and fix $\alpha \in T_{k+1} - \gamma(T_{k+2})$ such that $a \in \delta(\alpha)$. Then if $\alpha \in \mathcal{J}$, a is not $<^+$-maximal in its class. If $\alpha \notin \mathcal{J}$, then $[\alpha]_{\mathcal{J}} \in T^{-\lambda}_{/\mathcal{J}}$ and $[a]_{\mathcal{J}} \in \delta_{/\mathcal{J}}([\alpha]_{\mathcal{J}})$. In either case we get a contradiction. Thus there is no $\alpha \in T_{k+1}$ such that $a \in \delta(\alpha)$, and $\tilde{q}_k$ is onto indeed. $\Box$

Positive covers

Recall that the kernel of a collapsing morphism $q : Y \to X$ is the set $ker(q) = q^{-1}(1_X) \subseteq Y$. In more concrete terms, as q preserves codomains, we have $ker(q) = \{a \in Y : q(a) = 1_{q(a)}\}$.

We say that a collapsing morphism $q : Y \to X$ is a *(positive) cover* iff there is an ideal $\mathcal{J}$ in Y, and a monotone isomorphism $h : Y/_{\mathcal{J}} \longrightarrow X$ such that the triangle

$$\begin{tikzcd} & Y \arrow[dl, "p_{\mathcal{J}}"'] \arrow[dr, "q"] & \\ Y/_{\mathcal{J}} \arrow[rr, "h"'] & & X \end{tikzcd}$$

commutes.

Proposition 6.6 *Let $q : Y \to X$ be a collapsing morphism and $\mathcal{J}$ an ideal in Y, and $p_{\mathcal{J}} : Y \to Y/_{\mathcal{J}}$ a positive cover.*

1. *$ker(q)$ is an ideal iff $ker(q) \subseteq Y^u$.*
2. *$q : Y \to X$ is a positive cover iff q is onto and $ker(q)$ is an ideal.*
3. *If $ker(p_{\mathcal{J}}) \subseteq ker(q)$ then there is a unique collapsing morphism $r : Y/_{\mathcal{J}} \to X$ making the triangle*

$$\begin{tikzcd} & Y \arrow[dl, "p_{\mathcal{J}}"'] \arrow[dr, "q"] & \\ Y/_{\mathcal{J}} \arrow[rr, "r"'] & & X \end{tikzcd}$$

commutes.

Proof. Ad 1. Any ideal in Y is contained in Y^u. Thus we need to show that if $ker(q) \subseteq Y^u$ then $ker(q) \cap \gamma(Y) = \emptyset = ker(q) \cap \delta(ker(q))$.

Suppose there is $a \in ker(q) \cap \gamma(Y)$. Let $\alpha \in Y$ such that $\gamma(\alpha) = a$. Then

$$1_{\gamma(q(a))} = q(a) = q(\gamma(\alpha)) = \gamma(q(\alpha)) \in X$$

and we get a contradiction. Thus $ker(q) \cap \gamma(Y) = \emptyset$.

Now suppose that $a \in ker(q) \cap \delta(ker(q))$. Fix $\alpha \in ker(q)$ such that $a \in \delta(a)$. As $ker(q) \subseteq Y^u$ we have $a = \delta(\alpha)$. So we have

$$1_{\gamma(q(a))} = q(a) = q(\delta(\alpha)) = \delta(q(a)) = \delta(1_{\gamma(q(\alpha))}) = \gamma(q(\alpha)) \in X$$

and we get a contradiction again.

Ad 2. Clearly the conditions are necessary. To see that they are sufficient it is enough to note that they imply that the map $h : Y_{/ker(q)} \to X$ such that $h([a]) = q(a)$, for $a \in Y - \mathcal{J}$ is an isomorphism in **oFs**.

Ad 3. We put $r([y]) = q(y)$, for $y \in Y - \mathcal{J}$. Since $ker(p_{\mathcal{J}}) \subseteq ker(q)$, r is well defined. As

$$p_{\mathcal{J}}(y) = \begin{cases} [y] & \text{if } y \in Y - \mathcal{J}, \\ 1_{[\gamma(y)]} & \text{if } y \in \mathcal{J}. \end{cases}$$

we have $q = r \circ p_{\mathcal{J}}$. It remains to verify that r preserves γ, δ, and $<^{\sim}$.

Fix $y, y' \in Y - \mathcal{J}$. We have

$$[y] <^{Y/_{\mathcal{J}},\sim} [y'] \quad \text{iff} \quad y <^{Y,\sim} y' \quad \text{iff} \quad q(y) <^{X,\sim} q(y') \quad \text{iff} \quad r([y]) <^{X,\sim} r([y'])$$

i.e. r preserves $<^{\sim}$.

Now fix $y \in Y_{\geq 1} - \mathcal{J}$. We have

$$r(\gamma([y])) = r([\gamma(y)]) = q(\gamma(y)) = \gamma(q(y)) = \gamma(r([y]))$$

i.e. r preserves γ.

To see that r preserves δ we consider two cases: $\delta(y) \subseteq \mathcal{J}$ and $\delta(y) \not\subseteq \mathcal{J}$. If $\delta(y) \subseteq \mathcal{J}$ then we have

$$r(\delta([y])) = r(1_{[\gamma\gamma(y)]}) = 1_{r([\gamma\gamma(y)])} = 1_{q(\gamma\gamma(y))} = q(\delta(y)) \equiv_1 \delta(q(y)) = \delta(r([y]))$$

and if $\delta(y) \not\subseteq \mathcal{J}$ we have

$$r(\delta([y])) = r(\{[u] : u \in \delta(y) - \mathcal{J}\}) = \{q(u) : u \in \delta(y) - \mathcal{J}\} \equiv_1$$

$$\equiv_1 \{q(u) : u \in \delta(y)\} = q(\delta(y)) \equiv_1 \delta(q(y)) = \delta(r([y]).$$

Thus in both cases δ is preserved. $\Box$

7 Positive covers of ordered face structures

In this section we describe a kind of inverse construction to the quotient construction from previous section. We shall show that any ordered face structure S can be covered by a positive one $S^\dagger$. We begin with some notation and the construction. Then we shall prove few technical lemmas. Using these lemmas we shall describe the properties of the construction, in particular that $S^\dagger$ is a positive face structure and that $q_S : S^\dagger \to S$ is a quotient morphism. Finally, we will make some farther comments about this construction.

The construction of $S^\dagger$

S an ordered face structure fixed for the whole section. The construction of $S^\dagger$ is based on cuts, but this time we consider the cuts of initial faces in S not, as in section 5, of empty loops. We use essentially the same notation for both cuts of empty loops and cuts of initial faces. But, as we never use these different cuts in the same context so there is no risk to mix them.

Recall from section 3, that $\mathcal{I} = \mathcal{I}^S = S^\varepsilon - \gamma(S^{-\lambda})$ is the set of *initial faces* in S, and $\mathcal{I}_x = \mathcal{I}^S_x = \{a \in \mathcal{I} : \delta(a) = 1_x\}$ is the set of *initial faces based on* x. $\mathcal{I}_x$ is a linearly ordered by $<$ (we have, for $a, b \in \mathcal{I}_x$, that $a < b$ iff $\gamma(a) <^\sim \gamma(b)$). An *$x$-cut* is a triple (x, L, U) such that $L \cup U = \mathcal{I}_x$ and for $\alpha \in L$ and $\beta \in U$, $\alpha < \beta$. $\mathcal{C}(\mathcal{I}_x)$ is the set of all x-cuts. $\mathcal{C}(\mathcal{I}_X) = \mathcal{C}(\mathcal{I}^T_X) = \bigcup\{\mathcal{C}(\mathcal{I}_x) : x \in X\}$, where $X \subseteq S$, is the set of all X-cuts, i.e. all x-cuts with $x \in X$.

If (x, L, U) is a x-cut then L determines U and vice versa ($L = \mathcal{I}^X_x - U$ and $U = \mathcal{I}^X_x - L$). Therefore we sometimes denote this cut by describing only the lower cut $(x, L, -)$ *lower description of the cut* or only the upper cut $(x, -, U)$ *upper description of the cut*, whichever is easier to define.

For $a \in S$ and $x \in \dot{\delta}(a)$. We define the following sets:

$$\uparrow a \ = \{\alpha \in \mathcal{I}_{\gamma(a)} : a <^\sim \gamma(\alpha)\}, \qquad \downarrow_x a \ = \{\alpha \in \mathcal{I}_x : \gamma(\alpha) <^\sim a\}$$

and cuts $(\gamma(a), -, \uparrow a)$ and $(x, \downarrow_x a, -)$. In order to save the space we drop subscript x in the notation $\downarrow_x a$ inside x-cuts, i.e. we often write $(x, \downarrow a, -)$ instead of $(x, \downarrow_x a, -)$. Clearly, $(x, -, \uparrow b) = (x, \downarrow_x a, -)$ iff $\downarrow_x a \cup \uparrow b = \mathcal{I}_x$ and $\downarrow_x a \cap \uparrow b = \emptyset$.

We describe below the positive hypergraph $S^\dagger$. The set of faces of dimension k is

$$S^\dagger_k = \mathcal{C}(\mathcal{I}_{S_k}) \cup \overline{\mathcal{I}_{k+1}},$$

where $\overline{\mathcal{I}_{k+1}}$ is another copy of the set $\mathcal{I}_{k+1}$ whose elements have bars on it, i.e. $\overline{\mathcal{I}_{k+1}} = \{\overline{\alpha} : \alpha \in \mathcal{I}_{k+1}\}$. Thus the faces of each dimension are of two disjoint kinds: cuts and bars. The domains and codomains in $S^\dagger$ we define separately for bars and cuts. Fix $k > 0$. For $\overline{\alpha} \in \overline{\mathcal{I}_{k+1}}$ we have

$$\gamma^\dagger(\overline{\alpha}) = (\gamma\gamma(\alpha), -, \uparrow \gamma(\alpha)), \qquad \delta^\dagger(\overline{\alpha}) = (\gamma\gamma(\alpha), \downarrow \gamma(\alpha), -),$$

for $(a, L, U) \in \mathcal{C}(\mathcal{I}_a)$, with $a \in S_k$ we have

$$\gamma^\dagger(a, L, U) = (\gamma(a), -, \uparrow a), \qquad \delta^\dagger(a, L, U) = \overline{\mathcal{I}^{\leq^+ a}} \cup \{(x, \downarrow a, -) : x \in \dot{\delta}(a)\}.$$

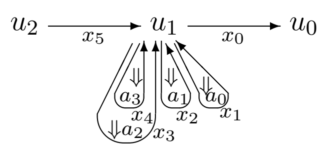
We have a map $q_S : S^\dagger \longrightarrow S$ such that

$$q_S(z) = \begin{cases} a & \text{if } z = (a, L, U) \in \mathcal{C}(\mathcal{I}_a), \\ 1_{\gamma\gamma(\alpha)} & \text{if } z = \overline{\alpha} \in \mathcal{I}. \end{cases}$$

i.e. it sends a-cuts to a, and any bar $\overline{\alpha}$ to an empty-face $1_{\gamma\gamma(\alpha)}$.

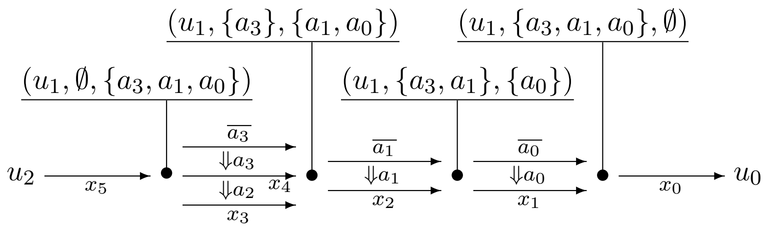
Example. The positive cover of the ordered face structure T as below

T



is the following positive face structure $T^\dagger$

$T^\dagger$



As before we use the convention that the empty cut in $T^\dagger$, say $(x_5, \emptyset, \emptyset)$, is identified with the corresponding face in T, x_5 in this case. All bullets $\bullet$ denote cuts and they are linked by a line to the descriptions of the cuts they denote.

An ideal $\mathcal{I}$ in an ordered face structure S is an *unary ideal* iff $\mathcal{I} \subseteq \delta(S^u)$. The following is a kind of inverse of the Theorem 6.1.

Theorem 7.1 *Let S be an ordered face structure. Then $S^\dagger$ is a positive face structure, $\overline{\mathcal{I}}$ is an unary ideal in $S^\dagger$, $q_S : S^\dagger \longrightarrow S$ is a positive cover with the kernel $\overline{\mathcal{I}}$.*

Some technical lemmas

Since the Lemmas stated below are very technical we will comments on them. Lemmas 7.2, 7.3, 7.4 are there to be used in the proofs of Lemmas 7.6, 7.7, 7.8. Lemma 7.4 is a suplement to the pencil linearity axiom and it says intuitively that if some faces are incident then some (other) faces are comparable. Lemmas 7.6, 7.7, 7.8 concern $\theta\delta(x)$-cuts. They express cuts determined by some faces in terms of cuts determined by some other faces. Lemma 7.6, is about the cuts determined by $\gamma(x)$, Lemma 7.7, is about the cuts determined by a faces $t \in \delta(x)$, and Lemma 7.8, is about the cuts determined by a faces in $\gamma(\mathcal{I}^{\leq^+ x})$.

Lemma 7.2 *Let S be an ordered face structure $x, a \in S$. If $a \in \mathcal{I}$ and $x \leq^+ \gamma(a)$ then $x = \gamma(a)$*

Proof. Suppose $x <^+ \gamma(a)$. Let $x, a_1, \ldots, a_k, \gamma(a)$ be an upper $(S - \gamma(S^{-\lambda}))$-path, $k \geq 1$. As $\gamma(a) = \gamma(a_k)$ and $a, a_k \in S - S^{-\lambda})$, we have $a = a_k$. But this is a contradiction, as $a \in S^{\varepsilon}$ and $a_k \in S^{-\varepsilon}$. $\Box$

Lemma 7.3 *Let S be an ordered face structure $t, t' \in S$, $z \in \mathcal{I}_{\gamma(t)}$, $t <^{\sim} t'$, $\gamma(t) \in \delta(t')$. Then either $t <^{\sim} \gamma(z)$ or $\gamma(z) <^{\sim} t'$.*

Proof. Suppose contrary that $t \not<^{\sim} \gamma(z) \not<^{\sim} t'$. Then, as $t <^{\sim} t'$, we also have $t' \not<^{\sim} \gamma(z) \not<^{\sim} t$. So $t \perp^+ \gamma(z) \perp^+ t'$. Thus, by Lemma 7.2, we have $\gamma(z) \leq^+ t, t'$. Hence, by Lemma 4.17, $t \perp^+ t'$ and we get a contradiction. $\Box$

Lemma 7.4 *Let S be an ordered face structure $u, x, y, z \in S$, $z \in \mathcal{I}_u$, $y \in \mathcal{I}_u^{\leq^+ x}$.*

1. *Let $x \in S^{-\varepsilon}$, $\gamma\gamma(x) = u$. If $\varrho(x) <^{\sim} \gamma(z)$ then either $z <^+ x$ or $\gamma(x) <^{\sim} \gamma(z)$.*
2. *Let $x \in S^{\varepsilon}$, $\gamma\gamma(x) = u$. If $\gamma(y) <^{\sim} \gamma(z)$ then either $z <^+ x$ or $\gamma(x) <^{\sim} \gamma(z)$.*
3. *Let $x \in S^{-\varepsilon}$, $u \in \dot{\delta}\gamma(x)$, $t = \inf_{\sim}\{t' \in \dot{\delta}(x) : u \in \delta(t')\}$. If $\gamma(z) <^{\sim} t$ then either $z <^+ x$ or $\gamma(z) <^{\sim} \gamma(x)$.*
4. *Let $x \in S^{\varepsilon}$, $u \in \dot{\delta}\gamma(x)$. If $\gamma(z) <^{\sim} \gamma(y)$ then either $z <^+ x$ or $\gamma(z) <^{\sim} \gamma(x)$.*

Proof. We use notation as above in the statement of Lemma. Recall that if $z \in \mathcal{I}$ then for no face z' we have either $z' <^{\sim} z$ or $z' <^+ z$.

Ad 1. As $\gamma\gamma(x) = \gamma\gamma(z)$ we have either $\gamma(x) \perp^+ \gamma(z)$ or $\gamma(x) \perp^{\sim} \gamma(z)$. In the later case, as assumption $\gamma(z) <^{\sim} \gamma(x)$ immediately leads to contradiction, we get $\gamma(x) <^{\sim} \gamma(z)$. In the former case we have either $z <^+ x$ or $z <^{\sim} x$. The later of these to is impossible, as we would have $\gamma(z) \leq^+ t \leq^{\sim} \varrho(x)$ and hence $\gamma(z) \leq \varrho(x)$ contrary to the supposition. Thus we get either $z <^+ x$ or $\gamma(x) <^{\sim} \gamma(z)$.

Ad 2. In this case again we have $\gamma\gamma(x) = \gamma\gamma(z)$ and hence either $\gamma(x) \perp^+ \gamma(z)$ or $\gamma(x) \perp^{\sim} \gamma(z)$. In the later case we again easily get that $\gamma(x) <^{\sim} \gamma(z)$. In the former case, as $z <^{\sim} x \in \mathcal{E}$ is impossible, we get $z <^+ x$. Thus again, we get that either $z <^+ x$ or $\gamma(x) <^{\sim} \gamma(z)$.

Ad 3. As $\gamma\gamma(z) = \dot{\delta}\gamma(x)$ we have either $\gamma(x) \perp^+ \gamma(z)$ or $\gamma(x) \perp^{\sim} \gamma(z)$. In the later case we easily get (otherwise $\gamma(x) <^{\sim} t$) that $\gamma(z) <^{\sim} \gamma(x)$. In the former case, as $z \in \mathcal{I}$, we get that either $z <^+ x$ or $z <^{\sim} x$. We shall show that $z <^{\sim} x$ is impossible. Suppose contrary, then there is $t' \in \delta(x)$ such that $\gamma(z) \leq^+ t'$. If we were to have $\gamma(z) = t'$ then, by definition of t, we would have $t \leq^{\sim} t' = \gamma(z)$. Thus $\gamma(z) <^+ t'$ and there is a flat upper path $\gamma(z), z_1, \ldots, z_k, t'$, with $k \geq 1$. If $u \not\in \theta(t')$ then, as $u = \gamma\gamma(z)$, there is $1 \leq i \leq k$ such that $u \in \iota(z_i)$. Hence $t <^+ \gamma(z_i) \leq^+ \gamma(z_k) = t'$ and we get a contradiction with local discreteness. If $u \in \theta(t')$ then, using the definition of t, we easily get that $t <^{\sim} t'$. As $\gamma(z) <^{\sim} t$ we get $\gamma(z) <^{\sim} t'$ contrary to the definition of t'. Thus the assumption $z <^{\sim} x$ leads to a contradiction.

Ad 4. As $\gamma\gamma(z) = \dot{\delta}\gamma(x)$ we have either $\gamma(x) \perp^+ \gamma(z)$ or $\gamma(x) \perp^{\sim} \gamma(z)$. In the later case we easily get that $\gamma(z) <^{\sim} \gamma(x)$. In the former case, as $z, x \in \mathcal{E}$ we cannot have $z \perp^{\sim} x$. As, $x <^+ z \in \mathcal{I}$ is also impossible, we have $z <^+ x$ in that case. Thus we get either $z <^+ x$ or $\gamma(z) <^{\sim} \gamma(x)$. $\Box$

The above Lemma had four parts with first two and second two having the same conclusions. The following Lemma contains in fact four statement with essentially the same conclusion. This is why we state it in a bit unusual form to emphasize it.

Lemma 7.5 *Let S be an ordered face structure $u, x, z \in S$, $z \in \mathcal{I}_u$, $u \in \theta\delta(x)$. Moreover, assume that one the following four conditions*

1. $t, t'' \in \dot{\delta}(x)$, $\gamma(t) = u \in \delta(t'')$,
2. $y \in \mathcal{I}_{\overline{u}}^{\leq^+ x}$, $t'' \in \dot{\delta}(x)$, $\gamma(y) = t$, $u \in \delta(t'')$,
3. $t \in \dot{\delta}(x)$, $y'' \in \mathcal{I}_{\overline{u}}^{\leq^+ x}$, $\gamma(t) = u$, $\gamma(y'') = t''$,
4. $y, y'' \in \mathcal{I}_{\overline{u}}^{\leq^+ x}$, $\gamma(y) = t$, $\gamma(y'') = t''$,

holds. If $t <^\sim \gamma(z) <^\sim t''$ then either $z <^+ x$ or there is $t' \in \delta^\lambda(x)$ such that $t <^\sim t' <^\sim t''$, $\gamma(z) \leq^+ t'$ and $\gamma(t') = u$.

Proof. We use notation as above in the statement of Lemma. Note that if $\gamma(z) \leq^+ t'$ and $t' \in S^\lambda$ then $\gamma(t') = u$.

First we shall show that any of the above four assumptions imply the claim: either $z <^+ x$ or $z <^\sim x$. Note that $\gamma\gamma(z) \in \theta\delta(x)$. If $\gamma\gamma(z) \in \iota(x)$ then the claim follows immediately from pencil linearity. If $\gamma\gamma(z) \in \theta\gamma(x)$ then by pencil linearity we get that either $\gamma(z) \perp^+ \gamma(x)$ or $\gamma(z) \perp^\sim \gamma(x)$. In the former case we get, again by pencil linearity, the claim. In the later case, as $t <^\sim \gamma(z) <^\sim t''$, we get either $t \perp^\sim \gamma(x)$ or $t'' \perp^\sim \gamma(x)$, i.e. a contradiction, as $t, t'' \leq^+ \gamma(x)$ under each of the four assumptions above. Thus we have the claim.

Now it remains to show that each of the following four assumptions imply that if $z <^\sim x$ then there is $t' \in \delta^\lambda(x)$ such that $t <^\sim t' <^\sim t''$, $\gamma(z) \leq^+ t'$. As all the arguments are very similar we shall show this for the assumption 1.

Assume $z <^\sim x$. Then there is $t' \in \delta(x)$ such that $\gamma(z) \leq^+ t'$. We need to show that $t <^\sim t' <^\sim t''$, and $t' \in S^\lambda$.

If $\gamma(z) = t'$ we are done. So assume that $\gamma(z) <^+ t'$ and then we have a flat upper path $\gamma(z), z_1, \ldots, z_k, t'$, with $k \geq 1$. If $\gamma\gamma(z) = u \notin \theta(t')$ then there is $1 \leq i \leq k$ that $u \in \iota(z_i)$. So $t, t'' <^+ \gamma(z_i) \leq^+ \gamma(z_k) = t'$ and we get a contradiction with local discreetness. Thus $u \in \theta(t')$ and we have $t \perp^\sim t' \perp^\sim t''$. If we were to have $t' \leq^\sim t$ then we would have $t' \leq^\sim \gamma(z)$ and if we were to have $t'' \leq^\sim t'$ then we would have $\gamma(z) <^\sim t'$. Thus we must have $t <^\sim t' <^\sim t''$. Therefore there are $u' \in \delta(t')$ and $u'' \in \delta(t'')$ such that $u = \gamma(t) \leq^+ u' \leq^+ \gamma(t') \leq^+ u''$. As $u, u'' \in \delta(t'')$ and $u \leq^+ u''$ we have $u = u''$. Hence $\gamma(t') \in \delta(t')$, i.e. $t' \in S^\lambda$. $\Box$

Lemma 7.6 *Let S be an ordered face structure $u, x \in S$, $u \in \dot{\delta}\gamma(x)$. We put*

$$t_{\sup} = \sup_{\sim}(\{\varrho(x)\} \cup \gamma(\mathcal{I}_{\gamma\gamma(x)}^{\leq^+ x})), \qquad t_{\inf} = \inf_{\sim}(\{t \in \dot{\delta}(x) : u \in \delta(t)\} \cup \gamma(\mathcal{I}_{\overline{u}}^{\leq^+ x})).$$

The elements $t_{\sup}$, $t_{\inf}$ are well defined and

1. $$(\gamma\gamma(x), -, \uparrow \gamma(x)) = (\gamma\gamma(x), -, \uparrow t_{\sup}),$$
2. $$(u, \downarrow \gamma(x), -) = (u, \downarrow t_{\inf}, -).$$

Proof. Ad 1. We consider two cases depending on whether $t_{\sup} = \varrho(x)$ or $t_{\sup} = \sup_\sim(\gamma(\mathcal{I}_{\gamma\gamma(x)}^{\leq^+ x}))$. Fix $z \in \mathcal{I}_{\gamma\gamma(x)}$.

Case $t_{\sup} = \varrho(x)$. Assume $\gamma(x) <^\sim \gamma(z)$. As $\varrho(x) <^+ \gamma(x)$, we have $\varrho(x) < \gamma(z)$. But if we were to have $\varrho(x) <^+ \gamma(z)$ we would have $\gamma(x) \perp^+ \gamma(z)$. Thus, as $\gamma\varrho(x) = \gamma\gamma(z)$, we have $\varrho(x) <^\sim \gamma(z)$. To see the converse, assume that $\varrho(x) <^\sim \gamma(z)$. So by Lemma 7.4.1 we have that either $\gamma(x) <^\sim \gamma(z)$ or $z <^+ x$. But $z <^+ x$ would contradict the choice of $t_{\sup}$. Thus $\gamma(x) <^\sim \gamma(z)$, and hence the equation 1. holds in this case.

Case $t_{\sup} = \sup_\sim(\gamma(\mathcal{I}^{\leq^+ x}_{\gamma\gamma(x)}))$. Fix $y_{\sup} \in \mathcal{I}^{\leq^+ x}_{\gamma\gamma(x)}$ such that $t_{\sup} = \gamma(y_{\sup})$. Assume that $\gamma(x) <^\sim \gamma(z)$. As $\gamma(y_{\sup}) \leq^+ \gamma(x)$ we have $\gamma(y_{\sup}) < \gamma(z)$. But $\gamma(y_{\sup}) \not<^+ \gamma(z)$, so $\gamma(y_{\sup}) <^\sim \gamma(z)$. For converse, assume that $\gamma(y_{\sup}) <^\sim \gamma(z)$. If $x \in S^{-\varepsilon}$ then $\varrho(x) <^\sim \gamma(y_{\sup})$ and again by Lemma 7.4.1 we have that either $\gamma(x) <^\sim \gamma(z)$ or $z <^+ x$. If $x \in S^\varepsilon$ then by Lemma 7.4.2 we get once more that either $\gamma(x) <^\sim \gamma(z)$ or $z <^+ x$. But $z <^+ x$ would contradict the choice of $t_{\sup}$. Thus $\gamma(x) <^\sim \gamma(z)$, and hence the equation 1. holds in this case as well.

Ad 2. We consider again two cases depending on the set $t_{\inf}$ is in. Fix $z \in \mathcal{I}_u$.

Case $t_{\inf} = \inf_\sim(\{t \in \dot{\delta}(x) : u \in \delta(t)\})$. Assume $\gamma(z) <^\sim t_{\inf}$. Then by Lemma 7.4.3 we have that either $\gamma(z) <^\sim \gamma(x)$ or $z <^+ x$. But $z <^+ x$ would contradict the choice of $t_{\inf}$. Thus $\gamma(z) <^\sim \gamma(x)$. To see the converse, assume that $\gamma(z) <^\sim \gamma(x)$. But then as other cases are easily excluded we must have $\gamma(z) <^\sim t_{\inf}$, and hence the equation 2. holds in this case.

Case $t_{\inf} = \inf_\sim(\gamma(\mathcal{I}^{\leq^+ x}_u))$. Fix $y_{\inf} \in \mathcal{I}^{\leq^+ x}_u$ such that $t_{\inf} = \gamma(y_{\inf})$. Assume that $\gamma(z) <^\sim \gamma(x)$. As $\gamma(y_{\inf}), \gamma(z) \in \mathcal{I}^{\leq^+ x}_u$ we have $\gamma(y_{\inf}) \perp^\sim \gamma(z)$. As $\gamma(y_{\inf}) <^\sim \gamma(z)$ leads immediately to a contradiction we have $\gamma(z) <^\sim \gamma(y_{\inf})$. To see the converse assume that $\gamma(z) <^\sim \gamma(y_{\inf})$. If $x \in S^{-\varepsilon}$ then by definition of $y_{\sup}$ we have $\gamma(y_{\sup}) <^\sim \inf_\sim\{t' \in \dot{\delta}(x) : u \in \delta(t')\}$ and again by Lemma 7.4.3 we have that either $\gamma(z) <^\sim \gamma(x)$ or $z <^+ x$. If $x \in S^\varepsilon$ then by Lemma 7.4.4 we get once more that either $\gamma(z) <^\sim \gamma(x)$ or $z <^+ x$. But $z <^+ x$ would contradict the choice of $t_{\inf}$. Thus $\gamma(z) <^\sim \gamma(x)$, and hence the equation 2. holds in this case as well. $\Box$

Lemma 7.7 *Let S be an ordered face structure $u, t, x \in S$, $u \in \dot{\delta}(t)$ and $t \in \dot{\delta}(x)$. We put*

$$t_{\sup} = \sup_\sim(\{t' \in \dot{\delta}(x) : t' <^\sim t,\, \gamma(t') = u\} \cup \gamma(\{y \in \mathcal{I}^{\leq^+ x}_u : \gamma(y) <^\sim t\})),$$

$$t_{\inf} = \inf_\sim(\{t' \in \dot{\delta}(x) : t <^\sim t'\} \cup \gamma(\{y \in \mathcal{I}^{\leq^+ x}_{\gamma(t)} : t <^\sim \gamma(y)\})).$$

The elements $t_{\sup}$, $t_{\inf}$ are not necessarily well defined, due to the fact that these set might be empty, but we have

3. $$(u, \downarrow t, -) = \begin{cases} (u, \downarrow \gamma(x), -) & \text{if } t_{\sup} \text{ is undefined,} \\ (u, -, \uparrow t_{\sup}) & \text{otherwise.} \end{cases}$$

4. $$(\gamma(t), -, \uparrow t) = \begin{cases} (\gamma\gamma(x), -, \uparrow \gamma(x)) & \text{if } t_{\inf} \text{ is undefined,} \\ (\gamma(t), \downarrow t_{\inf}, -) & \text{otherwise.} \end{cases}$$

Proof. Ad 3. First note that if $t_{\sup}$ is undefined then t is $t_{\inf}$ from Lemma 7.6. Thus the equation 3. holds in this case by Lemma 7.6.2. If $t_{\sup}$ is defined then we consider two cases depending on the set $t_{\sup}$ is in. However in either case, by Lemma 7.3, we have that $\downarrow_u t \cup \uparrow t_{\sup} = \mathcal{I}_u$.

Case $t_{\sup} = \sup_\sim(\{t' \in \dot{\delta}(x) : t' <^\sim t,\, \gamma(t') = u\})$. Suppose there is $z \in \mathcal{I}_u$ such that $t_{\sup} <^\sim \gamma(z) <^\sim t$. Then, as the assumption 1. of Lemma 7.5 holds, we have that either $z <^+ x$ or there is $t' \in \delta^\lambda(x)$ such that $t_{\sup} <^\sim t' <^\sim t$, $\gamma(z) \leq^+ t'$ and $\gamma(t') = u$. Both cases contradict the choice $t_{\sup}$. Thus $\downarrow_u t \cap \uparrow t_{\sup} = \emptyset$, and hence $(u, \downarrow t, -) = (u, -, \uparrow t_{\sup})$ i.e. the equation 3. holds in this case.

Case $t_{\sup} = \sup_\sim(\gamma(\{y \in \mathcal{I}^{\leq^+ x}_u : \gamma(y) <^\sim t\}))$. Fix $y_{\sup} \in \mathcal{I}^{\leq^+ x}_u$ such that $\gamma(y_{\sup}) = t_{\sup}$. Suppose there is $z \in \mathcal{I}_u$ such that $\gamma(y_{\sup}) <^\sim \gamma(z) <^\sim t$. Then, as the assumption 2. of Lemma 7.5 holds, we have that either $z <^+ x$ or there is $t' \in \delta^\lambda(x)$ such that $\gamma(y_{\sup}) <^\sim t' <^\sim t$, $\gamma(z) \leq^+ t'$ and $\gamma(t') = u$. Both cases contradict the choice $y_{\sup}$. Thus $\downarrow_u t \cap \uparrow \gamma(y_{\sup}) = \emptyset$, i.e. the equation 3. holds in this case, as well.

Ad 4. First note that if $t_{\rm inf}$ is undefined then t is $t_{\rm sup}$ from Lemma 7.6. Thus the equation 4. holds in this case by Lemma 7.6.1. If $t_{\rm inf}$ is defined then we consider two cases depending on the set $t_{\rm inf}$ is in. However in either case, by Lemma 7.3, we have that $\downarrow_{\gamma(t)} t_{\rm inf} \cup \uparrow t = \mathcal{I}_{\gamma(t)}$.

Case $t_{\rm inf} = \inf_\sim(\{t' \in \dot{\delta}(x) : t <^\sim t'\})$. Suppose there is $z \in \mathcal{I}_{\gamma(t)}$ such that $t <^\sim \gamma(z) <^\sim t_{\rm inf}$. Then, as the assumption 1. of Lemma 7.5 holds, we have that either $z <^+ x$ or there is $t' \in \delta^\lambda(x)$ such that $t <^\sim t' <^\sim t_{\rm inf}$, $\gamma(z) \leq^+ t'$ and $\gamma(t') = \gamma(t)$. Both cases contradict the choice $t_{\rm inf}$. Thus $\downarrow_{\gamma(t)} t_{\rm inf} \cap \uparrow t = \emptyset$, and hence $(\gamma(t), -, \uparrow t) = (\gamma(t), \downarrow t_{\rm inf}, -)$, i.e. the equation 4. holds in this case.

Case $t_{\rm inf} = \inf_\sim(\gamma(\{y \in \mathcal{I}^{\leq^+ x}_{\gamma(t)} : t <^\sim \gamma(y)\}))$. Fix $y_{\rm inf} \in \mathcal{I}^{\leq^+ x}_{\gamma(t)}$ such that $\gamma(y_{\rm inf}) = t_{\rm inf}$. Suppose there is $z \in \mathcal{I}_u$ such that $t <^\sim \gamma(z) <^\sim \gamma(y_{\rm inf})$. Then, as the assumption 3. of Lemma 7.5 holds, we have that either $z <^+ x$ or there is $t' \in \delta^\lambda(x)$ such that $\gamma(y_{\rm sup}) <^\sim t' <^\sim t$, $\gamma(z) \leq^+ t'$ and $\gamma(t') = \gamma(t)$. Both cases contradict the choice $y_{\rm inf}$. Thus $\downarrow_{\gamma(t)} t \cap \downarrow \gamma(y_{\rm inf}) = \emptyset$, i.e. the equation 4. holds in this case, as well. $\Box$

Lemma 7.8 *Let S be an ordered face structure $y, x \in S$, $y \in \mathcal{I}^{\leq^+ x}_{\gamma\gamma(y)}$. Then $\gamma\gamma(y) \in \theta\delta(x)$. We put*

$$t_{\rm sup} = \sup_\sim(\{t \in \dot{\delta}(x) : t <^\sim \gamma(y)\} \cup \gamma(\{y' \in \mathcal{I}^{\leq^+ x}_{\gamma\gamma(y)} : \gamma(y') <^\sim \gamma(y)\})),$$

$$t_{\rm inf} = \inf_\sim(\{t \in \dot{\delta}(x) : \gamma(y) <^\sim t\} \cup \gamma(\{y' \in \mathcal{I}^{\leq^+ x}_{\gamma\gamma(y)} : \gamma(y) <^\sim \gamma(y')\})).$$

The elements $t_{\rm sup}$,$t_{\rm inf}$ are not necessarily well defined, due to the fact that these set might be empty, but we have

5. $$(\gamma\gamma(y), \downarrow \gamma(y), -) = \begin{cases} (\gamma\gamma(y), \downarrow \gamma(x), -) & \text{if } t_{\rm sup} \text{ is undefined,} \\ (\gamma\gamma(y), -, \uparrow t_{\rm sup}) & \text{otherwise.} \end{cases}$$

6. $$(\gamma\gamma(y), -, \uparrow \gamma(y)) = \begin{cases} (\gamma\gamma(x), -, \uparrow \gamma(x)) & \text{if } t_{\rm inf} \text{ is undefined,} \\ (\gamma\gamma(y), \downarrow t_{\rm inf}, -) & \text{otherwise.} \end{cases}$$

Proof. Ad 5. First note that if $t_{\rm sup}$ is undefined then, with $u = \gamma\gamma(y)$, y is $y_{\rm inf}$ from (the proof of) Lemma 7.6. Thus the equation 5. holds in this case by Lemma 7.6.2. If $t_{\rm sup}$ is defined then we consider two cases depending on the set $t_{\rm sup}$ is in. However in either case, by Lemma 7.3, we have that $\downarrow_{\gamma\gamma(y)} \gamma(y) \cup \uparrow t_{\rm sup} = \mathcal{I}_{\gamma\gamma(y)}$.

Case $t_{\rm sup} = \sup_\sim(\{t \in \dot{\delta}(x) : t <^\sim \gamma(y)\})$. Suppose there is $z \in \mathcal{I}_{\gamma\gamma(y)}$ such that $t_{\rm sup} <^\sim \gamma(z) <^\sim \gamma(y)$. Then, as the assumption 3. of Lemma 7.5 holds, we have that either $z <^+ x$ or there is $t' \in \delta^\lambda(x)$ such that $t_{\rm sup} <^\sim t' <^\sim t$, $\gamma(z) \leq^+ t'$ and $\gamma(t') = \gamma\gamma(y)$. Both cases contradict the choice $t_{\rm sup}$. Thus $\downarrow_{\gamma\gamma(y)} \gamma(y) \cap \uparrow t_{\rm sup} = \emptyset$, i.e. the equation 5. holds in this case.

Case $t_{\rm sup} = \sup_\sim(\gamma(\{y' \in \mathcal{I}^{\leq^+ x}_{\gamma\gamma(y)} : \gamma(y') <^\sim \gamma(y)\}))$. Fix $y_{\rm sup} \in \mathcal{I}^{\leq^+ x}_{\gamma\gamma(y)}$ such that $\gamma(y_{\rm sup}) = t_{\rm sup}$. Suppose there is $z \in \mathcal{I}_{\gamma\gamma(y)}$ such that $\gamma(y_{\rm sup}) <^\sim \gamma(z) <^\sim \gamma(y)$. Then, as the assumption 4. of Lemma 7.5 holds, we have that either $z <^+ x$ or there is $t' \in \delta^\lambda(x)$ such that $\gamma(y_{\rm sup}) <^\sim t' <^\sim t$, $\gamma(z) \leq^+ t'$ and $\gamma(t') = \gamma\gamma(y)$. Both cases contradict the choice $y_{\rm sup}$. Thus $\downarrow_{\gamma\gamma(y)} \gamma(y) \cap \uparrow \gamma(y_{\rm sup}) = \emptyset$, i.e. the equation 5. holds in this case, as well.

Ad 6. First note that if $t_{\rm inf}$ is undefined then, $\gamma\gamma(y) = \gamma\gamma(x)$ and y is $y_{\rm sup}$ from (the proof of) Lemma 7.6. Thus the equation 6. holds in this case by Lemma 7.6.1. If $t_{\rm inf}$ is defined then consider two cases depending on the set $t_{\rm inf}$ is in. However in either case, by Lemma 7.3, we have that $\downarrow_{\gamma\gamma(y)} t_{\rm inf} \cup \uparrow \gamma(y) = \mathcal{I}_{\gamma\gamma(y)}$.

Case $t_{\inf} = \inf_{\sim}(\{t \in \dot{\delta}(x) : \gamma(y) <^{\sim} t\})$. Suppose there is $z \in \mathcal{I}_{\gamma\gamma(y)}$ such that $\gamma(y) <^{\sim} \gamma(z) <^{\sim} t_{\inf}$. Then, as the assumption 2. of Lemma 7.5 holds, we have that either $z <^{+} x$ or there is $t' \in \delta^{\lambda}(x)$ such that $t <^{\sim} t' <^{\sim} t_{\inf}$, $\gamma(z) \leq^{+} t'$ and $\gamma(t') = \gamma\gamma(y)$. Both cases contradict the choice $t_{\inf}$. Thus $\downarrow_{\gamma\gamma(y)} t_{\inf} \cap \uparrow \gamma(y) = \emptyset$, hence $(\gamma\gamma(y), -, \uparrow \gamma(y)) = (\gamma\gamma(y), \downarrow t_{\inf}, -)$, i.e. the equation 6. holds in this case.

Case $t_{\inf} = \inf_{\sim}(\gamma(\{y' \in \mathcal{I}^{\leq^{+}x}_{\gamma\gamma(y)} : \gamma(y) <^{\sim} \gamma(y')\}))$. Fix $y_{\inf} \in \mathcal{I}^{\leq^{+}x}_{\gamma\gamma(y)}$ such that $\gamma(y_{\inf}) = t_{\inf}$. Suppose there is $z \in \mathcal{I}_{\gamma\gamma(y)}$ such that $\gamma(y) <^{\sim} \gamma(z) <^{\sim} \gamma(y_{\inf})$. Then, as the assumption 4. of Lemma 7.5 holds, we have that either $z <^{+} x$ or there is $t' \in \delta^{\lambda}(x)$ such that $\gamma(y) <^{\sim} t' <^{\sim} \gamma(y_{\inf})$, $\gamma(z) \leq^{+} t'$ and $\gamma(t') = \gamma\gamma(y)$. Both cases contradict the choice $y_{\inf}$. Thus $\uparrow_{\gamma\gamma(y)} t \cap \downarrow \gamma(y_{\inf}) = \emptyset$, i.e. the equation 6. holds in this case, as well. $\Box$

The Proof

Proof of Theorem 7.1. Fix an ordered face structure S. Clearly for $a \in \mathcal{I}$, $\delta^{\dagger}(\overline{a}) \neq \emptyset$. Suppose that (x, L, U) is a cut in $S^{\dagger}_{k}$, with $k > 0$. Then either $\dot{\delta}(x) \neq \emptyset$ or $\dot{\delta}(x) = \emptyset$ and then by Lemma 4.3 we have that there is $y \in \mathcal{I}^{\leq^{+}x}$. In either case $\delta^{\dagger}(x, L, U) \neq \emptyset$. Thus $S^{\dagger}$ is a positive hypergraph. We shall check that $S^{\dagger}$ satisfies all four positive face structure axioms.

Globularity. We need to verify globularity for both kinds of faces in $S^{\dagger}$: bars and cuts. First we shall check globularity for bars. Fix $\alpha \in \mathcal{I}$. We have

$$\gamma^{\dagger}\gamma^{\dagger}(\overline{\alpha}) = (\gamma\gamma\gamma(\alpha), -, \uparrow \gamma\gamma(\alpha)) = \gamma^{\dagger}\delta^{\dagger}(\overline{\alpha}),$$

$$\delta^{\dagger}\gamma^{\dagger}(\overline{\alpha}) = \overline{\mathcal{I}^{\leq^{+}\gamma(\alpha)}} \cup \{(t, \downarrow \gamma\gamma(\alpha), -) : t \in \dot{\delta}\gamma\gamma(\alpha)\} = \delta^{\dagger}\delta^{\dagger}(\overline{\alpha}).$$

We need to show that

$$\gamma^{\dagger}\gamma^{\dagger}(\overline{\alpha}) \not\in \delta^{\dagger}\gamma^{\dagger}(\overline{\alpha}).$$

Suppose contrary that $\gamma^{\dagger}\gamma^{\dagger}(\overline{\alpha}) \in \delta^{\dagger}\gamma^{\dagger}(\overline{\alpha})$. Then, as $\gamma^{\dagger}\gamma^{\dagger}(\overline{\alpha})$ is a cut, we would have $\gamma\gamma\gamma(\alpha) \in \delta\gamma\gamma(\alpha)$, i.e. $\gamma\gamma(\alpha)$ is a loop. Thus by Lemma 4.3 there is $a \in \mathcal{I}_{\gamma\gamma\gamma(\alpha)}$ such that $\gamma(a) \leq \gamma\gamma(\alpha)$. But then $\uparrow \gamma\gamma(\alpha) \not\ni a \not\in \downarrow \gamma\gamma(\alpha)$ and hence

$$(\gamma\gamma\gamma(\alpha), -, \uparrow \gamma\gamma(\alpha)) \neq (\gamma\gamma\gamma(\alpha), \downarrow \gamma\gamma(\alpha), -),$$

which means that $\gamma^{\dagger}\gamma^{\dagger}(\overline{\alpha}) \not\in \delta^{\dagger}\gamma^{\dagger}(\overline{\alpha})$ after all. From this the globularity for bars follow easily.

Now we shall check globularity for cuts. Fix a cut (x, L, U) in $S^{\dagger}_{k}$, with $k > 1$. The sets involved in the globularity equations are sums of some sets. We shall spell these sets below giving names to their summands. We have

$$\gamma^{\dagger}\gamma^{\dagger}(x, L, U) = (\gamma\gamma(x), -, \uparrow \gamma(x)) = \psi,$$

$$\delta^{\dagger}\gamma^{\dagger}(x, L, U) = \overline{\mathcal{I}^{\leq^{+}\gamma(x)}} \cup \{(u, \downarrow \gamma(x), -) : u \in \dot{\delta}\gamma(x)\} = Z_1 \cup Z_2,$$

$$\gamma^{\dagger}\delta^{\dagger}(x, L, U) = \{(\gamma\gamma(y), -, \uparrow \gamma(y)) : y \in \mathcal{I}^{\leq^{+}x}\} \cup \{(\gamma(t), -, \uparrow t) : t \in \dot{\delta}(x)\} = Z_3 \cup Z_4$$

$$\delta^{\dagger}\delta^{\dagger}(x, L, U) = \{(\gamma\gamma(y), \downarrow \gamma(y), -) : y \in \mathcal{I}^{\leq^{+}x}\} \cup \{\overline{s} : s \in \mathcal{I}^{\leq^{+}t},\ t \in \dot{\delta}(x)\} \cup$$
$$\cup \{(u, \downarrow t, -) : t \in \dot{\delta}(x),\ u \in \dot{\delta}(t)\} = Z_5 \cup Z_6 \cup Z_7$$

In order to verify γ-globularity, i.e.

$$\gamma^{\dagger}\gamma^{\dagger}(x, L, U) = \gamma^{\dagger}\delta^{\dagger}(x, L, U) - \delta^{\dagger}\delta^{\dagger}(x, L, U),$$

we shall show:

(A) $\psi \in \gamma^\dagger\delta^\dagger(x, L, U)$,

(B) $\psi \notin \delta^\dagger\delta^\dagger(x, L, U)$,

(C) $\gamma^\dagger\delta^\dagger(x, L, U) - \psi \subseteq \delta^\dagger\delta^\dagger(x, L, U)$.

Ad A. By Lemma 7.6 ($\gamma(x)$-cuts) either $x \in S^{-\varepsilon}$ and $\psi = (\gamma\varrho(x), -, \uparrow \varrho(x)) \in Z_4$ or there is $y \in\in \mathcal{I}^{\leq^+ x}_{\gamma\gamma(x)}$ such that $\psi = (\gamma\gamma(y), -, \uparrow \gamma(y)) \in Z_3$. In either case $\psi \in \gamma^\dagger\delta^\dagger(x, L, U)$.

Ad B. As ψ is not a bar, we have $\psi \notin Z_6$.

Suppose $\psi \in Z_5$. Then there is $y \in \mathcal{I}^{\leq^+ x}_{\gamma\gamma(x)}$ such that $(\gamma\gamma(y), \downarrow \gamma(y), -) = \psi$. So $y \in \uparrow \gamma(x)$, i.e. $\gamma(x) <^\sim \gamma(y)$. But $y \leq^+ x$, so $\gamma(y) \leq^+ \gamma(x)$ and we have a contradiction with the disjointness. Thus $\psi \notin Z_5$.

Suppose now that $\psi \in Z_7$. So there is $t \in \dot{\delta}(x)$ such that $\gamma\gamma(x) \in \dot{\delta}(t)$ and $(\gamma\gamma(x), \downarrow t, -) = \psi$. As $t \in \dot{\delta}(x)$ we have $\gamma(t) \leq^+ \gamma\gamma(x)$. So t is a loop. Then, by Lemma 4.3, there is $y \in \mathcal{I}_{\gamma\gamma(x)}$ such that $\gamma(y) \leq^+ t$. As $y \in S^{-\lambda}$, we have $y <^\sim x$, and hence $\gamma(y) \leq^+ \gamma(x)$. Thus $y \not\in\downarrow_{\gamma\gamma(x)} t$ and $y \not\in\uparrow \gamma(x)$, i.e. $(\gamma\gamma(x), \downarrow t, -) \neq \psi$ after all. Thus $\psi \notin Z_5$ and hence $\psi \notin \delta^\dagger\delta^\dagger(x, L, U)$.

Ad C. Fix $\xi \in \gamma^\dagger\delta^\dagger(x, L, U)$, such that $\xi \neq \psi$. If $\xi \in Z_4$ then there is $t \in \dot{\delta}(x)$ such that $\xi = (\gamma(t), -, \uparrow t)$. We shall use Lemma 7.7 (t-cuts). As $\xi \neq \psi$

$$t_{\inf} = \inf_{\sim}(\{t' \in \delta(x) : t <^\sim t'\} \cup \gamma(\{y \in \mathcal{I}^{\leq^+ x}_{\gamma(t)} : t <^\sim \gamma(y)\})).$$

is well defined and then $\xi = (\gamma(t), \downarrow t_{\inf}, -)$. Now, if $t_{\inf} = \inf_\sim(\{t' \in \delta(x) : t <^\sim t'\})$ then $\xi \in Z_7$ and if $t_{\inf} = \inf_\sim(\gamma(\{y \in \mathcal{I}^{\leq^+ x}_{\gamma(t)} : t <^\sim \gamma(y)\}))$ then $\xi \in Z_5$.

If $\xi \in Z_3$ then there is $y \in \mathcal{I}^{\leq^+ x}$, so that $\xi = (\gamma\gamma(y), -, \uparrow \gamma(y))$. We shall use Lemma 7.8 ($\gamma(y)$-cuts). As $\xi \neq \psi$ then

$$t_{\inf} = \inf_{\sim}(\{t \in \delta(x) : \gamma(y) <^\sim t\} \cup \gamma(\{y' \in \mathcal{I}^{\leq^+ x}_{\gamma\gamma(y)} : \gamma(y) <^\sim \gamma(y')\}))$$

is well defined and $\xi = (\gamma\gamma(y), \downarrow t_{\inf}, -)$. Again, if $t_{\inf} = \inf_\sim(\{t \in \delta(x) : \gamma(y) <^\sim t\})$ then $\xi \in Z_7$ and if $t_{\inf} = \inf_\sim(\gamma(\{y' \in \mathcal{I}^{\leq^+ x}_{\gamma\gamma(y)} : \gamma(y) <^\sim \gamma(y')\}))$ then $\xi \in Z_5$. Thus C. holds. This ends verification of γ-globularity for $S^\dagger$.

Now we shall check δ-globularity for $S^\dagger$, i.e.

$$\delta^\dagger\gamma^\dagger(x, L, U) = \delta^\dagger\delta^\dagger(x, L, U) - \gamma^\dagger\delta^\dagger(x, L, U).$$

Both sides of this equation contains both bars and cuts. We show equalities for them separately.

First we shall show the equality for bars. We need to show that $Z_6 = Z_1$. Clearly, $Z_6 \subseteq Z_1$. We shall verify that $Z_1 \subseteq Z_6$. Let $\bar{t} \in Z_1$, i.e. $t \in \mathcal{I}$ and $t \leq^+ \gamma(x)$. As $t \in \mathcal{I}$ and $\mathcal{I} \cap \gamma(S^{-\lambda}) = \emptyset$ we have that either $t = \gamma(x) = \delta(x)$ or $t \neq \gamma(x)$. In the former case clearly $\bar{t} \in Z_6$. In the later case there is an upper $(S - \gamma(S^{-\lambda}))$-path $t, x_1, \ldots, x_k, \gamma(x)$ with $k \geq 1$. By pencil linearity $x_k \leq^+ x$. As $t \in \mathcal{I} \subseteq S - \gamma(S^{-\lambda})$ by Second Path Lemma, either $t \in \delta(x)$ or there is $1 \leq i < k$ such that $\gamma(x_i) \in \delta(x)$. In either case there is $s \in \delta(x)$ ($s = t$ or $s = \gamma(x_i)$) such that $t \leq^+ s$, i.e. $\bar{t} \in Z_6$. Thus the δ-globularity for bars holds.

Now we will show the δ-globularity for cuts. Clearly, it is enough to restrict ourself to cuts over $\dot{\theta}\delta(x)$ as other cuts cannot appear in the equation. Moreover, by Lemma 4.7 (atlas), we have $\dot{\theta}\delta(x) = \dot{\delta}\gamma(x) \cup \gamma\dot{\delta}^{-\lambda}(x)$. As in $\delta^\dagger\gamma^\dagger(x, L, U)$ can appear only cuts over $\dot{\delta}\gamma(x)$ we will split our proof farther by considering these two case separately. Let $u \in \dot{\theta}\delta(x)$. By X_{u-cuts} we mean u-cuts in the set X. To end the proof of δ-globularity we need to show:

(I) if $u \in \gamma\dot{\delta}^{-\lambda}(x)$ then $\delta^\dagger\delta^\dagger(x,L,U)_{u-cuts} \subseteq \gamma^\dagger\delta^\dagger(x,L,U)$,

(II) if $u \in \dot{\delta}\gamma(x)$ then the cut $\psi_u = (u, \downarrow\gamma(x), -)$ is the only u-cut in $\delta^\dagger\gamma^\dagger(x,L,U)$. Moreover we have:

(A$_u$) $\psi_u \in \delta^\dagger\delta^\dagger(x,L,U)$,

(B$_u$) $\psi_u \not\in \gamma^\dagger\delta^\dagger(x,L,U)$,

(C$_u$) $\delta^\dagger\delta^\dagger(x,L,U)_{u-cuts} - \psi_u \subseteq \gamma^\dagger\delta^\dagger(x,L,U)$.

Note the similarity of the conditions (A), (B), (C) with (A$_u$), (B$_u$), (C$_u$).

Ad I. Fix $u \in \gamma\dot{\delta}^{-\lambda}(x)$ and $t_u \in \dot{\delta}^{-\lambda}(x)$ such that $\gamma(t_u) = u$. Let $\varphi = (u, L', U') \in \delta^\dagger\delta^\dagger(x,L,U)$. We put

$$t_\varphi = \begin{cases} t & \text{if } \varphi \in Z_7 \text{ and } t \in \dot{\delta}(x) \text{ such that } L' = \downarrow_u t, \\ \gamma(y) & \text{if } \varphi \in Z_5, \text{ and } y \in \mathcal{I}_u^{\leq^+ x} \text{ such that } L' = \downarrow_u \gamma(y). \end{cases}$$

Thus $\varphi = (u, \downarrow t_\varphi, -)$. Put

$$t_{\mathrm{sup}} = \sup_\sim(\{t' \in \dot{\delta}(x) : t' <^\sim t_\varphi,\ \gamma(t') = u\} \cup \gamma(\{y' \in \mathcal{I}_u^{\leq^+ x} : \gamma(y') <^\sim t_\varphi\})).$$

As $t_u \in \{t' \in \delta(x) : t' <^\sim t_\varphi,\ \gamma(t') = u\} \neq \emptyset$ the face t_{sup} is well defined. Then, by Lemmas 7.7 and 7.8 we have that $\varphi = (u, -, \uparrow t_{\mathrm{sup}}) \in \gamma^\dagger\gamma^\dagger(x,L,U)$.

Ad II. The fact that ψ_u is the only u-cut in $\delta^\dagger\gamma^\dagger(x,L,U)$ is obvious from our description of this set as sum $Z_1 \cup Z_2$.

Ad A$_u$. Let $t_{\mathrm{inf}} = \inf_\sim(\{t \in \dot{\delta}(x) : u \in \delta(t)\} \cup \gamma(\mathcal{I}_u^{\leq^+}))$. If $\{t \in \dot{\delta}(x) : u \in \delta(t)\} = \emptyset$ then $x \in S^\varepsilon$ and hence, by Lemma 4.3, $\mathcal{I}_u^{\leq^+} \neq \emptyset$. Thus t_{inf} is well defined. By Lemma 7.6, we have $\psi_u = (u, \downarrow t_{\mathrm{inf}}, -) \in \delta^\dagger\delta^\dagger(x,L,U)$, as required.

Ad B$_u$. Suppose $\psi_u \in Z_3$. Then there is $y \in \mathcal{I}_u^{\leq^+ x}$ such that $\psi_u = (u, -, \uparrow\gamma(y))$. As $y \leq^+ x$ we have $\gamma(y) \leq^+ \gamma(x)$. Thus $\gamma(y) \not\leq^\sim \gamma(x)$. This means that $y \not\in \downarrow_u \gamma(x)$. Clearly $y \not\in \uparrow\gamma(y)$. Thus $\psi_u = (u, \downarrow\gamma(x), -) \neq (u, -, \uparrow\gamma(y))$, after all. This shows that $\psi_u \not\in Z_3$.

Suppose now that $\psi_u \in Z_4$. So there is $t \in \dot{\delta}(x)$ such that $\psi_u = (u, -, \uparrow t)$. As $\gamma(t) = u \in \delta\gamma(x)$, t is a loop. Then, by Lemma 4.3, there is $y \in \mathcal{I}_u$ such that $\gamma(y) \leq^+ t$ and, by transitivity of $<^+$, $\gamma(y) \leq^+ \gamma(x)$. Thus $y \not\in \uparrow t$ and $y \not\in \downarrow_u \gamma(x)$. Then $\psi_u = (u, \downarrow\gamma(x), -) \neq (u, -, \uparrow t)$. So $\psi \not\in Z_4$, and hence $\psi \not\in \gamma^\dagger\delta^\dagger(x,L,U)$.

Ad C$_u$. Fix $\xi = (u, L', U') \in \delta^\dagger\delta^\dagger(x,L,U)$, such that $\xi \neq \psi_u$. If $\xi \in Z_7$ then there is $t \in \dot{\delta}(x)$ such that $u \in \dot{\delta}(t)$ and $\xi = (u, \downarrow t, -)$. We shall use Lemma 7.7 (t-cuts). As $\xi \neq \psi_u$ the face

$$t_{\mathrm{sup}} = \sup_\sim(\{t' \in \dot{\delta}(x) : t' <^\sim t,\ \gamma(t') = u\} \cup \gamma(\{y \in \mathcal{I}_u^{\leq^+ x} : \gamma(y) <^\sim t\})),$$

is well defined and then $\xi = (u, -, \uparrow t_{\mathrm{sup}})$. Now, if $t_{\mathrm{sup}} = \sup_\sim(\{t' \in \delta(x) : t' <^\sim t,\ \gamma(t') = u\})$ then $\xi \in Z_4$ and if $t_{\mathrm{sup}} = \sup_\sim(\gamma(\{y \in \mathcal{I}_u^{\leq^+ x} : \gamma(y) <^\sim t\}))$ then $\xi \in Z_3$.

If $\xi \in Z_5$ then there is $y \in \mathcal{I}^{\leq^+ x}$, so that $\xi = (u, \downarrow\gamma(y), -)$. We shall use Lemma 7.8 ($\gamma(y)$-cuts).

As $\xi \neq \psi_u$ the face

$$t_{\mathrm{sup}} = \sup_\sim(\{t \in \dot{\delta}(x) : t <^\sim \gamma(y)\} \cup \gamma(\{y' \in \mathcal{I}_{\gamma\gamma(y)}^{\leq^+ x} : \gamma(y') <^\sim \gamma(y)\})),$$

is well defined and $\xi = (u, -, \uparrow t_{\mathrm{sup}})$. Again, if $t_{\mathrm{sup}} = \sup_\sim(\{t \in \dot{\delta}(x) : t <^\sim \gamma(y)\})$ then $\xi \in Z_4$ and if $t_{\mathrm{sup}} = \sup_\sim(\gamma(\{y' \in \mathcal{I}_{\gamma\gamma(y)}^{\leq^+ x} : \gamma(y') <^\sim \gamma(y)\}))$ then $\xi \in Z_3$. Thus C$_u$. holds. This ends verification of δ-globularity for $S^\dagger$.

Orders in $S^\dagger$. Before we verify the remaining axioms of positive face structures we shall describe the order in $S^\dagger$. Let $(x,L,U),(y,L',U') \in \mathcal{C}(\mathcal{I}_{S_k})$ be two cuts in $S^\dagger_k$ and $a,b \in \mathcal{I}_{k+1}$ so that $\overline{a},\overline{b}$ are two bars in $S^\dagger_k$.

For $k \geq 0$, the upper order $<^{\dagger,+}$ in $S^\dagger_k$ can be characterized as follows ($<^+$ is the upper order in S):

1. (cut,cut): $(x,L,U) <^{\dagger,+} (y,L',U')$ iff either $x <^+ y$ or $x = y$ and $L \subsetneq L'$;
2. (bar,cut): $\overline{a} <^{\dagger,+} (y,L',U')$ iff $\gamma(a) \leq^+ y$;
3. (cut,bar): $(x,L,U) <^{\dagger,+} \overline{b}$ never holds true;
4. (bar,bar): $\overline{a} <^{\dagger,+} \overline{b}$ never holds true.

For $k \geq 1$, the lower order $<^{\dagger,-}$ in $S^\dagger_k$ can be characterized as follows ($<^\sim$ is the lower order in S):

1. (cut,cut): $(x,L,U) <^{\dagger,-} (y,L',U')$ iff $x <^\sim y$;
2. (bar,cut): $\overline{a} <^{\dagger,-} (y,L',U')$ iff $\gamma(a) <^\sim y$;
3. (cut,bar): $(x,L,U) <^{\dagger,-} \overline{b}$ iff $x <^\sim \gamma(b)$;
4. (bar,bar): $\overline{a} <^{\dagger,-} \overline{b}$ iff $\gamma(a) <^\sim \gamma(b)$.

Strictness. The strictness is obvious from the above description of $<^{\dagger,+}$. Note that all faces in $S^\dagger_0$ are cuts. So $<^{\dagger,+}$ on $S^\dagger_0$ is a linear order since $<^+$ is.

Disjointness. With the description of $<^{\dagger,+}$ and $<^{\dagger,-}$ above the disjointness is a matter of a simple check using disjointness of $<^+$ and $<^\sim$.

Pencil linearity. Let $\overline{a}$, $\overline{b}$ be two different bars in $S^\dagger$ and (x,L,U), (y,L,U) be two different cuts in $S^\dagger$. To show γ-linearity we need to consider three cases:

1. $\gamma(x,L,U) = \gamma(y,L',U')$,
2. $\gamma(\overline{a}) = \gamma(x,L,U)$,
3. $\gamma(\overline{a}) = \gamma(\overline{b})$.

Ad 1. We have $(\gamma(x), -, \uparrow x) = (\gamma(y), -, \uparrow y)$. If $x = y$ then either $L \subsetneq L'$ or $L' \subsetneq L$. Thus $x \perp^+ y$. If $x \neq y$ and $\gamma(x) = \gamma(y)$ then either $x \perp^+ y$ or $x \perp^\sim y$. In case $x \perp^+ y$ we have $(x,L,U) \perp^+ (y,L,U)$. We shall show that $x \perp^\sim y$ is impossible. Suppose $x <^\sim y$. As $\gamma(x) = \gamma(y)$, it follows that y is a loop. Let $c \in \mathcal{I}_{\gamma(y)}$ be an initial face such that $\gamma(c) \leq^+ y$. Then $x <^\sim \gamma(c)$ and $y \not<^\sim \gamma(a)$, i.e. $\uparrow x \neq \uparrow y$, contrary to the supposition. Thus $x \perp^\sim y$ cannot hold true.

Ad 2. We have $(\gamma\gamma(a), -, \uparrow \gamma(a)) = (\gamma(x), -, \uparrow x)$. As $\gamma\gamma(a) = \gamma(x)$, we have either $\gamma(a) = x$ or $\gamma(a) \perp^+ x$ or $\gamma(a) \perp^\sim x$. If $\gamma(a) \leq^+ x$ then $\overline{a} <^+ (\gamma(a), \emptyset, -) \leq^+ (x,L,U)$. The other conditions are impossible. The condition $x <^+ \gamma(a)$ is impossible by Lemma 7.2, and the condition $\gamma(a) \perp^\sim x$ is impossible as it is easily seen that we were to have $\uparrow x \neq \uparrow \gamma(a)$.

Ad 3. We shall show that this case, i.e. $(\gamma\gamma(a), -, \uparrow \gamma(a)) = (\gamma\gamma(b), -, \uparrow \gamma(b))$ is impossible. As $a, b \in \mathcal{I}_{\gamma\gamma(a)}$ then $\gamma(a) \perp^\sim \gamma(b)$. Suppose $\gamma(a) <^\sim \gamma(b)$. Then $b \in \gamma(a)$ and $b \not\in \gamma(b)$. So we cannot have $\uparrow \gamma(a)) = \uparrow \gamma(b)$. This ends the proof of γ-linearity.

Finally, to verify δ-linearity we need to consider the following four cases:

1. $\overline{z} \in \delta(x,L,U) \cap \delta(y,L',U')$,
2. $(t,L'',U'') \in \delta(x,L,U) \cap \delta(y,L',U')$,

3. $\delta(\overline{a}) \in \delta(x, L, U)$,

4. $\delta(\overline{a}) = \delta(\overline{b})$.

Ad 1. In this case we have $z \leq^+ x$ and $z \leq^+ y$. Thus by Lemma 4.17, $x \perp^+ y$ or $x = y$. In both cases $(x, L, U) \perp^+ (y, L', U')$.

Ad 2. In this case $t \in \delta(x) \cap \delta(y)$ and $(t, L'', U'') = (t, \downarrow x, -) = (t, \downarrow y, -)$. Thus either $x \perp^+ y$ or $x \perp^\sim y$. In the former case we have $(x, L, U) \perp^+ (y, L', U')$. We shall show that the later case is impossible. Suppose $x <^\sim y$. Then $x \in S^\lambda$ and hence there is $a \in \mathcal{I}$ such that $\gamma(a) \leq^+ x$. So $a \in (\downarrow_t y - \downarrow_t x)$ and $(t, \downarrow x, -) \neq (t, \downarrow y, -)$ contrary to the supposition.

Ad 3. In this case $\gamma\gamma(a) \in \delta(x)$ and $(\gamma\gamma(a), \downarrow \gamma(a), -) = (\gamma\gamma(a), \downarrow x, -)$. Thus either $\gamma(a) \perp^+ x$ or $\gamma(a) \perp^\sim x$. If $\gamma(a) \leq^+ x$ then $\overline{a} <^+ (x, L, U)$. The remaining cases are impossible. $x <^+ \gamma(a)$ is impossible by Lemma 7.2, and if we were to have $\gamma(a) \perp^\sim x$ we would have $\downarrow_{\gamma\gamma(a)} \gamma(a) \neq \downarrow_{\gamma\gamma(a)} x$.

Ad 4. We shall show that this case $(\gamma\gamma(a), \downarrow \gamma(a), -) = (\gamma\gamma(b), \downarrow \gamma(b), -)$ is impossible. As $a, b \in \mathcal{I}_{\gamma\gamma(a)}$ we have $\gamma(a) \perp^\sim \gamma(b)$. Say $\gamma(a) <^\sim \gamma(b)$. Then $a \in \downarrow \gamma(b) - \downarrow \gamma(a)$ and $\delta(\overline{a}) \neq \delta(\overline{b})$ after all. This ends the proof of δ-linearity.

The fact that $q_S : S^\dagger \longrightarrow S$ is a positive cover with the kernel $\overline{\mathcal{I}}$. $\Box$

The theorem below show that if we take a positive cover of a quotient by an unary ideal then we get the ordered face structure back. Thus it shows that if we deal with unary ideals only the construction of taking quotient of a positive face structure and taking a positive cover of an ordered face structure are mutually inverse.

8 k-domains and k-codomains of ordered face structures

For any $k \in \omega$, we introduce two operations

$$\mathbf{d}^{(k)}, \mathbf{c}^{(k)} : Ob(\mathbf{oFs}) \longrightarrow Ob(\mathbf{oFs}_k)$$

of the k-th *domain* and the k-th *codomain*.

For a given ordered face structure T the we shall define $\mathbf{d}^{(k)}T$ and $\mathbf{c}^{(k)}T$ via convex subhypergraphs $d^{(k)}T$ and $c^{(k)}T$ of T. Then we shall put

$$\mathbf{d}^{(k)}T = [d^{(k)}T], \qquad \mathbf{c}^{(k)}T = [c^{(k)}T].$$

The operations $d^{(k)}X$ and $c^{(k)}X$ are defined for any convex subset of any ordered face structure T. We put, for $l \in \omega$,

$$(d^{(k)}X)_l = \begin{cases} \emptyset & \text{if } l > k, \\ X_k - \gamma(X^{-\lambda}_{k+1}) & \text{if } l = k, \\ X_l & \text{if } l < k, \end{cases}$$

and

$$(c^{(k)}X)_l = \begin{cases} \emptyset & \text{if } l > k, \\ X_k - \delta(X^{-\lambda}_{k+1}) & \text{if } l = k, \\ X_{k-1} - \iota(X_{k+1}) & \text{if } l = k-1, \\ X_l & \text{if } l < k-1. \end{cases}$$

Example. Here is an example of an ordered face structure T and its 1-domain and 1-codomain:

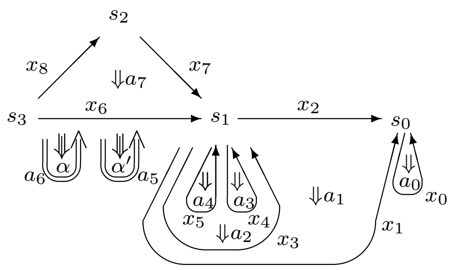
T $\qquad$ $\mathbf{d}^{(1)}T$ $\qquad$ $c^{(1)}T$ $\qquad$ $\mathbf{c}^{(1)}T$

s

$$(s, \emptyset, \{x\}) \xrightarrow{x} (s, \{x\}, \emptyset)$$

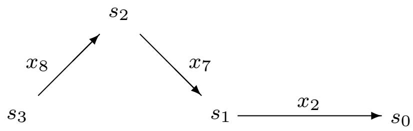
The following is a more involved example. With ordered face structure S as below

S



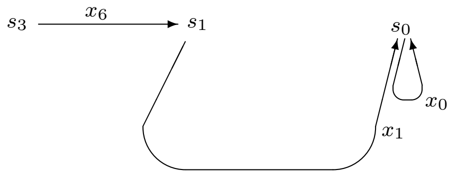
its 1-domain is

$\mathbf{d}^{(1)}S$



the convex subset of S defining 1-codomain is

$c^{(1)}S$



and finally the 1-codomain of S is

$\mathbf{c}^{(1)}S$

$$s_3 \xrightarrow{x_6} s_1 \xrightarrow{x_1} (s_0,\emptyset,\{x_0\}) \xrightarrow{x_0} (s_0,\{x_0\},\emptyset)$$

We have

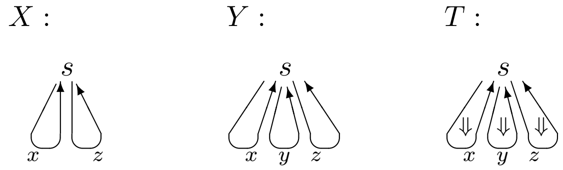
Lemma 8.1 *Let T be an ordered face structure, X a convex subset in T. The subhypergraphs $d^{(k)}X$ and $c^{(k)}X$ of T are convex. Moreover, for $X = T$, $\mathcal{E}^{d^{(k)}T}$ is empty, i.e. there are no empty loops in $d^{(k)}T$ (hence $d^{(k)}T = \mathbf{d}^{(k)}T$) and all empty loops in $\mathcal{E}^{c^{(k)}T}$ have dimension k.*

Proof. The fact that $d^{(k)}X$ and $c^{(k)}X$ are convex sets is an easy consequence Lemmas 4.11 and 4.16. $\mathcal{E}^{d^{(k)}T}$ is empty by loop-filling. The empty loops in $\mathcal{E}^{c^{(k)}T}$ have dimension k by globularity. $\Box$

Thus the ordered face structures $\mathbf{d}^{(k)}T$ and $\mathbf{c}^{(k)}T$ are well defined. We denote $\nu_{d^{(k)}T}$ by $\mathbf{d}^{(k)}_T$ and $\nu_{c^{(k)}T}$ by $\mathbf{c}^{(k)}_T$. Thus we have defined a diagram in **oFs**:

$$\begin{tikzcd} & T & \\ \mathbf{d}^{(k)}T \arrow[ur, "\mathbf{d}^{(k)}_T"] & & \mathbf{c}^{(k)}T \arrow[ul, "\mathbf{c}^{(k)}_T"'] \end{tikzcd}$$

Example. Let $X \subset Y$ be convex subsets of an ordered face structure T as shown on the diagram below.



Clearly $X \subseteq Y$. And the stretching of X and Y gives

$[X]$:

$$(s, \emptyset, -) \xrightarrow{x} (s, \{x\}, -) \xrightarrow{z} (s, \{x, z\}, -)$$

and

$[Y]$:

$$(s, \emptyset, -) \xrightarrow{x} (s, \{x\}, -) \xrightarrow{y} (s, \{x, y\}, -) \xrightarrow{z} (s, \{x, y, z\}, -)$$

respectively. Clearly there is no natural map from $[X]$ to $[Y]$.

This shows that there might be no natural comparison map between stretchings even if one of the convex subset is contained in the other. The Lemma below says that however in some important cases we do have such comparison maps.

Lemma 8.2 *Let T be an ordered face structure, X a convex subset in T. The embeddings $d^{(k)}X \longrightarrow X$ and $c^{(k)}X \longrightarrow X$ induce monotone morphisms $d^{(k)}_X : [d^{(k)}X] \longrightarrow [X]$ and $c^{(k)}_X : [c^{(k)}X] \longrightarrow [X]$ so that the triangles*

$$\begin{tikzcd}
 & T & \\
{[d^{(k)}X]} \arrow[ur, "\nu"] \arrow[r, "d^{(k)}_X"] & {[X]} \arrow[u, "\nu"] & {[c^{(k)}X]} \arrow[ul, "\nu"'] \arrow[l, "c^{(k)}_X"'] \\
\mathbf{d}^{(k)}[X] \arrow[u, "f"] \arrow[ur, "\mathbf{d}^{(k)}_{[X]}"'] & & \mathbf{c}^{(k)}[X] \arrow[u, "g"'] \arrow[ul, "\mathbf{c}^{(k)}_{[X]}"]
\end{tikzcd}$$

commute, where f and g are monotone isomorphisms.

Proof. The morphisms ν send cuts over a face to that face. The commutation of the upper triangles comes to the observation (see below) that both $d^{(k)}_X$ and $c^{(k)}_X$ sends cuts over a to cuts over a for any a in $d^{(k)}X$ and $c^{(k)}X$, respectively.

Next we deal with the left lower triangle

$$\begin{tikzcd}
{[d^{(k)}X]} \arrow[r, "d^{(k)}_X"] & {[X]} \\
\mathbf{d}^{(k)}[X] \arrow[u, "f"] \arrow[ur, "\mathbf{d}^{(k)}_{[X]}"'] &
\end{tikzcd}$$

In dimensions $l < k$, we have $[X]_l = [d^{(k)}X]_l = \mathbf{d}^{(k)}[X]_l$ and

$$f_l = (\mathbf{d}^{(k)}_{[X]})_l = (d^{(k)}_X)_l = id_{[X]_l}$$

In dimension k, we have

$$\mathbf{d}^{(k)}[X]_k = \{(a, \emptyset, \mathcal{E}^X_a) : a \in X_k - \gamma(X^{-\lambda}_{k+1})\},$$

$$[d^{(k)}X]_k = \{(a, \emptyset, \emptyset) : a \in X_k - \gamma(X^{-\lambda}_{k+1})\}$$

and

$$\begin{tikzcd}
(a,\emptyset,\emptyset) \arrow[r, mapsto] & (a,\emptyset,\mathcal{E}_a^X) \\
(a,\emptyset,\mathcal{E}_a^X) \arrow[u, mapsto] \arrow[ur, mapsto] &
\end{tikzcd}$$

From the description it is clear that both triangles commute and that f is an iso.

Now we shall describe the lower right triangle

$$\begin{tikzcd}
{[X]} & {[c^{(k)}X]} \arrow[l, "c_X^{(k)}"'] \\
& \mathbf{c}^{(k)}[X] \arrow[ul, "\mathbf{c}^{(k)}_{[X]}"] \arrow[u, "g"']
\end{tikzcd}$$

In this case we need to look at the cells of both dimensions k and $k-1$. In lower dimensions this triangle is, as in the previous case, the triangle of identities. To describe the above diagram, we shall describe the diagram

$$\begin{tikzcd}
X & c^{(k)}X \arrow[l] & {[c^{(k)}X]} \arrow[l, "\nu"'] \\
{[X]} \arrow[u, "\nu"] & c^{(k)}[X] \arrow[l] \arrow[u] & {[c^{(k)}[X]]} \arrow[l, "\nu"] \arrow[u, "g"']
\end{tikzcd}$$

As the horizontal arrows in the left hand square are inclusions we need to describe only the right hand square. In dimension k, we have

$$(c^{(k)}X)_k = X_k - \delta(X_{k+1}^{-\lambda})$$

$$[c^{(k)}X]_k = \{(a,\emptyset,\emptyset) \in \mathcal{C}(\mathcal{E}_a^{c^{(k)}X}) : a \in X_k - \delta(X_{k+1}^{-\lambda})\}$$

$$c^{(k)}[X]_k = [X] - \delta([X]_{k+1}^{-\lambda}) = \{(a,-,\emptyset) \in \mathcal{C}(\mathcal{E}_a^X) : a \in X_k - \delta(X_{k+1}^{-\lambda})\}$$

$$\mathbf{c}^{(k)}[X]_k = [c^{(k)}[X]]_k = \{((a,-,\emptyset),\emptyset,\emptyset) \in \mathcal{C}(\mathcal{E}_{(a,-,\emptyset)}^{c^{(k)}[X]}) : (a,-,\emptyset) \in c^{(k)}[X]_k\}$$

and the commutation of the square is

$$\begin{tikzcd}
a & (a,\emptyset,\emptyset) \arrow[l, mapsto] \\
(a,\emptyset,\emptyset) \arrow[u, mapsto] & ((a,-,\emptyset),\emptyset,\emptyset) \arrow[l, mapsto] \arrow[u, mapsto]
\end{tikzcd}$$

So the diagram in dimension k commutes and g_k is a bijection.

In dimension $k-1$ we have

$$(c^{(k)}X)_{k-1} = X_{k-1} - \iota(X_{k+1})$$

$$[c^{(k)}X]_{k-1} = \{(x,L_0,U_0) \in \mathcal{C}(\mathcal{E}_x^{c^{(k)}X}) : x \in X_{k-1} - \iota(X_{k+1})\}$$

$$\begin{aligned}
c^{(k)}[X]_{k-1} = \{(x,L_1,U_1) \in \mathcal{C}(\mathcal{E}_x^X) :\ & \text{there is no } \alpha \in X \text{, such that} \\
& \exists_{a,b\in\delta(\alpha)}\, \gamma(a) = x \in \delta(b),\ (a,\downarrow\alpha,-),(b,\downarrow\alpha,-) \in [X]^{-\lambda} \\
& \text{and } (\gamma(a),-,\uparrow a) = (x,L_1,U_1) = (x,\downarrow b,-)\}
\end{aligned}$$

$$\begin{aligned}
\mathbf{c}^{(k)}[X]_{k-1} &= [c^{(k)}[X]]_{k-1} = \\
&= \{((x,L_1,U_1),L_2,U_2) \in \mathcal{C}(\mathcal{E}_{(x,L_1,U_1)}^{c^{(k)}[X]}) : (x,L_1,U_1) \in c^{(k)}[X]_{k-1}\}
\end{aligned}$$

and the commutation of the square is

$$\begin{tikzcd}
x & (x, L_0, U_0) \arrow[l, maps to] \\
(x, L_1, U_1) \arrow[u, maps to] & ((x, L_1, U_1), L_2, U_2) \arrow[l, maps to] \arrow[u, maps to]
\end{tikzcd}$$

where the bijective correspondence between cuts (x, L_0, U_0) in $\mathcal{C}(\mathcal{E}_x^{c^{(k)}X})$ and the cuts of cuts $((x, L_1, U_1), L_2, U_2)$ in $[c^{(k)}[X]]_{k-1}$ is described below.

First we introduce a piece of notation. We denote the faces $a_{in}, a_{out} \in X_k^{-\lambda} - \delta(X_{k+1}^{-\lambda})$ such that $\gamma(a_{in}) = x \in \delta(a_{out})$. Such faces do not need to exists but if they do they are unique. We have

$$L_0 = \{l \in \mathcal{E}_x^{c^{(k)}X} : (l, -, \emptyset) \in L_2 \text{ or } \exists_{a \in L_1}\, a \leq^+ l\},$$

$$U_0 = \{l \in \mathcal{E}_x^{c^{(k)}X} : (l, -, \emptyset) \in U_2 \text{ or } \exists_{a \in U_1}\, a \leq^+ l\},$$

$$L_1 = \{a \in \mathcal{E}_x^{X} : \exists_{l \in L_0}\, a \leq^+ l \text{ or } a_{in} \text{ exists and } a <^+ a_{in}\},$$

$$U_1 = \{a \in \mathcal{E}_x^{X} : \exists_{l \in U_0}\, a \leq^+ l \text{ or } a_{out} \text{ exists and } a <^+ a_{out}, \}$$

$$L_2 = \{(l, -, \emptyset) \in \mathcal{E}_{(x,-,\uparrow l)}^{c^{(k)}[X]} : l \in L_0\},$$

$$U_2 = \{(l, -, \emptyset) \in \mathcal{E}_{(x,-,\uparrow l)}^{c^{(k)}[X]} : l \in U_0\}.$$

It is a matter of a check to see that this correspondence is bijective and that g is indeed an iso. Note that in this notation the map $c_X^{(k)} : [c^{(k)}X] \longrightarrow [X]$ is given by

$$(x, L_0, U_0) \mapsto (x, L_1, U_1).$$

□

Proposition 8.3 *Let $q : S \to T$ be a positive cover, $\mathcal{I} = ker(q)$ be an ideal in T determining this cover. Then we have positive covers $\mathbf{d}^{(k)}(q) : \mathbf{d}^{(k)}S \to \mathbf{d}^{(k)}T$ and $\mathbf{c}^{(k)}(q) : \mathbf{c}^{(k)}S \to \mathbf{c}^{(k)}T$, with kernels $\mathcal{I} \cap \mathbf{d}^{(k)}S$ and $\mathcal{I} \cap \mathbf{c}^{(k)}S$, respectively, making both squares*

$$\begin{tikzcd}
\mathbf{d}^{(k)}S \arrow[r, "\mathbf{d}_S^{(k)}"] \arrow[d, "\mathbf{d}^{(k)}(q)"'] & S \arrow[d, "q"] & \mathbf{c}^{(k)}S \arrow[l, "\mathbf{c}_S^{(k)}"'] \arrow[d, "\mathbf{c}^{(k)}(q)"] \\
\mathbf{d}^{(k)}T \arrow[r, "\mathbf{d}_T^{(k)}"'] & T & \mathbf{c}^{(k)}T \arrow[l, "\mathbf{c}_T^{(k)}"]
\end{tikzcd}$$

commute.

Proof. To see that $d^{(k)}q$ exists, we shall show that if $a \in S_k$ and $q(a) \in \gamma(T_{k+1}^{-\lambda})$ then $a \in \gamma(S_{k+1})$. So pick $\alpha \in T_{k+1}^{-\lambda}$ such that $\gamma(\alpha) = q(a)$. As q is a cover, there is $\beta \in S_{k+1}$ such that $q(\beta) = \alpha$. Hence there is a $\mathcal{I}$-path from a to $\gamma(\alpha)$ or from $\gamma(\alpha)$ to a. In the latter case $a \in \gamma(S_{k+1})$ and we are done. So assume that there is an upper $\mathcal{I}$-path $a, \alpha_1, \ldots, \alpha_n, \gamma(\beta)$. As $q(\beta)$ is not a loop and $q(\alpha_i) = 1_{\gamma(\beta)}$, we have $\alpha_i <^+ \beta$ and $a \in \iota(S)$. In particular, we have $a \in \gamma(S_{k+1})$, as required.

Similarly we can show that we have a hypergraph morphism q' as in the diagram

$$\begin{tikzcd}
S \arrow[d, "q"'] & \mathbf{c}^{(k)}S \arrow[l] \arrow[d, "q'"'] \arrow[dr, "c^{(k)}q"] & \\
T & c^{(k)}T \arrow[l] & {[c^{(k)}T]} \arrow[l, "\nu"]
\end{tikzcd}$$

making the square commutes. We shall show that q' can be lifted to $c^{(k)}q : \mathbf{c}^{(k)}S \longrightarrow [\mathbf{c}^{(k)}T]$. As the only empty loops in $c^{(k)}T$ have dimension k we need to define the function

$$(c^{(k)}q)_{k-1} : S_{k-1} - \iota(S_{k+1}) \longrightarrow \bigcup\{\mathcal{C}(\mathcal{E}_x^{c^{(k)}T}) : x \in T_{k-1} - \iota(T_{k+1})\}$$

only. For $x \in S_{k-1} - \iota(S_{k+1})$, we put

$$(c^{(k)}q)_{k-1}(x) = \begin{cases} 1_{(\gamma(q'(x)),\emptyset,\emptyset)} & \text{if } q'(x) \in 1_{T_{k-2}}, \\ (x,\emptyset,\emptyset) & \mathcal{E}_{q'(x)}^{c^{(k)}T} = \emptyset, \\ (x,\downarrow a,\emptyset) & a \in S_k - \delta(S_{k+1}) \text{ and } x \in \delta(a), \\ (x,\emptyset,\uparrow b) & b \in S_k - \delta(S_{k+1}) \text{ and } x = \gamma(b). \end{cases}$$

As for any $x \in S_{k-1} - \iota(S_{k+1})$ if $q'(x) \in T_{k-1}$ and $\mathcal{E}_{q'(x)}^{c^{(k)}T} \neq \emptyset$ then either there is a unique $a \in S_k - \delta(S_{k+1})$ such that $x \in \delta(a)$ or there is a unique $b \in S_k - \delta(S_{k+1})$ such that $x = \gamma(b)$, $(c^{(k)}q)_{k-1}$ is well defined. The remaining details are left for the reader. $\Box$

In particular, from this Proposition and Theorem 7.1, we have

Corollary 8.4 *Let S be an ordered face structure, $q_S : S^\dagger \longrightarrow S$ it's positive cover, with kernel $\overline{\mathcal{I}}$, as defined in section 7. Then, $dim(\overline{\mathcal{I}}) < dim(S)$, $\overline{\mathcal{I}} \cap \mathbf{c}(S^\dagger) = \overline{\mathcal{I}}_{\leq n-2}$ and*

$$\mathbf{c}(S^\dagger)_{/\overline{\mathcal{I}}_{\leq n-2}} \cong \mathbf{c}S, \qquad \mathbf{d}(S^\dagger)_{/\overline{\mathcal{I}}} \cong \mathbf{d}S.$$

The globularity equations for ordered face structures can be deduced from the above Proposition.

Proposition 8.5 *Let S be an ordered face structure $k, l \in \omega$, $k < l \leq dim(S)$. Then the diagram*

$$\begin{tikzcd} & S & \\ \mathbf{d}^{(l)}S \arrow[ur, "\mathbf{d}_S^{(l)}"] & & \mathbf{c}^{(l)}S \arrow[ul, "\mathbf{c}_S^{(l)}"'] \\ \mathbf{d}^{(k)}S \arrow[u, "\mathbf{d}^{(k)}_{\mathbf{d}^{(l)}S}"] \arrow[urr, "\mathbf{d}^{(k)}_{\mathbf{c}^{(l)}S}"' near end] & & \mathbf{c}^{(k)}S \arrow[u, "\mathbf{c}^{(k)}_{\mathbf{c}^{(l)}S}"'] \arrow[ull, "\mathbf{c}^{(k)}_{\mathbf{d}^{(l)}S}" near end] \end{tikzcd}$$

commutes.

Proof. Having Theorem 7.1 and Proposition 8.3 we see that the above diagram commutes as a consequence of the same diagram being commutative for the positive face structure. $\Box$

9 k-tensor squares of ordered face structures

Let S and T be ordered face structures such that $\mathbf{c}^{(k)}S = \mathbf{d}^{(k)}T$. In that case we define the *k-tensor* $S \otimes_k T$ of S and T and the *k-tensor square* in **oFs**

$$\begin{tikzcd} S \arrow[r, "\kappa_S"] & S \otimes_k T \\ \mathbf{c}^{(k)}S = \mathbf{d}^{(k)}T \arrow[u, "\mathbf{c}_S^{(k)}"] \arrow[r, "\mathbf{d}_T^{(k)}"'] & T \arrow[u, "\kappa_T"'] \end{tikzcd}$$

The local part of $S \otimes_k T$ is defined so that the square

$$\begin{tikzcd} |S| \arrow[r, "\kappa_S"] & |S \otimes_k T| \\ |\mathbf{c}^{(k)} S| \arrow[u, "\mathbf{c}_S^{(k)}"] \arrow[r, "\mathbf{d}_T^{(k)}"'] & |T| \arrow[u, "\kappa_T"'] \end{tikzcd}$$

is a pushout in **lFs**, so the faces of $S \otimes_k T$ are as in the following table:

dim	$\mathbf{c}^{(k)} S$	$\mathbf{d}^{(k)} T$	$S \otimes_k T$
$l > k$	$\emptyset$	$\emptyset$	$S_l + T_l$
k	$S_k - \delta(S^{-\lambda}_{k+1})$	$T_k - \gamma(T^{-\lambda}_{k+1})$	$S_k + \gamma(T^{-\lambda}_{k+1}) = T_k + \delta(S^{-\lambda}_{k+1})$
$k-1$	$\mathcal{C}(S_{k-1} - \iota(S_{k+1}))$	T_{k-1}	S_{k-1}
$l < k-1$	S_l	T_l	S_l

By the assumption the first and the second columns are equal and the third describes the faces of $S \otimes_k T$. To simplify the description of $S \otimes_k T$, we assume that

$$S_k - \delta(S^{-\lambda}_{k+1}) = T_k - \gamma(T^{-\lambda}_{k+1}) = S_k \cap T_k,$$

and we introduce the notation for the function

$$[-] = (\mathbf{c}_S^{(k)})_{k-1} : (\mathbf{c}^{(k)} S)_{k-1} = T_{k-1} \longrightarrow S_{k-1}.$$

that sends t-cuts in S, (with $t \in S_{k-1}$, i.e. elements of T_{k-1}) to t. All the components of the maps $\kappa_S : S \longrightarrow S \otimes_k T$ and $\kappa_T : T \longrightarrow S \otimes_k T$ are inclusions except for $(\kappa_T)_{k-1}$ which is $[-]$. The domain and codomain maps in $S \otimes_k T$, denoted $\gamma^\otimes$ and $\delta^\otimes$ for short, are obvious except for k-faces in $\gamma(T^{-\lambda}_{k+1})$. If $t \in \gamma(T^{-\lambda}_{k+1})$ we put:

$$\gamma^\otimes(t) = [\gamma^T(t)], \qquad \delta^\otimes(t) = \{[u] : u \in \delta^T(t)\}.$$

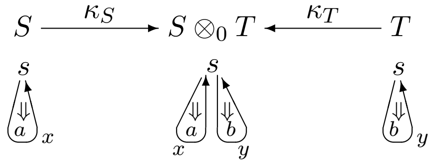
To finish off the definition of $S \otimes_k T$, it is enough to define $<^{\otimes,l,\sim}$, for $l \geq 1$. For $l < k$, $<^{(S\otimes_k T)_l,\sim}$ is $<^{S_l,\sim}$ and $<^{(S\otimes_k T)_l,\sim}$ is $<^{S_l,\sim} + <^{T_l,\sim}$, for $l > k+1$. Thus, it remains to define the orders $<^{(S\otimes_k T)_k,\sim}$ and $<^{(S\otimes_k T)_{k+1},\sim}$. The order $<^{(S\otimes_k T)_{k+1},\sim}$ is defined for $a, b \in (S \otimes_k T)_{k+1} = S_{k+1} + T_{k+1}$ we put

$$a <^{(S\otimes_k T)_{k+1},\sim} b \text{ iff } \begin{cases} \text{either } a, b \in S_{k+1} \text{ and } a <^{S_{k+1},\sim} b, \\ \text{or } a, b \in T_{k+1} \text{ and } a <^{T_{k+1},\sim} b, \\ \text{or } a \in S_{k+1},\ b \in T_{k+1} \text{ and } a <^{S\otimes_k T,-} b. \end{cases}$$

i.e. it is $<^{S,\sim}$ on S_{k+1}, $<^{T,\sim}$ on is T_{k+1}, and moreover if the faces comes from different parts and are $<^{S\otimes_k T,-}$ related, then faces from S comes before the faces from T. The last clause of this definition is the only reason $S \otimes_k T$ is not a pushout in **oFs**, in general. It may cause a face a from S to be $<^\sim$-smaller than a face b from T even if there is no $\sim$-relation between a and b, whatsoever. By Lemma 4.28, to define the order $<^{(S\otimes_k T)_k,\sim}$ it is enough to say that it agrees with $<^{T_k,\sim}$ on the set $(S \otimes_k T)_k - \delta((S \otimes_k T)^{-\lambda}_{k+1}) = (T_k - \delta(T^{-\lambda}_{k+1}))$. However we give below the full, but more involved, definition of the order $<^{(S\otimes_k T)_k,\sim}$. We have $(S \otimes_k T)_k = S_k + \gamma(T^{-\lambda}_{k+1}) = T_k + \delta(S^{-\lambda}_{k+1})$. We define $<^{(S\otimes_k T)_k,\sim}$ to be $<^{S,\sim}$ on S_k, and to be $<^{T,\sim}$ on T_{k+1}. The essential case is if $x \in \delta(S^{-\lambda}_{k+1})$ and $y \in \gamma(T^{-\lambda}_{k+1})$. In that case there is a unique $x' \in S_k \cap T_k$ that $x <^+ x'$. We put $x <^{(S\otimes_k T)_k,\sim} y$ iff $x' <^{T_k,\sim} y$ and $y <^{(S\otimes_k T)_k,\sim} x$ iff $y <^{T_k,\sim} x'$ and $y <^{(S\otimes_k T)_k,-} x$. In other words for $x, y \in (S \otimes_k T)_k$, we have:

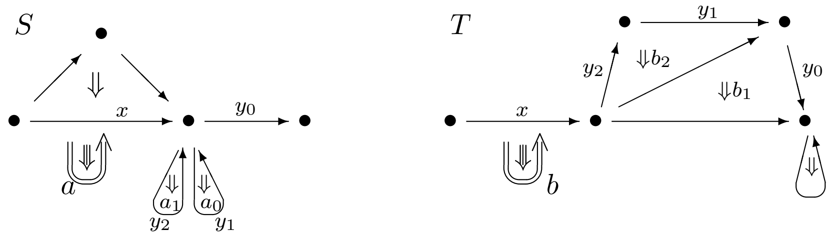
$$x <^{(S\otimes_k T)_k,\sim} y \text{ iff } \begin{cases} \text{either } x, y \in S \text{ and } x <^{S,\sim} y, \\ \text{or } x, y \in T \text{ and } x <^{T,\sim} y, \\ \text{or } x \in \delta(S^{-\lambda}_{k+1}),\ y \in \gamma(T^{-\lambda}_{k+1}) \text{ and } \exists_{z\in S_k\cap T_k}\ x \leq^{S,+} z \text{ and } z <^{T,\sim} y, \\ \text{or } x \in \gamma(T^{-\lambda}_{k+1}),\ y \in \delta(S^{-\lambda}_{k+1}) \text{ and } x <^{(S\otimes_k T)_k,-} y \\ \quad \text{and } \exists_{z\in S_k\cap T_k}\ x \leq^{T,+} z \text{ and } x <^{S,\sim} z. \end{cases}$$

Examples. Before we prove some properties of the above construction let us look at some examples of k-tensors:



In this case the only relation that is not coming from the fact that $S \otimes_0 T$ is a pushout locally is $x <^{\sim} y$. We have that x comes before y as in 'case of doubt' faces from S comes before those from T.

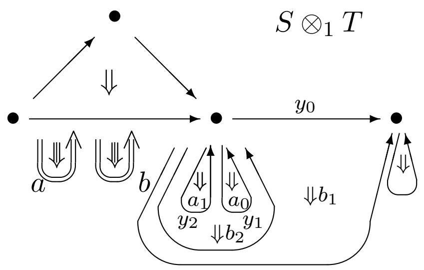
The next example is a bit more involved. For the ordered face structures



we have $\mathbf{c}^{(1)} S = \mathbf{d}^{(1)} T$

$$\bullet \xrightarrow{x} \bullet \xrightarrow{y_2} \bullet \xrightarrow{y_1} \bullet \xrightarrow{y_0} \bullet$$

and their 1-tensor $S \otimes_1 T$ square is



with $a <^{\sim} b$ as the only additional data not following from the fact that $S \otimes_1 T$ is a pushout locally.

Proposition 9.1 *Let S and T be ordered face structures, $k \in \omega$, and $\mathbf{c}^{(k)} S = \mathbf{d}^{(k)} T$. Then $S \otimes_k T$ is an ordered face structure, and the k-tensor square*

$$\begin{tikzcd} S \arrow[r, "\kappa_S"] & S \otimes_k T \\ \mathbf{c}^{(k)} S \arrow[u, "\mathbf{c}^{(k)}_S"] \arrow[r, "\mathbf{d}^{(k)}_T"'] & T \arrow[u, "\kappa_T"'] \end{tikzcd}$$

commutes in **oFs**. *Moreover the functor* $|-| : \mathbf{oFs} \longrightarrow \mathbf{lFs}$ *sends the k-tensor squares to pushouts.*

Proof. The whole proof is a matter of a check. We shall discuss globularity leaving the verification of other axioms of ordered face structure for the reader.

The globularity condition for faces in $S \otimes_k T$ for other faces than those in T_k and T_{k+1} holds as a direct consequence of globularity for S and T. A simple check shows that in fact globularity for T_k is also a consequence of globularity for T. Thus we need to verify the globularity for $a \in T_{k+1} \subseteq (S \otimes_k T)_{k+1}$. We will write γ for γ^T

and $\gamma^{\otimes}$ for $\gamma^{S\otimes_k T}$. For empty-domain faces the globularity is obvious so we assume that $a \in T^{-\varepsilon}_{k+1}$. Put

$$L = \{a \in \gamma^u(T^{-\lambda}_{k+1}) : \text{there is a } S^\lambda \cap T_k - \text{path (possibly empty) from } \delta(a) \text{ to } \gamma a\}.$$

We have

$$(S \otimes_k T)^\lambda_l = \begin{cases} S^\lambda_l + T^\lambda_l & \text{for } l > k, \\ S^\lambda_k + L & \text{for } l = k, \\ S^\lambda_l & \text{for } l < k. \end{cases}$$

We shall describe the sets involved in the globularity conditions:

$$\delta^{\otimes}(a) = \delta(a), \quad \dot{\delta}^{\otimes,-\lambda}(a) = \dot{\delta}^{-\lambda}(a) - L,$$

$$\gamma^{\otimes}\gamma^{\otimes}(a) = [\gamma\gamma(a)], \quad \delta^{\otimes}\gamma^{\otimes}(a) = \{[t] : t \in \delta\gamma(a)]\},$$

$$\gamma^{\otimes}\delta^{\otimes}(a) = \{[\gamma(x)] : x \in \delta(a)\}, \quad \delta^{\otimes}\delta^{\otimes}(a) = \{[t] : t \in \delta\delta(a)]\},$$

$$\gamma^{\otimes}\delta^{\otimes,-\lambda}(a) = \{[\gamma(x)] : x \in \delta(a) - L\}, \quad \delta^{\otimes}\delta^{\otimes}(a) = \{[t] : \exists_{x \in \delta(a)-L}\, t \in \delta(x)]\}.$$

By assumption on T we have $\gamma\gamma(a) = \gamma\delta(a) - \delta\dot{\delta}^{-\lambda}(a)$. Thus to show the γ-globularity

$$\gamma^{\otimes}\gamma^{\otimes}(a) = \gamma^{\otimes}\delta^{\otimes}(a) - \delta^{\otimes}\dot{\delta}^{\otimes,-\lambda}(a)$$

we need to show

1. $\gamma^{\otimes}\gamma^{\otimes}(a) \not\in \delta^{\otimes}\dot{\delta}^{\otimes,-\lambda}(a)$
2. $\gamma^{\otimes}\delta^{\otimes}(a) \subseteq \gamma^{\otimes}\gamma^{\otimes}(a) \cup \delta^{\otimes}\dot{\delta}^{\otimes,-\lambda}(a)$.

Ad 1. Suppose 1. does not hold and fix face $x \in \delta(a) - L$, $t \in \dot{\delta}(x)$ such that there is an upper $(S^\lambda \cap T)$-path $t, x_0, \ldots, x_k, \gamma\gamma(a)$ from t to $\gamma\gamma(a)$. In particular, this is $T_k - \gamma(T^{-\lambda}_{k+1})$-path. As $x \in T_k$, we have $x_0 \leq^+ x$.As $\gamma(x) \leq^+ \gamma\gamma(a)$, by Path Lemma and the definition of L, we have that $x \in L$, contrary to the assumption.

Ad 2. Fix $x \in \delta(a)$. Let $\gamma(x), x_1, \ldots, x_k, \gamma\gamma(a)$ be the flat upper $(T_k - \gamma(T^{-\lambda}_{k+1}))$-path. If this path is S^λ-path then $[\gamma(x)] = [\gamma\gamma(a)]$, if it is not then $[\gamma\gamma(x)] \in \delta^{\otimes}\dot{\delta}^{\otimes,-\lambda}(a)$, as required.

For δ-globularity we consider only the case $\gamma(a) \in T^{-\varepsilon}$. The other case is easy. For empty-faces in T we have $\gamma\gamma\delta^{\varepsilon}(a) \subseteq \theta\delta\gamma(a)$ and hence passing to equivalence classes we also have $\gamma^{\otimes}\gamma^{\otimes}\delta^{\otimes,\varepsilon}(a) \subseteq \theta^{\otimes}\delta^{\otimes}\gamma(a)$. Moreover as $\delta\gamma(a) = \dot{\delta}\delta(a) - \gamma\dot{\delta}^{-\lambda}(a)$ holds in T to show $\delta^{\otimes}\gamma^{\otimes}(a) = \dot{\delta}^{\otimes}\delta(a) - \gamma^{\otimes}\dot{\delta}^{\otimes,-\lambda}(a)$ we need to show again two things

3. $\delta^{\otimes}\gamma^{\otimes}(a) \cap \gamma^{\otimes}\dot{\delta}^{\otimes,-\lambda}(a) = \emptyset$,
4. $\delta^{\otimes}\delta^{\otimes}(a) \subseteq \delta^{\otimes}\gamma^{\otimes}(a) \cup \gamma^{\otimes}\dot{\delta}^{\otimes,-\lambda}(a)$

Ad 3. Suppose contrary, that 3. does not hold. Fix $t \in \delta\gamma(a)$ such that $[t] \in \gamma^{\otimes}\dot{\delta}^{\otimes,-\lambda}(a)$. So there is $x \in \delta^{-\lambda}(a) - L$ and upper $(S^\lambda_k \cap T_k)$-path $t, x_1, \ldots, x_k, \gamma(x)$, with $k \geq 1$. If $t \in \delta(x)$ or $\gamma(x_i) \in \delta(x)$, for some $i = 1, \ldots, k-1$ then $x \in L$, contrary to the supposition. If $t \not\in \delta(x)$ and $x_i <^+ x$, for some $i = 1, \ldots, k$, then, by Lemma 4.8.1 and Path Lemma, $\gamma(a) <^+ x \in \delta(a)$ which is again a contradiction. Thus 3. holds.

Ad 4. Fix $t \in \delta(x)$ such that $x \in \delta(a)$. Let $x_1, \ldots, x_k, t$ be the maximal flat upper $(T_k - \gamma(T^{-\lambda}_{k+1}))$-path ending at t. By Path Lemma either there is $t' \in \delta\gamma(a)$ such that $t' \in \dot{\delta}(x_1)$ or $t' = \gamma(x_i)$, for some $i = 1, \ldots, k-1$, or $x_i <^+ \gamma(a)$, for some $i = 1, \ldots, k$, $x_i \in T^{\varepsilon}$ and $\gamma\gamma(x) \in \theta\delta\gamma(a)$. In the former case, if the path $t', x_j, \ldots, x_k, t$ is an S^λ-path then $[t] \in \delta^{\otimes}\gamma^{\otimes}(a)$, if not then using again Path

Lemma we get that $[t] \in \gamma^{\otimes}\dot{\delta}^{\otimes,-\lambda}(a)$. In the later case we can also easily show that $[t] \in \gamma^{\otimes}\dot{\delta}^{\otimes,-\lambda}(a)$, as required. $\Box$

The following propositions establish a connection between tensor squares of ordered faces structures and special pushouts of positive face structures.

Proposition 9.2 *Let X and Y be positive face structures, $k \in \omega$, $\mathbf{c}^{(k)}(X) = \mathbf{d}^{(k)}(Y)$, and $\mathcal{J}$ an ideal in the special pushout*[8] *$X +_k Y$. The quotient by ideal $\mathcal{J}$ of the special pushout being the top of the following cube*

$$\begin{tikzcd}
& X \arrow[rr, "\kappa"] \arrow[dd, "p"] & & X +_k Y \arrow[dd, "p"] \\
\mathbf{c}^{(k)}(X) \arrow[ur, "\mathbf{c}^{(k)}"] \arrow[rr, "\mathbf{d}^{(k)}"] \arrow[dd, "p"] & & Y \arrow[ur, "\kappa"] \arrow[dd, "p"] & \\
& X/_{\mathcal{J}^X} \arrow[rr, "\kappa"] & & (X +_k Y)/_{\mathcal{J}} \\
\mathbf{c}^{(k)}(X)/_{\mathcal{J}^k} \arrow[ur, "\mathbf{c}^{(k)}"] \arrow[rr, "\mathbf{d}^{(k)}"] & & Y/_{\mathcal{J}^Y} \arrow[ur, "\kappa"] &
\end{tikzcd}$$

is a k-tensor square on the bottom of the following cube, where $\mathcal{J}^X$, $\mathcal{J}^Y$, and $\mathcal{J}^k$ are the ideals that arise by intersecting $\mathcal{J}$ with X, Y, and $\mathbf{c}^{(k)}(X)$, respectively. In the cube all squares commutes, and all vertical maps are covers.

Proof. This is a matter of a simple check. $\Box$

Proposition 9.3 *Let S and T be ordered face structures, $k \in \omega$, and $\mathbf{c}^{(k)}S = \mathbf{d}^{(k)}T$ and a positive cover $p : (S \otimes_k T)^{\ddagger} \longrightarrow S \otimes_k T$ with the kernel $\mathcal{J}$. Then there are covers $S^{\ddagger} \to S$ and $T^{\ddagger} \to T$, such that the top square of the following cube*

$$\begin{tikzcd}
& S^{\ddagger} \arrow[rr, "\kappa"] \arrow[dd, "p"] & & (S \otimes_k T)^{\ddagger} \arrow[dd, "p"] \\
\mathbf{c}^{(k)}(S^{\ddagger}) \arrow[ur, "\mathbf{c}^{(k)}"] \arrow[rr, "\mathbf{d}^{(k)}"] \arrow[dd, "p"] & & T^{\ddagger} \arrow[ur, "\kappa"] \arrow[dd, "p"] & \\
& S \arrow[rr, "\kappa"] & & S \otimes_k T \\
\mathbf{c}^{(k)}(S) \arrow[ur, "\mathbf{c}^{(k)}"] \arrow[rr, "\mathbf{d}^{(k)}"] & & T \arrow[ur, "\kappa"] &
\end{tikzcd}$$

is a special pushout in $\mathbf{Fs}^{+/1}$, and the bottom square is the quotient k-tensor square of the top by the kernel $\mathcal{J}$.

Proof. We denote $(S \otimes_k T)^{\ddagger}$ by P. We shall define the positive face structures $S^{\ddagger}$, $T^{\ddagger}$, and the morphisms from them in the diagram

$$\begin{tikzcd}
S^{\ddagger} \arrow[r, "\kappa_S^{\ddagger}"] \arrow[d, "q_S"] & P \arrow[d, "q"] & T^{\ddagger} \arrow[l, "\kappa_T^{\ddagger}"'] \arrow[d, "q_T"] \\
S \arrow[r, "\kappa_S"'] & S \otimes_k T & T \arrow[l, "\kappa_S"]
\end{tikzcd}$$

S can be identified with a subset of $S \otimes_k T$ (via κ_S). We define $S^{\ddagger}$ as the inverse image of S i.e. $S^{\ddagger} = q^{-1}(S + 1_S)$. $S^{\ddagger}$ is a positive face structure as a convex subset of a positive face structure P. q_S is the restriction of q to $S^{\ddagger}$. It is onto since q is. It is also easy to see that the kernel of q_S is $\mathcal{J} \cap S^{\ddagger}$.

The description of faces of $T^{\ddagger}$ is more involved.

[8] By this we mean the pushouts, in the category of positive face structures $\mathbf{Fs}^{+/1}$, of a special kind that have been described in [Z].

1. $T^\ddagger_{>k} = q^{-1}(T_{>k} \cup 1_{T_{\geq k}})$,

2. $T^\ddagger_k = P_k - \delta(S^\ddagger_{k+1})$,

3. $T^\ddagger_{k-1} = P_k - \iota(S^\ddagger_{k+1})$,

4. $T^\ddagger_{<k-1} = S^\ddagger_{<k-1} (= P_{<k-1})$.

q_T is the restriction of q to $T^\ddagger$.

The verification that q_T is a cover and $\mathbf{c}^{(k)} S^\ddagger = \mathbf{d}^{(k)} T^\ddagger$, which comes to verification of two equalities

$$\mathbf{c}^{(k)} S^\ddagger_k = q^{-1}(S_k \cup 1_{S_{k-1}}) - \delta(q^{-1}(S_{k+1} \cup 1_{S_{k+1}})) =$$

$$= P_k - \delta(S^\ddagger_{k+1}) - \gamma(T_{k+1} \cup 1_{T_{k+1}}) = \mathbf{d}^{(k)} T^\ddagger_k$$

$$\mathbf{c}^{(k)} S^\ddagger_{k-1} = S_{k-1} - \iota(S^\ddagger_{k+1}) = P_{k-1} - \iota(S^\ddagger_{k+1}) = \mathbf{d}^{(k)} T^\ddagger_{k-1}$$

is left for the reader. $\Box$

The following proposition describe explicitly the abstract properties of k-domain, k-codomain, and k-tensor operations in **oFs**. For more abstract treatment of these properties in terms of the notion of a graded tensor category see [Z1].

Proposition 9.4 *The k-tensor operation* **oFs** *is functorial, compatible with the k-domain and k-codomain operations, associative, and satisfy the middle exchange law.*

Proof. In the course of the proof I will explain precisely what I mean by this statement in details. Roughly speaking, it means that the all local morphisms form al objects of **oFs** into a single ordered faces structure S has a natural structure of an ω-category S^*, with domains, codomains, and compositions in S^* defined in terms of k-domain, k-codomain, and k-tensor operations in **oFs**.

The operations will be defined the operations on the skeleton of **oFs**. If X and Y are isomorphic ordered face structures there is a unique isomorphism between them and in fact it is the only monotone morphism between them. We shall identify two morphisms $f : X \to Y$ and $f' : X' \to Y'$ in **oFs** iff there are isomorphisms making the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\cong"'] & Y \arrow[d, "\cong"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

commutes. As these identifications are harmless we shall work in **oFs** recalling the identifications if needed.

To explain the functoriality of k-tensor we define the category **oFs** $\times_k$ **oFs** as follows. The objects of **oFs** $\times_k$ **oFs** are pairs of ordered face structures (S, S') such that $\mathbf{c}^{(k)} S = \mathbf{d}^{(k)} S'$ and whose maps are pairs of monotone morphisms $(f, f') : (S, S') \longrightarrow (T, T')$ such that the diagram

$$\begin{tikzcd} S \arrow[d, "f"'] & \mathbf{c}^{(k)} S \arrow[l, "c^{(k)}_S"'] \arrow[r, "\mathbf{d}^{(k)}_{S'}"] \arrow[d, "f''"] & S' \arrow[d, "f'"] \\ T & \mathbf{c}^{(k)} T \arrow[l, "\mathbf{c}^{(k)}_T"] \arrow[r, "\mathbf{d}^{(k)}_{T'}"'] & T' \end{tikzcd}$$

commutes, where f'' is the restriction of f' to $\mathbf{d}^{(k)}S'$. We have four functors

$$\pi^0, \pi^1, \pi, \otimes_k : \mathbf{oFs} \times_k \mathbf{oFs} \longrightarrow \mathbf{oFs}.$$

The three first functors are defined on objects as follows $\pi^0(S, S') = S$, $\pi^1(S, S') = S'$, $\pi(S, S') = \mathbf{c}^{(k)}S$ for (S, S') in $\mathbf{oFs} \times_k \mathbf{oFs}$, and on morphisms they are defined in the obvious way. The functor $\otimes_k$ is defined on object and morphisms in the obvious way but we need to verify that the local morphisms we get between local pushouts are in fact monotone. This we leave for the reader. Moreover we have four obvious natural transformations making the square

$$\begin{tikzcd}
\pi^0 \arrow[r, "\kappa^0"] & \otimes_k \\
\pi \arrow[u, "\mathbf{c}^{(k)}"] \arrow[r, "\mathbf{d}^{(k)}"'] & \pi^1 \arrow[u, "\kappa^1"']
\end{tikzcd}$$

commutes, in $Nat(\mathbf{oFs} \times_k \mathbf{oFs}, \mathbf{oFs})$.

By compatibility of k-tensor operation with the k-domain and k-codomain operations, we mean that for any ordered face structure X the squares

$$\begin{tikzcd}
X \arrow[r, "1_X"] & X \\
\mathbf{d}^{(k)}X \arrow[u, "\mathbf{d}^{(k)}_X"] \arrow[r, "1_{\mathbf{d}^{(k)}X}"'] & \mathbf{d}^{(k)}X \arrow[u, "\mathbf{d}^{(k)}_X"']
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "1_X"] & X \\
\mathbf{c}^{(k)}X \arrow[u, "\mathbf{c}^{(k)}_X"] \arrow[r, "1_{\mathbf{c}^{(k)}X}"'] & \mathbf{c}^{(k)}X \arrow[u, "\mathbf{c}^{(k)}_X"']
\end{tikzcd}$$

are k-tensor squares. Moreover, for $k > l$, there are isomorphism making the triangles

$$\begin{tikzcd}
& X \otimes_l X' & \\
\mathbf{d}^{(k)}X \otimes_l \mathbf{d}^{(k)}X' \arrow[ur, "\mathbf{d}^{(k)}_X \otimes_l \mathbf{d}^{(k)}_{X'}"] \arrow[rr, "\cong"'] & & \mathbf{d}^{(k)}(X \otimes_l X') \arrow[ul, "\mathbf{d}^{(k)}_{X \otimes_l X'}"']
\end{tikzcd}
\quad
\begin{tikzcd}
& X \otimes_l X' & \\
\mathbf{c}^{(k)}X \otimes_l \mathbf{c}^{(k)}X' \arrow[ur, "\mathbf{c}^{(k)}_X \otimes_l \mathbf{c}^{(k)}_{X'}"] \arrow[rr, "\cong"'] & & \mathbf{c}^{(k)}(X \otimes_l X') \arrow[ul, "\mathbf{c}^{(k)}_{X \otimes_l X'}"']
\end{tikzcd}$$

commute, and for $k \leq l$, there are isomorphism making the triangles

$$\begin{tikzcd}
& X \otimes_l X' & \\
\mathbf{d}^{(k)}X \arrow[ur, "\kappa^1_X \circ \mathbf{d}^{(k)}_X"] \arrow[rr, "\cong"'] & & \mathbf{d}^{(k)}(X \otimes_l X') \arrow[ul, "\mathbf{d}^{(k)}_{X \otimes_l X'}"']
\end{tikzcd}
\quad
\begin{tikzcd}
& X \otimes_l X' & \\
\mathbf{c}^{(k)}X' \arrow[ur, "\kappa^2_{X'} \circ \mathbf{c}^{(k)}_X"] \arrow[rr, "\cong"'] & & \mathbf{c}^{(k)}(X \otimes_l X') \arrow[ul, "\mathbf{c}^{(k)}_{X \otimes_l X'}"']
\end{tikzcd}$$

commute.

The associativity isomorphisms come from the fact that for any ordered face structure X, Y, Z such that $\mathbf{c}^{(k)}X = \mathbf{d}^{(k)}Y$ and $\mathbf{c}^{(k)}Y = \mathbf{d}^{(k)}Z$ both objects

$$(X \otimes_k Y) \otimes_k Z, \qquad\qquad X \otimes_k (Y \otimes_k Z)$$

are locally colimits of the diagram

$$\begin{tikzcd}
X & & Y & & Z \\
& \mathbf{c}^{(k)}X \arrow[ul, "\mathbf{c}^{(k)}_X"] \arrow[ur, "\mathbf{d}^{(k)}_Y"'] & & \mathbf{c}^{(k)}Y \arrow[ul, "\mathbf{c}^{(k)}_Y"] \arrow[ur, "\mathbf{d}^{(k)}_Z"'] &
\end{tikzcd}$$

and the local isomorphism between them is in fact a monotone morphism. This easily follows from Proposition 4.27.

Similarly, the interchange isomorphism between objects

$$(X \otimes_k Y) \otimes_l (Z \otimes_k T), \qquad\qquad (X \otimes_l Z) \otimes_k (Y \otimes_l T)$$

where $k < l$, is defined as the local isomorphism between two colimits of the diagram

$$\begin{tikzcd}
X & & & Y \\
\mathbf{c}^{(l)}X \arrow[u, "\mathbf{c}^{(l)}_X"] \arrow[d, "\mathbf{d}^{(l)}_Z"'] & \mathbf{c}^{(k)}X \arrow[l, "\mathbf{c}^{(k)}_X"'] \arrow[rr, "\mathbf{d}^{(k)}_Y"] & & \mathbf{c}^{(l)}Y \arrow[u, "\mathbf{c}^{(l)}_Y"'] \arrow[d, "\mathbf{d}^{(l)}_T"] \\
Z & & & T
\end{tikzcd}$$

which is in fact a monotone isomorphism. □

Remark. It may seem that the k-tensor operation is a bit arbitrary, as only part the order $<^\sim$ in the ordered face structure $S\otimes_k S'$ is determined by the fact that it is a pushout locally and that the embeddings $\kappa^S : S \to S\otimes_k S'$ and $\kappa^{S'} : S' \to S\otimes_k S'$ are monotone morphisms. If this structure determine uniquely the order $<^\sim$ in $S \otimes_k S'$ (and hence the whole structure of $S \otimes_k S'$) then we shall call such a k-tensor *locally determined*. It is not hard to see that the k-tensor $S \otimes_k S'$ is locally determined iff there are no 'free' loops of dimension $k+1$ over the same k-face $x \in \mathbf{c}^{(k)}S$ that came from both S and S', i.e. there are no $l \in S^{\lambda}_{k+1} - \delta(S^{-\lambda}_{k+2})$ and $l' \in S'^{\lambda}_{k+1} - \delta(S'^{-\lambda}_{k+2})$ such that $\gamma(l) = \gamma(l')$ (as usually in such cases we assume that $\mathbf{c}^{(k)}S = \mathbf{d}^{(k)}S'$). However if we ask for an operation which is both pushout locally and functorial (in the sense explained above) then the k-tensor operation is the only possible one.

Proposition 9.5 *The k-tensor operation is the unique functor $\otimes_k : \mathbf{oFs}\times_k\mathbf{oFs} \longrightarrow \mathbf{oFs}$ which is a pushout functor locally, i.e. the square*

$$\begin{tikzcd}
\pi^0 \arrow[r, "\kappa^0"] & \otimes_k \\
\pi \arrow[u, "\mathbf{c}^{(k)}"] \arrow[r, "\mathbf{d}^{(k)}"'] & \pi^1 \arrow[u, "\kappa^1"']
\end{tikzcd}$$

evaluated at any object of $\mathbf{oFs} \times_k \mathbf{oFs}$ *is a pushout in* $\mathbf{lFs}$.

Proof. Assume that for any $(X, X') \in \mathbf{oFs} \times_k \mathbf{oFs}$ the square

$$\begin{tikzcd}
X \arrow[r, "\kappa^0_X"] & X \otimes_k X' \\
\mathbf{c}^{(k)}X \arrow[u, "\mathbf{c}^{(k)}_X"] \arrow[r, "\mathbf{d}^{(k)}_{X'}"'] & X' \arrow[u, "\kappa^1_{X'}"']
\end{tikzcd}$$

is a pushout in $\mathbf{lFs}$. This condition determines the functor $\otimes_k$ uniquely on all the objects (Y, Y') of $\mathbf{oFs}\times_k\mathbf{oFs}$ for which k-tensor $Y\otimes_k Y'$ is locally determined. However every object (X, X') can be embedded in $\mathbf{oFs}\times_k\mathbf{oFs}$ into a locally determined object (Y, Y'), i.e. we have morphism $(f, f') : (X, X') \longrightarrow (Y, Y')$ in $\mathbf{oFs} \times_k \mathbf{oFs}$. As the morphism $f \otimes f' : X \otimes X' \longrightarrow Y \otimes Y'$ is monotone the order $<^\sim$ in $X \otimes X'$ is uniquely determined by the order $<^\sim$ in $Y \otimes Y'$, i.e. $\otimes_k$ is indeed the unique functor satisfying the above requirements. □

Thus the above proposition says that $\otimes_k$ is the only operation which is at the same time functorial and locally determined as a pushout.

10 ω-categories generated by local face structures

Now we shall describe an ω-category T^* generated by an ordered face structure T, i.e. we shall describe a functor

$$(-)^* : \mathbf{oFs} \longrightarrow \omega Cat$$

however to prove some properties of $(-)^*$ it is more convenient to describe this functor on a larger category $\mathbf{lFs}$, i.e. we shall describe in fact a functor

$$(-)^* : \mathbf{lFs} \longrightarrow \omega Cat$$

We have forgetful functors, for $n \in \omega$,

$$\boldsymbol{\pi}_n : \mathbf{oFs}_n \longrightarrow \mathbf{lFs}$$

For a local face structure T, and for $n \in \omega$, the set T^*_n of n-cells of T^* is the set of isomorphisms classes of objects of the comma category $\boldsymbol{\pi}_n \downarrow T$.

For $k \leq n$, the *domain* and *codomain* operations in T^*

$$d^{(k),T^*}, c^{(k),T^*} : T^*_n \longrightarrow T^*_k$$

of the k-th *domain* and the k-th *codomain* are defined by composition. For an object X of $\mathbf{oFs}_n$ and a cell $x : X \longrightarrow T$ in T^*_n, we define

$$d^{(k),T^*}(x) = \mathbf{d}^{(k)}_X; x : \mathbf{d}^{(k)}X \longrightarrow T, \qquad c^{(k),T^*}(x) = \mathbf{c}^{(k)}_X; x : \mathbf{c}^{(k)}X \longrightarrow T$$

The *identity* operation

$$\mathbf{i}^{(n)} : T^*_k \longrightarrow T^*_n$$

is an inclusion. The composition map

$$\mathbf{m}_{n,k,n} : T^*_n \times_{T^*_k} T^*_n \longrightarrow T^*_n$$

is given by the k-tensor, i.e. for two n-cells $x : X \to T$, $y : Y \to T$ in T^*_n such that $\mathbf{c}^{(k)}_X; x = \mathbf{d}^{(k)}_Y; y$ then

$$\mathbf{m}_{n,k,n}(x, y) = [x, y]$$

where $[x, y]$ is the unique map making the following diagram

$$\begin{tikzcd}
 & & T \\
X \arrow[urr, "x"] \arrow[r] & X \otimes_k Y \arrow[ur, "{[x,y]}"] & \\
\mathbf{c}^{(k)}X = \mathbf{d}^{(k)}Y \arrow[u, "\mathbf{c}^{(k)}_X"] \arrow[r, "\mathbf{d}^{(k)}_Y"'] & Y \arrow[u] \arrow[uur, "y"'] &
\end{tikzcd}$$

commutes. Note that $[x, y]$ exists and is unique since the forgetful functor $|-| : \mathbf{oFs} \longrightarrow \mathbf{lFs}$ sends $X \otimes_k Y$ to a pushout. We often write $x;_k y$ for $\mathbf{m}_{n,k,n}(x, y)$.

Proposition 10.1 *Let T be a local face structure. Then T^* is an ω-category. In fact, we have a functor $(-)^* : \mathbf{lFs} \longrightarrow \omega Cat$.*

Proof. All the properties in question of T^* follows more or less in the same way from the fact that **oFs** is a monoidal globular category and the the tensors in **oFs** are pushouts locally. To see how it goes we shall check the associativity of the compositions. So suppose we have local morphisms $x : X \to T$, $y : Y \to T$, $z : Z \to T$ such that $c^{(k)}(x) = d^{(k)}(y)$ and $c^{(k)}(y) = d^{(k)}(z)$, i.e. the diagram

$$\begin{tikzcd}
 & & T & & \\
X \arrow[urr, "x"] & & Y \arrow[u, "y"'] & & Z \arrow[ull, "z"'] \\
 & \mathbf{c}^{(k)}X \arrow[ul] \arrow[ur] & & \mathbf{c}^{(k)}Y \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

commutes. Hence the two compositions of these cells are isomorphic via the canonical (local) isomorphism of pushouts

$$\begin{tikzcd}
 & T & \\
(X \otimes_k Y) \otimes_k Z \arrow[ur, "{[[x,y],z]}"] \arrow[rr, "\cong"'] & & X \otimes_k (Y \otimes_k Z) \arrow[ul, "{[x,[y,z]]}"']
\end{tikzcd}$$

But as we shown in Proposition 9.4 these isomorphisms are in fact monotone morphisms. Thus the morphisms $[[x, y], z]$ and $[x, [y, z]]$ represent the same cell in T^*.

The verification that T^* satisfy also the remaining condition of the definition of the ω-category is left for the reader. It should be also obvious that any local morphism between local face structures $f : S \to T$ induces an ω-functor $f^* : S^* \to T^*$ by composition. $\Box$

The k-truncation $S_{\leq k}$ of an ordered face structure S need not to be an ordered face structure, however it gives rise to a local face structure of dimension k, i.e. for $k \in \omega$ we have a truncation functor

$$tr_k : \mathbf{oFs} \longrightarrow \mathbf{lFs}_k$$

sending $(S, <^{S_k,\sim})_{k\in\omega}$ to $(S, <^{\sim}_a)_{a\in S_{>1,\leq k}}$, where $<^{\sim}_a$ is the restriction of $<^{\sim}$ to $\dot{\delta}(a)$, for $a \in S_{>1,\leq k}$. Here $\mathbf{lFs}_k$ denotes the full subcategory of **lFs** whose object have dimension at most k. Clearly, we have a commuting square

$$\begin{tikzcd}
\mathbf{oFs} \arrow[r, "tr_k"] \arrow[d, "(-)^*"'] & \mathbf{lFs}_k \arrow[d, "(-)^*"] \\
\omega Cat \arrow[r, "tr_k"'] & kCat
\end{tikzcd}$$

Thus we have a functor

$$(-)^*_{\leq k} : \mathbf{oFs} \longrightarrow kCat$$

which is defined as either of the above compositions. $kCat$ is the category of k-categories.

11 Principal and Normal ordered face structures

We recall few notions form section 3. Let N be an ordered face structure. N is k-normal iff $dim(N) \leq k$ and $size(N)_l = 1$, for $l < k$. N is k-principal iff $size(N)_l = 1$, for $l \leq k$. N is principal iff $size(N)_l \leq 1$, for $l \leq \omega$. N is principal of dimension k iff N is principal and $dim(N) = k$.

Notation for a k-normal N: $\{\mathbf{p}^N_l\} = \{\mathbf{p}_l\} = N_l - \delta(N_{l+1})$, for $l < k$.

Lemma 11.1 *Let P, Q, N, T be an ordered face structures, $k \in \omega$, P, Q principal, N k-normal.*

1. *If the map $f : P \to T$ is local then it is a monotone morphism.*

2. *If the map $f : N \to T$ is local then f is monotone iff f_k preserves $<^{\sim}$.*

3. *If $dim(P) = dim(Q)$ and the maps $f : P \to T$ and $g : Q \to T$ are local such that $f(\mathbf{p}^P) = g(\mathbf{p}^Q)$ then there is a unique monotone isomorphism $h : P \to Q$ making the triangle*

$$\begin{tikzcd} P \arrow[rr, "h"] \arrow[dr, "f"'] & & Q \arrow[dl, "g"] \\ & T & \end{tikzcd}$$

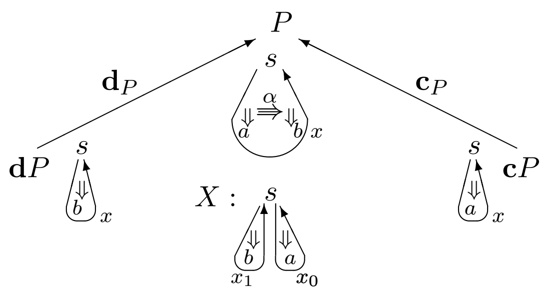
   commutes, where $\mathbf{p}^P$, $\mathbf{p}^Q$ are the unique faces of dimension k in P and in Q, respectively.

4. *If $dim(P) = n > dim(Q)$ then any monotone morphism $x : Q \to P$ factorizes either via $\mathbf{d}_P : \mathbf{d}P \to P$ or $\mathbf{c}_P : \mathbf{c}P \to P$.*

Proof. 1. and 2. follows immediately from Lemma 4.27. 4. follows easily from 3.

Ad 3. Let $dim(P) = dim(Q) = k$. By 1. we need to construct a local isomorphism only. The argument is by induction on k. For $k = 0$ the claim is obvious. For $k > 0$, we have by induction hypothesis the local morphism $h' : \mathbf{c}^{(k-1)}P \longrightarrow \mathbf{c}^{(k-1)}Q$. Then we note that the bijections $f : \delta(\mathbf{p}^P) \longrightarrow \delta(f(\mathbf{p}^P))$, $g : \delta(\mathbf{p}^Q) \longrightarrow \delta(g(\mathbf{p}^Q))$ preserves order. As $f(\mathbf{p}^P) = g(\mathbf{p}^Q)$ we get easily the local morphism $h : P \longrightarrow Q$. $\Box$

Example. Note that in Lemma 11.1.4 it is essential that Q is principal and not any ordered face structure. In the example as below



with morphism $f : X \to P$ sending x_i to x and other cells to the same cell we clearly cannot factor f via neither $\mathbf{d}_P$ nor $\mathbf{c}_P$.

Lemma 11.2 *Let T be an ordered face structure, $l, k \in \omega$, $l < k$, and $\alpha \in T_k$. We have*

1. *$\{\alpha\}$ is a convex set and $[\alpha]$ is a principal ordered face structure,*

2. *$\delta(\alpha)$ is a convex set and $[\delta(\alpha)]$ is a $(k-1)$-normal ordered face structure,*

3. *Moreover, if $k > 0$, then*

$$\mathbf{c}^{(l)}[\alpha] = [\gamma^{(l)}(\alpha)] \qquad \mathbf{d}^{(l)}[\alpha] = [\delta^{(l)}(\alpha)].$$

Proof. We shall prove 1. The rest is left as an exercise.

The proof goes by induction of the $dim(\alpha) = k$. If $k = 0$ then the thesis is obvious. So assume that $k > 0$, the thesis holds for $\gamma(\alpha)$ and we shall prove it for α. If $\alpha \in T^{\lambda} \cup T^{\varepsilon}$ then $< \alpha >$ has faces as follows

dim	faces
k	α
$k-1$	$\gamma(\alpha)$
$k-2$	$\dot{\delta}\gamma(\alpha) \cup \gamma\gamma(\alpha)$

and hence the thesis is obvious. So assume that $\alpha \in T^{-\lambda\varepsilon}$. Then $< \alpha >$ has faces as follows

dim	faces
k	α
$k-1$	$\delta(\alpha) \cup \gamma(\alpha)$
$k-2$	$\delta\dot{\delta}^{-\lambda}(\alpha) \cup \gamma\gamma(\alpha)$
$k-3$	$\delta\dot{\delta}^{-\lambda}\gamma(\alpha) \cup \gamma\gamma\gamma(\alpha)$

For the faces of dimension k and $k-1$ the thesis is obvious, for dimensions $k-3$ and lower the thesis holds by inductive assumption on $\gamma(\alpha)$. We need to check that $<^{S,+}$ and $<^{<\alpha>,+}$ agree on $< \alpha >_{k-2}$. First note that by Lemma 4.2.3, if $x \in \delta\dot{\delta}^{-\lambda}(\alpha)$ then $x <^{<\alpha>,+} \gamma\gamma(\alpha)$. So assume that $x \in \delta(a)$ and $y \in \delta(b)$, $a, b \in \dot{\delta}(\alpha)$, $x <^{S,+} y$. Let $x, a_1, \ldots, a_n, y$ be a flat upper $T - \gamma(T^{-\lambda})$-path from x to y and $x, b_1, \ldots, b_l, \gamma\gamma(\alpha)$ be a flat upper $\dot{\delta}(\alpha)$-path from x to $\gamma\gamma(\alpha)$. If $y = \gamma(b_{l_0})$ for some $l_0 \leq l$, we are done. So assume contrary. Then by Path Lemma, there is $l_1 \leq l$ such that $a_n <^{+} b_{l_1}$, $\gamma(a_n) = y \neq \gamma(b_{l_1})$. Thus we have a flat upper path $a_n, \beta_1, \ldots, \beta_r, b_{l_1}$ and, as $y \neq \gamma(b_{l_1})$, there is $r_0 \leq r$ such that $y \in \iota(\beta_{r_0})$. Hence $\delta(b) \cap \iota(\beta_{r_0}) \neq \emptyset$ and by Lemma 4.8.1, $b <^{+} \gamma(b_{r_0}) \leq b_{l_1}$. But $b, b_{l_1} \in \delta(\alpha)$ and we get a contradiction with discreetness. $\Box$

Let $k \in \omega$, N be a k-normal ordered face structure. We define a $(k+1)$-hypergraph $N^{\bullet}$, that contains two additional faces: $\mathbf{p}_{k+1}^{N^{\bullet}}$ of dimension $k+1$, and $\mathbf{p}_{k}^{N^{\bullet}}$ of dimension k. We shall drop superscripts if it does not lead to confusions. We also put

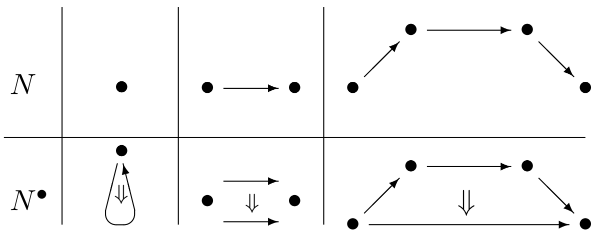
$$\gamma(\mathbf{p}_{k+1}) = \mathbf{p}_k, \quad \gamma(\mathbf{p}_k) = \mathbf{p}_{k-1},$$

$$\delta(\mathbf{p}_{k+1}) = \begin{cases} N_k & \text{if } N_k \neq \emptyset, \\ 1_{\mathbf{p}_{k-1}} & \text{otherwise.} \end{cases} \qquad \delta(\mathbf{p}_k) = \begin{cases} \delta(N_k) - \gamma(N_k) & \text{if } N_k \neq \emptyset, \\ 1_{\mathbf{p}_{k-2}} & \text{otherwise.} \end{cases}$$

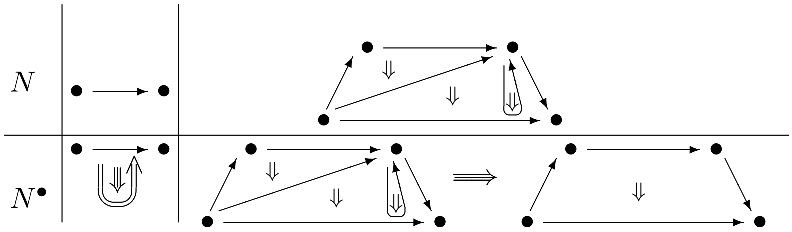
Clearly, $\gamma(\mathbf{p}_k)$ and $\delta(\mathbf{p}_k)$ are defined only if $k > 0$.

As N is k-normal, $N_{k+1} = \emptyset$, so N_k cannot contain loops. Thus, if $N_k \neq \emptyset$ then $\delta(N_k) - \gamma(N_k) \neq \emptyset$ and $\delta(\mathbf{p}_k)$ is well defined. This determines $N^{\bullet}$ uniquely. $N^{\bullet}$ is called *the principal extension of* N.

Examples. Here are some examples of 1-normal ordered face structures N and their principal extensions $N^{\bullet}$:



and some examples of 2-normal ordered face structures N and their principal extensions $N^{\bullet}$:



Clearly, $\bullet$ the 'empty' 1-normal, and $\bullet \longrightarrow \bullet$ is 'empty' 2-normal ordered face structure.

Proposition 11.3 *Let N be a k-normal ordered face structure. Then*

1. *$N^\bullet$ is a principal ordered face structure of dimension $k+1$.*
2. *We have $\mathbf{d}(N^\bullet) \cong N$, $\mathbf{c}(N^\bullet) \cong (\mathbf{d}N)^\bullet$.*
3. *If N is a principal, then $N \cong (\mathbf{d}N)^\bullet$.*

Proof. Exercise. $\Box$

12 Decomposition of ordered face structures

As positive face structures are easier and we understand well their decompositions we define decomposition of ordered face structures via positive ones. This will simplify the proof of properties of the decompositions, as they will be easy consequences of the analogous properties of decompositions of positive face structures. However to get a better insight how the ordered face structures are decomposed we shall characterize the decompositions using convex subsets and stretching empty loops. We decompose along an $\mathcal{I}$-cut rather than a face.

NB. We write $\check{a}$ instead of (a, L, U) if we don't need to specify explicitly which cut over a we consider.

The *k-decomposition* of X is any presentation of X as a k-tensor $X = X_1 \otimes_k X_2$ of two other ordered face structures. X_1 is *the lower part of the decomposition* and X_2 is *the upper part of the decomposition.* The k-decomposition of $X = X_1 \otimes_k X_2$ is said to be *proper* iff $size(X_1), size(X_2) < size(X)$.

Let X be an ordered faces structure, $\check{a} \in X^\dagger$, $\mathcal{J}$ the kernel of the standard positive cover $q : X^\dagger \to X$. We define the decomposition of X along $\check{a}$ as the bottom square of the following cube

$$\begin{tikzcd}
& X^{\dagger\downarrow\check{a}} \arrow[rr, "\kappa"] \arrow[dd, "p"' near start] & & X^\dagger \arrow[dd, "p"] \\
\mathbf{c}^{(k)}(X^{\dagger\downarrow\check{a}}) \arrow[ur, "\mathbf{c}^{(k)}"] \arrow[rr, "\mathbf{d}^{(k)}" near start, crossing over] \arrow[dd, "p"'] & & X^{\dagger\uparrow\check{a}} \arrow[ur, "\kappa"] & \\
& X^{\downarrow\check{a}} \arrow[rr, "\kappa" near start] & & X \\
\mathbf{c}^{(k)}(X^{\downarrow\check{a}}) \arrow[ur, "\mathbf{c}^{(k)}"] \arrow[rr, "\mathbf{d}^{(k)}"'] & & X^{\uparrow\check{a}} \arrow[ur, "\kappa"'] \arrow[from=uu, "p" near start, crossing over] &
\end{tikzcd}$$

where the top square is the decomposition of the positive face structure $X^\dagger$ along $\check{a}$, and the bottom square

$$
\begin{tikzcd}
X^{\downarrow\check{a}} \arrow[r, "\kappa_X^{\downarrow\check{a}}"] & X \\
\mathbf{c}^{(k)}(X^{\downarrow\check{a}}) \arrow[u, "\mathbf{c}^{(k)}"] \arrow[r, "\mathbf{d}^{(k)}"'] & X^{\uparrow\check{a}} \arrow[u, "\kappa_X^{\uparrow\check{a}}"']
\end{tikzcd}
$$

is obtained from the top square by dividing by $\mathcal{J}$.

Lemma 12.1 *We note for the record*

$$\mathbf{d}^{(k)}(X^{\downarrow\check{a}}) = \mathbf{d}^{(k)}(X), \qquad \mathbf{c}^{(k)}(X^{\uparrow\check{a}}) = \mathbf{c}^{(k)}(X),$$

$$\mathbf{c}^{(k)}(X^{\downarrow\check{a}}) = \mathbf{d}^{(k)}(X^{\uparrow\check{a}}), \qquad X^{\downarrow\check{a}} \otimes_k X^{\uparrow\check{a}} = X.$$

Proof. Exercise. $\Box$

Lemma 12.2 *Let S, T be ordered face structures, $k \in \omega$, and $\check{a} = (a, L, U) \in (T_k^\dagger - \iota(T_{k+2}^\dagger))$. Then*

1. *$\check{a} \in Sd(T)$ iff there are $\alpha, \beta \in T_{k+1}$ such that $(\gamma^{(k)}(\alpha), -, \uparrow \gamma^{(k+1)}(\alpha)) \leq^+ \check{a}$ and $(\gamma^{(k)}(\beta), -, \uparrow \gamma^{(k+1)}(\beta)) \not\leq^+ \check{a}$.*
2. *$Sd(T) = Sd(T^\dagger)$.*
3. *$size(T) = size(T^\dagger)$.*
4. *if $\mathbf{c}^{(k)}(S) = \mathbf{d}^{(k)}(T)$ then, for $l \in \omega$,*

$$size(S \otimes_k T)_l = \begin{cases} size(S)_l + size(T)_l & \text{if } l > k, \\ size(T)_l & \text{if } l \leq k. \end{cases}$$

5. *$size(T)_k \geq 1$ iff $k \leq dim(T)$.*
6. *$Sd(T)_k \neq \emptyset$ iff $size(T)_{k+1} \geq 2$.*
7. *T is principal iff $Sd(T)$ is empty.*

Proof. Easy. $\Box$

Before we shall establish the important properties of this decomposition we shall show another way of constructing this decomposition. Let Y be a convex subset of an ordered face structure X. We define two subhypergraphs $Y^{\Downarrow\check{a}}$ and $Y^{\Uparrow\check{a}}$ of X:

$$Y_l^{\Downarrow\check{a}} = \begin{cases} \{\alpha \in Y_l : (\gamma^{(k)}(\alpha), -, \uparrow \gamma^{(k+1)}(\alpha)) \leq^+ (a, L, U)\} & \text{for } l > k, \\ \{b \in Y_k : b \leq^+ a \text{ or } b \not\in \gamma(Y_{k+1}^{-\lambda})\} & \text{for } l = k \\ X_l & \text{for } l < k. \end{cases}$$

$$Y_l^{\Uparrow\check{a}} = \begin{cases} \{\alpha \in Y_l : (\gamma^{(k)}(\alpha), -, \uparrow \gamma^{(k+1)}(\alpha)) \not\leq^+ (a, L, U)\} & \text{for } l > k, \\ \{b \in Y_k : b \not\leq^+ a\} & \text{for } l = k \\ Y_{k-1} - \iota(Y_{k+1}^{\Downarrow\check{a}}) & \text{for } l = k-1 \\ Y_l & \text{for } l < k-1. \end{cases}$$

Lemma 12.3 *With the notation as above $Y^{\Downarrow\check{a}}$ and $Y^{\Uparrow\check{a}}$ are convex subhypergraphs of X, $c^{(k)}(Y^{\Downarrow\check{a}}) = d^{(k)}(Y^{\Uparrow\check{a}})$. Moreover $\mathcal{E}^{Y^{\Downarrow\check{a}}} = \emptyset$ and*

$$\mathcal{E}^{Y^{\Uparrow\check{a}}} = \begin{cases} \{a\} & \text{if } a \text{ is a loop,} \\ \emptyset & \text{otherwise.} \end{cases}$$

Proof. Easy. □

Lemma 12.4 *With the notation as above there are monotone isomorphisms $h_\downarrow$ and $h_\uparrow$ making the triangles*

$$\begin{tikzcd}
X^{\downarrow\check{a}} \arrow[r, "\kappa"] & X \arrow[r, "\kappa"] & X^{\downarrow\check{a}} \\
{[X^{\Downarrow\check{a}}]} \arrow[u, "h_\downarrow"] \arrow[ur, "\nu"'] & & {[X^{\Uparrow\check{a}}]} \arrow[u, "h_\uparrow"'] \arrow[ul, "\nu"]
\end{tikzcd}$$

commute. $\kappa_X^{\uparrow\check{a}}$

Proof. By Lemma 5.3 it is enough to show that the image of the monotone morphism $\kappa_X^{\uparrow\check{a}} : X^{\uparrow\check{a}} \longrightarrow X$ is $X^{\Uparrow\check{a}}$ and the image of the monotone morphism $\kappa_X^{\downarrow\check{a}} : X^{\downarrow\check{a}} \longrightarrow X$ is $X^{\Downarrow\check{a}}$. The remaining details are left for the reader. □

Note that, by the above Lemma $X^{\Downarrow\check{a}}$ is isomorphic to $X^{\downarrow\check{a}}$ and if a is not a loop in X, $X^{\Uparrow\check{a}}$ is isomorphic to $X^{\uparrow\check{a}}$. However the ordered face structure $X^{\uparrow\check{a}}$ is not that complicated even if a is a loop. We shall describe it now. Thus $\mathcal{E}^{X^{\Uparrow\check{a}}} = \{a\}$. In this case, up to isomorphism, the underlying hypergraph of $X^{\uparrow\check{a}}$ can be describe as follows.

$$X_l^{\uparrow\check{a}} = \begin{cases} X_l^{\Uparrow\check{a}} & \text{if } l \neq k-1, \\ (X_{k-1}^{\Uparrow\check{a}} - \{\gamma(a)\}) \cup \{\gamma(a)^-, \gamma(a)^+\} & \text{if } l = k-1. \end{cases}$$

$\gamma^{X^{\uparrow\check{a}}}$ and $\delta^{X^{\uparrow\check{a}}}$ are as in $X^{\Uparrow\check{a}}$ (and X) except for $\gamma_{k-1}^{X^{\uparrow\check{a}}}$ and $\delta_{k-1}^{X^{\uparrow\check{a}}}$. For $c \in X_k^{\uparrow\check{a}}$ we put (γ and δ stands for γ^X and δ^X, respectively)

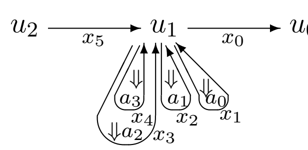
$$\gamma^{X^{\uparrow\check{a}}}(c) = \begin{cases} \gamma(c) & \text{if } \gamma(c) \neq \gamma(a), \\ \gamma(a)^- & \text{if } \gamma(c) = \gamma(a) \text{ and } c <^\sim a, \\ \gamma(a)^+ & \text{otherwise.} \end{cases}$$

$$\delta^{X^{\uparrow\check{a}}}(c) = \begin{cases} \delta(c) & \text{if } \gamma(b) \not\in \delta(c), \\ (\delta(c) - \{\gamma(a)\}) \cup \{\gamma(a)^+\} & \text{if } \gamma(a) \in \delta(c) \text{ and } a <^\sim c, \\ (\delta(c) - \{\gamma(a)\}) \cup \{\gamma(a)^-\} & \text{otherwise.} \end{cases}$$

The order $<^\sim$ in $X^{\downarrow\check{a}}$ and $X^{\uparrow\check{a}}$ is uniquely determined by the fact that it is reflected from X via $\kappa_X^{\downarrow\check{a}}$ and $\kappa_X^{\uparrow\check{a}}$.

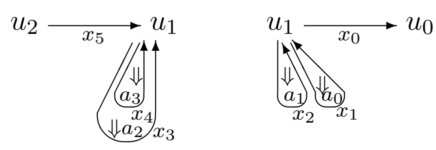
Examples. For the ordered face structure T as below

T

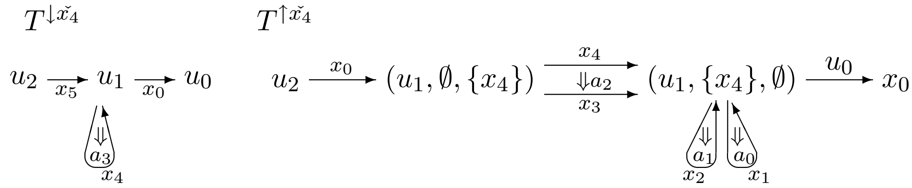


and a cut $\check{u}_1 = (u_1, \{a_3\}, \{a_2, a_1\})$ we have the following decomposition

$T^{\downarrow\check{u}_1}$ $\quad$ $T^{\uparrow\check{u}_1}$



and for the cut $\check{x}_4 = (x_4, \emptyset, \emptyset)$ we have the following decomposition



The following Lemma establishes some properties of the double decompositions. The double decomposition is meant in the sense of convex set decomposition, i.e. when we write $X^{\downarrow\check{x}\uparrow\check{a}}$ we mean $[X^{\Downarrow\check{x}\Uparrow\check{a}}]$.

Lemma 12.5 *Let X be an ordered face structure, $\check{a} = (a, L, U), \check{x} = (x, L', U') \in (X^\dagger - \iota(X^\dagger))$, $k = dim(x) < dim(a) = m$.*

1. *We have the following equations of ordered face structures:*
$$X^{\downarrow\check{x}\downarrow\check{a}} = X^{\downarrow\check{a}\downarrow\check{x}}, \quad X^{\downarrow\check{x}\uparrow\check{a}} = X^{\uparrow\check{a}\downarrow\check{x}}, \quad X^{\uparrow\check{x}\downarrow\check{a}} = X^{\downarrow\check{a}\uparrow\check{x}}, \quad X^{\uparrow\check{x}\uparrow\check{a}} = X^{\uparrow\check{a}\uparrow\check{x}},$$
i.e. x-decompositions and a-decompositions commute.
2. *If $\check{x} \in Sd(X)$ then $\check{x} \in Sd_{\kappa^{\downarrow\check{a}}}(X^{\downarrow\check{a}}) \cap Sd_{\kappa^{\uparrow\check{a}}}(X^{\uparrow\check{a}})$.*
3. *Moreover, we have the following equations concerning domains and codomains*
$$\mathbf{c}^{(k)}(X^{\downarrow\check{x}\downarrow\check{a}}) = \mathbf{c}^{(k)}(X^{\downarrow\check{x}\uparrow\check{a}}) = \mathbf{d}^{(k)}(X^{\uparrow\check{x}\downarrow\check{a}}) = \mathbf{d}^{(k)}(X^{\uparrow\check{x}\uparrow\check{a}})$$
$$\mathbf{c}^{(m)}(X^{\downarrow\check{x}\downarrow\check{a}}) = \mathbf{d}^{(m)}(X^{\downarrow\check{x}\uparrow\check{a}}), \quad \mathbf{c}^{(m)}(X^{\uparrow\check{x}\downarrow\check{a}}) = \mathbf{d}^{(m)}(X^{\uparrow\check{x}\uparrow\check{a}}).$$
4. *Finally, we have the following equations concerning compositions*
$$X^{\downarrow\check{x}\downarrow\check{a}} \otimes_m X^{\downarrow\check{x}\uparrow\check{a}} = X^{\downarrow\check{x}}, \quad X^{\uparrow\check{x}\downarrow\check{a}} \otimes_m X^{\uparrow\check{x}\uparrow\check{a}} = X^{\uparrow\check{x}},$$
$$X^{\downarrow\check{x}\downarrow\check{a}} \otimes_k X^{\uparrow\check{x}\downarrow\check{a}} = X^{\downarrow\check{a}}, \quad X^{\downarrow\check{x}\uparrow\check{a}} \otimes_k X^{\uparrow\check{x}\uparrow\check{a}} = X^{\uparrow\check{a}}.$$

Proof. We need to verify the above equations for arrows $\Downarrow$ and $\Uparrow$ instead of $\downarrow$ and $\uparrow$. $\Box$

Lemma 12.6 *Let T be ordered face structure, X convex subhypergraph of T, and $a, b \in X$, $\check{a} = (a, L, U), \check{b} = (b, L', U') \in T^\dagger - \iota(T^\dagger)$, $dim(a) = dim(b) = m$.*

1. *We have the following equations of ordered face structures:*
$$X^{\downarrow\check{a}\downarrow\check{b}} = X^{\downarrow\check{b}\downarrow\check{a}}, \quad X^{\uparrow\check{a}\uparrow\check{b}} = X^{\uparrow\check{b}\uparrow\check{a}},$$
i.e. the same direction a-decompositions and b-decompositions commute.
2. *Assume $\check{a} <^+ \check{b}$. Then we have the following farther equations of ordered face structures:*
$$X^{\uparrow\check{b}} = X^{\uparrow\check{a}\uparrow\check{b}}, \quad X^{\downarrow\check{a}} = X^{\downarrow\check{a}\downarrow\check{b}}, \quad X^{\downarrow\check{b}\uparrow\check{a}} = X^{\uparrow\check{a}\downarrow\check{b}}.$$
Moreover, if $\check{a}, \check{b} \in Sd(X)$ then $\check{a} \in Sd_{\kappa^{\uparrow\check{b}}}(X^{\downarrow\check{b}})$ and $\check{b} \in Sd_{\kappa^{\uparrow\check{a}}}(X^{\uparrow\check{a}})$.
3. *Assume $\check{a} <^-_l \check{b}$, for some $l < m$. Then $X^{\uparrow\check{b}\downarrow\check{a}}$, $X^{\uparrow\check{a}\downarrow\check{b}}$, are ordered face structures, and*
$$X^{\downarrow\check{a}} \otimes_m X^{\uparrow\check{a}\downarrow\check{b}} = X^{\downarrow\check{b}} \otimes_m X^{\uparrow\check{b}\downarrow\check{a}}$$
Moreover, if $a, b \in Sd(X)$ then either there is k such that $l - 1 \leq k < m$ and $(\gamma^{(k)}(a), -, \uparrow \gamma^{(k+1)}(a)) \in Sd(X)$ or $\check{a} \in Sd_{\kappa^{\uparrow\check{b}}}(X^{\uparrow\check{b}})$ and $\check{b} \in Sd_{\kappa^{\uparrow\check{a}}}(X^{\uparrow\check{a}})$.

Proof. Easy. □

The following properties of ordered face structures are inherited from the corresponding properties of positive face structures.

Lemma 12.7 *Let T be ordered face structures of dimension n, $l < n-1$, $\breve{a} = (a, L, U) \in Sd(T)_l$. Then*

1. $\breve{a} \in Sd(\mathbf{c}T) \cap Sd(\mathbf{d}T)$;
2. $\mathbf{d}(T^{\downarrow\breve{a}}) = (\mathbf{d}T)^{\downarrow\breve{a}}$;
3. $\mathbf{d}(T^{\uparrow\breve{a}}) = (\mathbf{d}T)^{\uparrow\breve{a}}$;
4. $\mathbf{c}(T^{\downarrow\breve{a}}) = (\mathbf{c}T)^{\downarrow\breve{a}}$;
5. $\mathbf{c}(T^{\uparrow\breve{a}}) = (\mathbf{c}T)^{\uparrow\breve{a}}$.

Proof. See the the corresponding properties of positive face structures in [Z]. □

Lemma 12.8 *Let T, T_1, T_2 be ordered face structures, $dim(T_1), dim(T_2) > k$, such that $\mathbf{c}^{(k)}(T_1) = \mathbf{d}^{(k)}(T_2)$ and $T = T_1 \otimes_k T_2$, and let $Z = \gamma((T_1)_{k+1}) - \delta((T_1^{-\lambda})_{k+1})$. Then $\emptyset \neq Z \subseteq \mathbf{c}^{(k)}(T_1)_k$. For any face $a \in Z$, the cut $\breve{a} = (a, \mathcal{I}_a \cap (T_1)_{k+2}, \mathcal{I}_a \cap (T_2)_{k+2}) \in Sd(T)$ and one of the following conditions holds:*

1. *either* $T_1 = T^{\downarrow\breve{a}}$ *and* $T_2 = T^{\uparrow\breve{a}}$;
2. *or* $\breve{a} \in Sd(T_1)_k$, $T^{\downarrow\breve{a}} = T_1^{\downarrow\breve{a}}$ *and* $T^{\uparrow\breve{a}} = T_1^{\uparrow\breve{a}} \otimes_k T_2$;
3. *or* $\breve{a} \in Sd(T_2)_k$, $T^{\uparrow\breve{a}} = T_2^{\uparrow\breve{a}}$ *and* $T^{\downarrow\breve{a}} = T_1 \otimes_k T_2^{\downarrow\breve{a}}$.

Proof. See the the corresponding properties of positive face structures in [Z]. □

13 T^* is a many-to-one computad

Proposition 13.1 *Let T be an ordered face structure. Then T^* is a many-to-one computad, whose indets correspond to the faces of T.*

Proof. In fact, to be able to carry on the induction we need to prove more. Let T be an ordered face structure, $n \in \omega$.

Inductive Hypothesis for n. For any ordered face structure T, the n-truncation $T^*_{\leq n}$ of T^* is a many-to-one computad whose n-indets are in the image of the embedding $\nu : T_n \longrightarrow T^*_n$, sending $a \in T_n$ to the local morphism $\nu_a : [a] \longrightarrow T$ in $T^*_{\leq n}$.

The proof proceeds by induction on n. The Inductive Hypothesis for $n = 0, 1$ is obvious.

So assume that the Inductive Hypothesis holds already for some $n \geq 1$. Suppose that T is an ordered face structure. We shall show that $T^*_{\leq n+1}$ is a many-to-one computad whose $n+1$-indets are in the image of $\nu : T_{n+1} \longrightarrow T^*_{n+1}$.

We need to verify that for any ω-functor $f : T^*_{\leq n} \longrightarrow C$ to any ω-category C, and any function $|f| : T_{n+1} \longrightarrow C_{n+1}$ such that for $a \in T_{n+1}$, and $\nu_a : [a] \to T$

$$d_C(|f|(a)) = f(\mathbf{d}(\nu_a)), \qquad c_C(|f|(a)) = f(\mathbf{c}(\nu_a)),$$

there is a unique ω-functor $F : T^*_{\leq n+1} \longrightarrow C$, such that

$$F_{n+1}(\nu_a) = |f|(a), \qquad F_{\leq n} = f$$

as in the diagram

$$\begin{tikzcd}
 & & C \\
T_{n+1} \arrow[rru, "|f|"] \arrow[r, "\nu_{n+1}"'] \arrow[d, shift right, "\delta"'] \arrow[d, shift left, "\gamma"] & T^*_{\leq n+1} \arrow[ru, "F"'] \arrow[d, shift right, "\mathbf{d}"'] \arrow[d, shift left, "\mathbf{c}"] & \\
T_{\leq n} \arrow[r, "\nu_{\leq n}"'] & T^*_{\leq n} \arrow[ruu, "f"'] &
\end{tikzcd}$$

We need some notation for decompositions of cells in T^*. If $\varphi : X \to T \in T^*$ and $\check{a}$ is a cut in $X^\dagger$ then $\varphi^{\downarrow\check{a}} = \kappa^{\downarrow\check{a}}; \varphi : X^{\downarrow\check{a}} \longrightarrow T$, and $\varphi^{\uparrow\check{a}} = \kappa^{\uparrow\check{a}}; \varphi : X^{\downarrow\check{a}} \longrightarrow T$.

We define F_{n+1} as follows. For $\varphi : X \to T \in T^*_{n+1}$

$$F_{n+1}(\varphi) = \begin{cases} id_{f(\varphi)} & \text{if } dim(X) \leq n, \\ |f|(a) & \text{if } \varphi = \nu_a : [a] \to T, \text{ for some } a \in T_{n+1}, \\ F_{n+1}(\varphi^{\downarrow\check{a}});_l F_{n+1}(\varphi^{\uparrow\check{a}}) & \text{if } dim(X) = n+1,\ \check{a} \in Sd(X)_l. \end{cases}$$

Clearly $F_k = f_k$, for $k \leq n$. The above morphism, if well defined, clearly preserves identities. We need to verify, for any $\varphi : X \to T$ in T^*_{n+1}, the following three conditions:

I F is well defined, i.e. for $\check{a}, \check{b} \in Sd(X)$ we have $F_{n+1}(\varphi^{\downarrow\check{a}});_l F_{n+1}(\varphi^{\uparrow\check{a}}) = F_{n+1}(\varphi^{\downarrow\check{b}});_l F_{n+1}(\varphi^{\uparrow\check{b}})$,

II F preserves the domains and codomains i.e.we have $F(\mathbf{d}\varphi) = d(F(\varphi))$ and $F(\mathbf{c}\varphi) = c(F(\varphi))$,

III F preserves compositions i.e., we have $F(\varphi) = F(\varphi_1);_k F(\varphi_2)$ whenever $\varphi_i : X_i \to T \in T^*_{n+1}$ for $i = 1, 2$, $c^{(k)}(\varphi_1) = d^{(k)}(\varphi_2)$, and $\varphi = \varphi_1;_k \varphi_2$.

Assume that $\varphi : X \to T \in T^*_{k+1}$, and for faces $y : Y \to T$ of T^* of size less than $size(X)$ the conditions [**I**], [**II**], [**III**] holds. We shall show that [**I**], [**II**], [**III**] hold for φ, as well. For X such that $size(X)_{n+1} = 0$ all three conditions are obvious.

If X is principal of dimension $n+1$, [**I**] is trivially true as $Sd(X) = \emptyset$, [**III**] is true as if $\varphi = \varphi_1;_k \varphi_2$, with X principal then either φ_1 or φ_2 is an identity. So we need to check [**II**]. We have that $X_{n+1} = \{m^X\}$ and $\varphi(m^X) = a \in T_{k+1}$. By Lemma 11.1.3, there is a unique isomorphism $h : [a] \to X$ making the triangle

$$\begin{tikzcd}
{[\varphi(m^X)]} \arrow[r, "h"] \arrow[dr, "\nu_{\varphi(m^X)}"'] & X \arrow[d, "\varphi"] \\
 & T
\end{tikzcd}$$

commutes, i.e. $\nu_{\varphi(m^X)}$ and φ represent the same cell in T^*, and hence [**II**] follows immediately from the properties of f.

Now assume that X is not principal and $dim(X) = n+1$.

Ad **I**. First we will consider two saddle cuts $\check{a}, \check{x} \in Sd(X)$ of different dimension $k = dim(x) < dim(a) = m$. Using Lemma 12.5 we have

$$\begin{aligned}
& F(\varphi^{\downarrow\check{a}});_m F(\varphi^{\uparrow\check{a}}) = && ind.\ hyp.\ III \\
&= (F(\varphi^{\downarrow\check{a}\downarrow\check{x}});_k F(\varphi^{\downarrow\check{a}\uparrow\check{x}}));_m (F(\varphi^{\uparrow\check{a}\downarrow\check{x}});_k F(\varphi^{\uparrow\check{a}\uparrow\check{x}})) = && MEL \\
&= (F(\varphi^{\downarrow\check{a}\downarrow\check{x}});_m F(\varphi^{\uparrow\check{a}\downarrow\check{x}}));_k (F(\varphi^{\downarrow\check{a}\uparrow\check{x}});_m F(\varphi^{\uparrow\check{a}\uparrow\check{x}})) = && \\
&= (F(\varphi^{\downarrow\check{x}\downarrow\check{a}});_m F(\varphi^{\downarrow\check{x}\uparrow\check{a}}));_k (F(\varphi^{\uparrow\check{x}\downarrow\check{a}});_m F(\varphi^{\uparrow\check{x}\uparrow\check{a}})) = && ind.\ hyp.\ III \\
&= F(\varphi^{\downarrow\check{x}});_m F(\varphi^{\uparrow\check{x}}) &&
\end{aligned}$$

Now we will consider two saddle cuts $\check{a}, \check{b} \in Sd(X)$ of the same dimension $dim(a) = dim(b) = m$. We shall use Lemma 12.6. Assume that $\check{a} <_l^- \check{b}$, for some $l < m$. If $\check{x} = (\gamma^{(k)}(a), -, \uparrow \gamma^{(k+1)}(a)) \in Sd(X)$, for some $k < m$, then this case reduces to the previous one for two pairs $\check{a}, \check{x} \in Sd(X)$ and $\check{b}, \check{x} \in Sd(X)$. Otherwise $\check{a} \in Sd(X^{\uparrow \check{b}})$, $\check{a} \in Sd(X^{\uparrow \check{b}})$, and we have

$$
\begin{aligned}
F(\varphi^{\downarrow \check{a}});_k F(\varphi^{\uparrow \check{a}}) = & \quad \textit{ind. hyp III} \\
= F(\varphi^{\downarrow \check{a}});_k (F(\varphi^{\uparrow \check{a} \downarrow \check{b}});_k F(\varphi^{\uparrow \check{a} \uparrow \check{b}})) = & \\
= (F(\varphi^{\downarrow \check{a}});_k F(\varphi^{\uparrow \check{a} \downarrow \check{b}}));_k F(\varphi^{\uparrow \check{b} \uparrow \check{a}}) = & \quad \textit{ind hyp III} \\
= F(\varphi^{\downarrow \check{a}};_k \varphi^{\uparrow \check{a} \downarrow \check{b}});_k F(\varphi^{\uparrow \check{b} \uparrow \check{a}}) = & \\
= F(\varphi^{\downarrow \check{b}};_k \varphi^{\uparrow \check{b} \downarrow \check{a}});_k F(\varphi^{\uparrow \check{b} \uparrow \check{a}}) = & \quad \textit{ind hyp III} \\
= (F(\varphi^{\downarrow \check{b}});_k F(\varphi^{\uparrow \check{b} \downarrow \check{a}}));_k F(\varphi^{\uparrow \check{b} \uparrow \check{a}}) = & \\
= F(\varphi^{\downarrow \check{b}});_k (F(\varphi^{\uparrow \check{b} \downarrow \check{a}});_k F(\varphi^{\uparrow \check{b} \uparrow \check{a}})) = & \quad \textit{ind hyp III} \\
= F(\varphi^{\downarrow \check{b}});_k F(\varphi^{\uparrow \check{b}}) &
\end{aligned}
$$

Finally, we consider the case $a <^+ b$. We have

$$
\begin{aligned}
F(\varphi^{\downarrow \check{a}});_k F(\varphi^{\uparrow \check{a}}) = & \quad \textit{ind. hyp III} \\
= F(\varphi^{\downarrow \check{a}});_k (F(\varphi^{\uparrow \check{a} \downarrow \check{b}});_k F(\varphi^{\uparrow \check{a} \uparrow \check{b}})) = & \\
= (F(\varphi^{\downarrow \check{b} \downarrow \check{a}});_k F(\varphi^{\downarrow \check{b} \uparrow \check{a}}));_k F(\varphi^{\uparrow \check{b}}) = & \quad \textit{ind hyp III} \\
= F(\varphi^{\downarrow \check{b}});_k F(\varphi^{\uparrow \check{b}}) &
\end{aligned}
$$

This shows that $F(\varphi)$ is well defined.

Ad **II**. We shall show that the domains are preserved. The proof that, the codomains are preserved, is similar.

The fact that if $Sd(X) = \emptyset$ then F preserves domains and codomains follows immediately from the assumption on f and $|f|$ and Lemma 12.2. So assume that $Sd(X) \neq \emptyset$ and let $\check{a} \in Sd(X)_k$. We use Lemma 12.7. We have to consider two cases: $k < n$, and $k = n$. If $k < n$ then

$$
\begin{aligned}
F_n(d(\varphi)) = F_n(d(\varphi)^{\downarrow \check{a}});_k F_n(d(\varphi)^{\uparrow \check{a}}) = & \\
= F_n(d(\varphi^{\downarrow \check{a}}));_k F_n(d(\varphi^{\uparrow \check{a}})) = & \quad \textit{ind hyp II} \\
= d(F_{n+1}(\varphi^{\downarrow \check{a}}));_k d(F_{n+1}(\varphi^{\uparrow \check{a}})) = & \\
= d(F_{n+1}(\varphi^{\downarrow \check{a}});_k F_{n+1}(\varphi^{\uparrow \check{a}})) = & \quad \textit{ind hyp I} \\
= d(F_{n+1}(\varphi)) &
\end{aligned}
$$

If $k = n$ then

$$
\begin{aligned}
F_n(d(\varphi)) = F_n(d(\varphi^{\downarrow \check{a}};_n \varphi^{\uparrow \check{a}})) = & \\
= F_n(d(\varphi^{\downarrow \check{a}})) = & \quad \textit{ind hyp II} \\
= d(F_{n+1}(\varphi^{\downarrow \check{a}})) = & \quad \textit{ind hyp II} \\
= d(F_{n+1}(\varphi^{\downarrow \check{a}});_n F_{n+1}(\varphi^{\uparrow \check{a}})) = & \quad \textit{ind hyp I} \\
= d(F_{n+1}(\varphi)) &
\end{aligned}
$$

Ad **III**. Suppose that $\varphi = \varphi_1;_k \varphi_2$. We shall show that F preserves this composition. If $dim(X_1) = k$ then $\varphi_2 = \varphi$, $\varphi_1 = \mathbf{d}^{(k)}(\varphi)$. We have

$$
F_{n+1}(\varphi) = F_{n+1}(\varphi_2) = 1^{(n+1)}_{F_k(\mathbf{d}^{(k)}(\varphi_2))};_k F_{n+1}(\varphi_2) =
$$

$$= 1^{(n+1)}_{F_k(\varphi_1)};_k F_{n+1}(\varphi_2) = F_{n+1}(\varphi_1);_k F_{n+1}(\varphi_2)$$

The case $dim(X_2) = k$ is similar. So now assume that $dim(X_1), dim(X_2) > k$. We shall use Lemma 12.8. Let us fix a face $a \in \gamma((X_1)_{k+1}) - \delta((X_1^{-\lambda})_{k+1})$, and a cut $\breve{a} = (a, \mathcal{I}_a \cap (X_1)_{k+2}, \mathcal{I}_a \cap (X_2)_{k+2}) \in Sd(X)$.

If $X_1 = X^{\downarrow\breve{a}}$ and $X_2 = X^{\uparrow\breve{a}}$ then we have

$$F(\varphi) = F(\varphi^{\downarrow\breve{a}});_k F(\varphi^{\uparrow\breve{a}}) = F(\varphi_1);_k F(\varphi_2).$$

If $\breve{a} \in Sd(T_1)_k$, $T^{\downarrow\breve{a}} = T_1^{\downarrow\breve{a}}$ and $T^{\uparrow\breve{a}} = T_1^{\uparrow\breve{a}} \otimes_k T_2$

$$\begin{aligned} F(\varphi) = F(\varphi^{\downarrow\breve{a}});_k F(\varphi^{\uparrow\breve{a}}) = &\quad ind\ hyp\ III \\ = F(\varphi^{\downarrow\breve{a}});_k (F(\varphi_1^{\uparrow\breve{a}});_k F(\varphi_2)) = & \\ = (F(\varphi_1^{\downarrow\breve{a}});_k F(\varphi_1^{\uparrow\breve{a}}));_k F(\varphi_2) = &\quad ind\ hyp\ III \\ F(\varphi_1);_k F(\varphi_2) & \end{aligned}$$

If $\breve{a} \in Sd(T_2)_k$, $T^{\uparrow\breve{a}} = T_2^{\uparrow\breve{a}}$ and $T^{\downarrow\breve{a}} = T_1 \otimes_k T_2^{\downarrow\breve{a}}$

$$\begin{aligned} F(\varphi) = F(\varphi^{\downarrow\breve{a}});_k F(\varphi^{\uparrow\breve{a}}) = &\quad ind\ hyp\ III \\ = (F(\varphi_1);_k F(\varphi_2^{\downarrow\breve{a}}));_k F(\varphi_2^{\uparrow\breve{a}})) = & \\ = F(\varphi_1^{\downarrow\breve{a}});_k (F(\varphi_2^{\uparrow\breve{a}});_k F(\varphi_2^{\uparrow\breve{a}})) = &\quad ind\ hyp\ III \\ F(\varphi_1);_k F(\varphi_2) & \end{aligned}$$

So in any case the composition is preserved. This ends the proof of the Lemma.
□

For $n \in \omega$, we have truncation functors

$$(-)^{\sharp,n} : \mathbf{oFs}_{loc} \longrightarrow \mathbf{Comma}_n^{m/1}, \qquad (-)^{*,n} : \mathbf{oFs}_{loc} \longrightarrow \mathbf{Comp}_n^{m/1}$$

such that, for S in **oFs**

$$S^{\sharp,n} = (S_n, S^*_{<n}, [\delta], [\gamma]), \qquad (S)^{*,n} = S^*_{\leq n}$$

and for $f : S \to T$ in $\mathbf{oFs}_{loc}$ we have

$$f^{\natural,n} = (f_n, (f_{<n})^*), \qquad (f)^{*,n} = f^*_{\leq n}.$$

Corollary 13.2 *For every $n \in \omega$, the functors $(-)^{\sharp,n}$ and $(-)^{*,n}$ are well defined, full, faithful, and they send all tensor squares to pushouts. Moreover, for S in* **oFs** *we have $S^* = \overline{S^{\sharp,n}}^n$.*

Proof. The functor $\overline{(-)}^n : \mathbf{Comma}_n^{m/1} \longrightarrow \mathbf{Comp}_n^{m/1}$, which is an equivalence of categories, is described in the Appendix.

Fullness and faithfulness of $(-)^{\sharp,n}$ is left for the reader. We shall show simultaneously that for every $n \in \omega$, both functors $(-)^{\sharp,n}$ and $(-)^{*,n}$ send n-truncations of tensor squares to pushouts. For $n = 0, 1$ this is obvious. So assume that $n \geq 1$ and that $(-)^{*,n}$ and $(-)^{\sharp,n}$ send n-truncations of k-tensor squares to pushouts. Let

$$\begin{tikzcd} S \arrow[r] & S \otimes_k T \\ \mathbf{c}^{(k)}S \arrow[u] \arrow[r] & T \arrow[u] \end{tikzcd}$$

be a tensor squares in **oFs**. The fact that the functor $(-)^{\sharp,n+1}$ sends this square to a pushout in $\mathbf{Comma}^{m/1}_{n+1}$ can be verified in each dimension separately. In dimensions lower or equal to n this follows from the fact that the functor $(-)^{*,n}$ sends n-truncations of tensor squares to pushouts. In dimension $n+1$ we need to check that the square in Set

$$\begin{tikzcd} S_{n+1} \arrow[r] & (S\otimes_k T)_{n+1} \\ (\mathbf{c}^{(k)}S)_{n+1} \arrow[u] \arrow[r] & T_{n+1} \arrow[u] \end{tikzcd}$$

is a pushout. But this easily follows from the the description of the tensor square given earlier. So the whole square

$$\begin{tikzcd} S^{\sharp,n+1} \arrow[r] & (S\otimes_k T)^{\sharp,n+1} \\ \mathbf{c}^{(k)}S^{\sharp,n+1} \arrow[u] \arrow[r] & T^{\sharp,n+1} \arrow[u] \end{tikzcd}$$

is a pushout in $\mathbf{Comma}^{m/1}_{n+1}$, i.e. $(-)^{\sharp,n+1}$ send $(n+1)$-truncations of k-tensor squares to pushouts. As $(-)^{*,n+1}$ is a composition of $(-)^{\sharp,n+1}$ with an equivalence of categories it send $(n+1)$-truncations of k-tensor squares to pushouts, as well. □

Corollary 13.3 *The functor*

$$(-)^* : \mathbf{oFs}_{loc} \longrightarrow \mathbf{Comp}^{m/1}$$

is full and faithful and sends tensor squares to pushouts.

Proof. This follows from the previous Corollary and the fact that the functor $\overline{(-)}^n$: $\mathbf{Comma}^{m/1}_n \longrightarrow \mathbf{Comp}^{m/1}_n$ (see Appendix) is an equivalence of categories. □

Let P be a many-to-one computad, a a k-cell in P. A *description of the cell* a is a pair

$$< T_a, \tau_a : T^*_a \longrightarrow P >$$

where T_a is an ordered face structure and τ_a is a computad map such that

$$\tau_a(id_{T_a}) = a.$$

14 The terminal many-to-one computad

In this section we shall describe the terminal many-to-one computad $\mathcal{T}$.

The set of n-cell $\mathcal{T}_n$ consists of (isomorphisms classes of) ordered face structures of dimension less than or equal to n. For $n > 0$, the operations of domain and codomain $d^{\mathcal{T}}, c^{\mathcal{T}} : \mathcal{T}_n \to \mathcal{T}_{n-1}$ are given, for $S \in \mathcal{T}_n$ by

$$d(S) = \begin{cases} S & \text{if } dim(S) < n, \\ \mathbf{d}S & \text{if } dim(S) = n, \end{cases}$$

and

$$c(S) = \begin{cases} S & \text{if } dim(S) < n, \\ \mathbf{c}S & \text{if } dim(S) = n. \end{cases}$$

and, for $S, S' \in \mathcal{T}_n$ such that $c^{(k)}(S) = d^{(k)}(S')$ the composition is just the k-tensor of S and S' as ordered face structures i.e. $S \otimes_k S'$

The identity $id_{\mathcal{T}} : T_{n-1} \to \mathcal{T}_n$ is the inclusion map.

The n-indets in $\mathcal{T}$ are the principal ordered n-face structures.

Proposition 14.1 *$\mathcal{T}$ described above is the terminal many-to-one computad.*

Proof. The fact that $\mathcal{T}$ is an ω-category is easy. The fact that $\mathcal{T}$ is free with free n-generators being principal n-face structures can be shown much like the freeness of S^* before. The fact that $\mathcal{T}$ is terminal follows from the following observation:

Observation. For every parallel pair of ordered face structures N and B (i.e. $\mathbf{d}N = \mathbf{d}B$ and $\mathbf{c}N = \mathbf{c}B$) such that N is normal and B is principal, there is a unique (up to an iso) principal ordered face structure $N^\bullet$ such that $\mathbf{d}N^\bullet = N$ and $\mathbf{c}N^\bullet = B$. $\Box$

Lemma 14.2 *Let S be an ordered face structure and $! : S^* \longrightarrow \mathcal{T}$ the unique computad map from S^* to $\mathcal{T}$. Then, for $x : X \to S \in S^*_k$ we have*

$$!_k(x) = X.$$

Proof. The proof is by induction on $k \in \omega$ and the size of X in S^*_k. For $k = 0, 1$ the lemma is obvious. Let $k > 1$ and assume that lemma holds for $i < k$.

If $dim(X) = l < k$ then, using the inductive hypothesis and the fact that ! is an ω-functor, we have

$$!_k(x) = !_k(1^{(k)}_x) = 1^{(k)}_{!_l(x)} = 1^{(k)}_X = X$$

Suppose that $dim(X) = k$ and X is principal. As ! is a computad map $!_k(x)$ is an indet, i.e. it is principal, as well. We have, using again the inductive hypothesis and the fact that ! is an ω-functor,

$$d(!_k(x)) = !_{k-1}(\mathbf{d}x) = \mathbf{d}X$$

$$c(!_k(x)) = !_{k-1}(\mathbf{c}x) = \mathbf{c}X$$

As X is the only (up to a unique iso) ordered face structure with the domain $\mathbf{d}X$ and the codomain $\mathbf{c}X$, it follows that $!_k(x) = X$, as required.

Finally, suppose that $dim(X) = k$, X is not principal, and for the ordered face structures of size smaller than the size of X the lemma holds. Thus there are $l \in \omega$ and $\check{a} \in Sd(X)_l$ so that

$$!_k(x) = !_k(x^{\uparrow\check{a}} ;_l x^{\downarrow\check{a}}) = !_k(x^{\uparrow\check{a}}) \otimes_l !_k(x^{\downarrow\check{a}}) = X^{\uparrow\check{a}} \otimes_l X^{\downarrow\check{a}} = X,$$

as required $\Box$

15 A description of the many-to-one computads

In this section we shall describe all the cells in many-to-one computads using ordered face structures, in other words we shall describe in concrete terms the functor:

$$\overline{(-)} : \mathbf{Comma}^{m/1}_n \longrightarrow \mathbf{Comp}^{m/1}_n$$

More precisely, the many-to-one computads of dimension 1 (and all computads as well) are free computads over graphs and they are well understood. So suppose that $n > 1$, and we are given an object of $\mathbf{Comma}^{m/1}_n$, i.e. a quadruple $(|\mathcal{P}|_n, \mathcal{P}, d, c)$ such that

1. a many-to-one $(n-1)$-computad $\mathcal{P}$;
2. a set $|P|_n$ with two functions $c : |\mathcal{P}|_n \longrightarrow |\mathcal{P}|_{n-1}$ and $d : |\mathcal{P}|_n \longrightarrow \mathcal{P}_{n-1}$ such that for $x \in |\mathcal{P}|_n$, $cc(x) = cd(x)$ and $dc(x) = dd(x)$.

If the maps d and c in the object $(|\mathcal{P}|_n, \mathcal{P}, d, c)$ are understood from the context we can abbreviate notation to $(|\mathcal{P}|_n, \mathcal{P})$.

For an ordered face structure S, we denote by $S^{\sharp,n}$ the object $(S_n, (S_{<n})^*, [\delta], [\gamma])$ in $\mathbf{Comma}_n^{m/1}$. In fact, we have an obvious functor

$$(-)^{\sharp,n} : \mathbf{oFs}_{loc} \longrightarrow \mathbf{Comma}_n^{m/1}$$

such that

$$S \mapsto S^{\sharp,n} = (S_n, (S_{<n})^*, [\delta], [\gamma])$$

Any many-to-one computad $\mathcal{P}$ can be restricted to its part in $\mathbf{Comma}_n^{m/1}$. So we have an obvious forgetful functor

$$(-)^{\natural,n} : \mathbf{Comp}^{m/1} \longrightarrow \mathbf{Comma}_n^{m/1}$$

such that

$$\mathcal{P} \mapsto \mathcal{P}^{\natural,n} = (|\mathcal{P}|_n, \mathcal{P}_{<n}, d, c)$$

We shall describe the many-to-one n-computad $\overline{\mathcal{P}}$ whose $(n-1)$-truncation is $\mathcal{P}$ and whose n-indets are $|\mathcal{P}|_n$ with the domains and codomains given by c and d.

n-cells of $\overline{\mathcal{P}}$. An n-cell in $\overline{\mathcal{P}}_n$ is a(n equivalence class of) pair(s) (S, f) where

1. S is an ordered face structure, $dim(S) \leq n$;
2. $f : S^{\sharp,n} \longrightarrow \mathcal{P}^{\natural,n}$ is a morphism in $\mathbf{Comma}_n^{m/1}$, i.e.

$$\begin{tikzcd} S_n \arrow[r, "|f|_n"] \arrow[d, shift right, "{[\delta]}"'] \arrow[d, shift left, "{[\gamma]}"] & {|\mathcal{P}|_n} \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] \\ S^*_{n-1} \arrow[r, "f_{n-1}"'] & \mathcal{P}_{n-1} \end{tikzcd}$$

   commutes.

We identify two pairs (S, f), (S', f') if there is a monotone isomorphism $h : S \longrightarrow S'$ such that the triangles of sets and of $(n-1)$-computads

$$\begin{tikzcd} S_n \arrow[rr, "h_n"] \arrow[dr, "f_n"'] & & S'_n \arrow[dl, "f'_n"] \\ & {|\mathcal{P}|_n} & \end{tikzcd} \qquad \begin{tikzcd} (S_{<n})^* \arrow[rr, "(h_{<n})^*"] \arrow[dr, "f_{<n}"'] & & (S'_{<n})^* \arrow[dl, "f'_{<n}"] \\ & \mathcal{P} & \end{tikzcd}$$

commute. Clearly, such an h, if exists, is unique. Even if formally cells in $\mathcal{P}_n$ are equivalence classes of triples we will work on triples themselves as if they were cells understanding that equality between such cells is an isomorphism in the sense defined above.

Domains and codomains in $\overline{\mathcal{P}}$. The domain and codomain functions

$$d^{(k)}, c^{(k)} : \overline{\mathcal{P}}_n \longrightarrow \overline{\mathcal{P}}_k$$

are defined for an n-cell (S, f) as follows:

$$d^{(k)}(S, f) = \begin{cases} (S, f) & \text{if } dim(S) \leq k, \\ (\mathbf{d}^{(k)}S, d^{(k)}f) & \text{otherwise.} \end{cases}$$

where, for $x \in (\mathbf{d}^{(k)}S)_k$ (and hence $\nu_x : [x] \to \mathbf{d}^{(k)}S$),

$$(d^{(k)}f)_k(x) = f_k(\nu_x)(x)$$

(i.e. we take the cell $\nu_x : [x] \to \mathbf{d}^{(k)}S$ of S^*, then value of f on it, and then we evaluate the map in $\mathbf{Comma}^{m/1}_n$ on x the only element of $[x]_k$), and

$$(\mathbf{d}^{(k)}f)_{<k} = (\mathbf{d}^k_S; f_{<n})_{<k}.$$

$$c^{(k)}(S,f) = \begin{cases} (S,f) & \text{if } dim(S) \leq k, \\ (\mathbf{c}^{(k)}S, c^{(k)}f) & \text{otherwise.} \end{cases}$$

where, for $x \in (\mathbf{c}^{(k)}S)_k$ (and hence $\nu_x : [x] \to \mathbf{c}^{(k)}S$),

$$(c^{(k)}f)_k(x) = f_k(\nu_x)(x)$$

and

$$(c^{(k)}f)_{<k} = (\mathbf{c}^k_S; f_{<n})_{<k}.$$

i.e. we calculate the k-th domain and k-th codomain of an n-cell (S,f) by taking $\mathbf{d}^{(k)}$ and $\mathbf{c}^{(k)}$ of the domain S of the cell f, respectively, and by restricting the maps f accordingly.

Identities in $\overline{\mathcal{P}}$. The identity function

$$\mathbf{i} : \overline{\mathcal{P}}_{n-1} \longrightarrow \overline{\mathcal{P}}_n$$

is defined for an $(n-1)$-cell $((S,f)$ in $\mathcal{P}_{n-1}$, as follows:

$$\mathbf{i}(S,f) = \begin{cases} (S,f) & \text{if } dim(S) < n-1, \\ (S,\overline{f}) & \text{if dim(S)=n-1} \end{cases}$$

Note that $\overline{f}$ is the map $\mathbf{Comp}^{m/1}_{n-1}$ which is the value of the functor $\overline{(-)}$ on a map f from $\mathbf{Comma}^{m/1}_{n-1}$. So it is in fact defined as 'the same $(n-1)$-cell' but considered as an n-cell.

Compositions in $\overline{\mathcal{P}}$. Suppose that (S^i, f^i) are n-cells for $i = 0, 1$, such that

$$c^{(k)}(S^0, f^0) = d^{(k)}(S^1, f^1).$$

Then their k-composition in $\mathbf{Comma}^{m/1}_n$ is defined as

$$(S^0, f^0);_k (S^1, f^1) = (S^0 \otimes_k S^1, [f^0, f^1])$$

i.e.

$$\begin{tikzcd} (S^0 \otimes_k S^1)_n \arrow[rr, "{[f^0_n, f^1_n]}"] \arrow[d, shift right, "{[\delta]}"'] \arrow[d, shift left, "{[\gamma]}"] & & {|\mathcal{P}|_n} \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] \\ (S^0 \otimes_k S^1)^*_{n-1} \arrow[rr, "{[f^0_{n-1}, f^1_{n-1}]}"'] & & \mathcal{P}_{n-1} \end{tikzcd}$$

This ends the description of the computad $\overline{\mathcal{P}}$.

Now let $h : \mathcal{P} \to \mathcal{Q}$ be a morphism in $\mathbf{Comma}^{m/1}_n$, i.e. a function $h_n : |\mathcal{P}|_n \longrightarrow |\mathcal{Q}|_n$ and a $(n-1)$-computad morphism $h_{<n} : \mathcal{P}_{<n} \longrightarrow \mathcal{Q}_{<n}$ such that the square

$$\begin{tikzcd} {|\mathcal{P}|_n} \arrow[r, "h_n"] \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] & {|Q|_n} \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] \\ \mathcal{P}_{n-1} \arrow[r, "h_{n-1}"'] & \mathcal{Q}_{n-1} \end{tikzcd}$$

commutes serially. We define

$$\overline{h} : \overline{\mathcal{P}} \longrightarrow \overline{Q}$$

by putting $\overline{h}_k = h_k$, for $k < n$, and for $(S, f) \in \overline{\mathcal{P}}_n$, we put

$$\overline{h}(S, f) = (S, h \circ f).$$

Embedding $\eta_{\mathcal{P}} : |\mathcal{P}|_n \longrightarrow \overline{\mathcal{P}}_n$ is defined in the Proposition below.

Notation. Let $x = (X, f : X^{\sharp,n} \to \mathcal{P}^{\natural,n})$ be a cell in $\overline{\mathcal{P}}_n$ as above, and $\check{a} \in Sd(X)_k$. Then by $x^{\uparrow\check{a}} = (X^{\uparrow\check{a}}, f^{\uparrow\check{a}})$ and $x^{\downarrow\check{a}} = (X^{\downarrow\check{a}}, f^{\downarrow\check{a}})$ we denote the cells in $\overline{\mathcal{P}}_n$ that are the obvious restrictions of x. Clearly, we have $c^{(k)}(x^{\uparrow\check{a}}) = d^{(k)}(x^{\downarrow\check{a}})$ and that $x = x^{\uparrow\check{a}};_k x^{\downarrow\check{a}}$, where $k = dim(a)$.

The following Proposition contains several statements. We have put all of the together since they have to be proved simultaneously, i.e. to prove them for n we need to know all of them for $n-1$.

Proposition 15.1 *Let $n \in \omega$. We have*

1. *Let $\mathcal{P}$ be an object of* $\mathbf{Comma}_n^{m/1}$. *We define the function*

$$\eta_{\mathcal{P}} : |\mathcal{P}|_n \longrightarrow \overline{\mathcal{P}}_n$$

as follows. Let $x \in |\mathcal{P}|_n$. As $c(x)$ is an indet $d(x)$ is a normal cell of dimension $n-1$. Thus there is a unique descriptions of the cells $d(x)$ and $c(x)$

$$< T_{d(x)}, \tau_{d(x)} : T^*_{d(x)} \longrightarrow \mathcal{P}_{<n} >, \quad < T_{c(x)}, \tau_{c(x)} : T^*_{c(x)} \longrightarrow \mathcal{P}_{<n} >$$

with $T_{d(x)}$ being $(n-1)$-normal ordered face structure and $T_{c(x)}$ being principal ordered face structure of dimension $n-1$. Then we have a unique n-cell in $\overline{\mathcal{P}}$:

$$\bar{x} =< T^\bullet_{d(x)},\ \ |\overline{\tau}_x|_n : \{1_{T^\bullet_{d(x)}}\} \to |\mathcal{P}|_n,\ \ (\overline{\tau}_x)_{<n} : (T^\bullet_{d(x)})^*_{<n} \to \mathcal{P}_{<n} >$$

(note: $|T^\bullet_{d(x)}|_n = \{1_{T^\bullet_{d(x)}}\}$) such that

$$|\overline{\tau}_x|_n(1_{T^\bullet_{d(x)}}) = x$$

and, for $y : Y \to T^\bullet_{d(x)} \in (T^\bullet_{d(x)})^*_{<n}$

$$(\overline{\tau}_x)_{n-1}(y) = \begin{cases} (\tau_{c(x)})_{n-1}(y') & \text{if } Y \text{ is principal} \\ & \quad \text{and } y = y'; \mathbf{c}_{(T^\bullet_{d(x)})}, \\ (\tau_{d(x)})_{n-1}(y'') & \text{if } Y \text{ is principal} \\ & \quad \text{and } y = y''; \mathbf{d}_{(T^\bullet_{d(x)})}, \\ (\overline{\tau}_x)_{n-1}(y^{\downarrow\check{a}});_k (\overline{\tau}_x)_{n-1}(y^{\uparrow\check{a}}) & \text{if } \check{a} \in Sd(Y)_k \end{cases}$$

and $(\overline{\tau}_x)_{<(n-1)} = (\tau_{dx})_{<(n-1)}$. We put $\eta_{\mathcal{P}}(x) = \bar{x}$.

Then $\overline{\mathcal{P}}$ is a many-to-one computad with $\eta_{\mathcal{P}}$ the inclusion of n-indeterminates. Moreover, any many-to-one n-computad Q is equivalent to a computad $\overline{\mathcal{P}}$, for some $\mathcal{P}$ in $\mathbf{Comma}_n^{m/1}$.

2. *Let $\mathcal{P}$ be an object of* $\mathbf{Comma}_n^{m/1}$, *$! : \overline{\mathcal{P}} \longrightarrow \mathcal{T}$ the unique morphism into the terminal object $\mathcal{T}$ and $f : S^{\sharp,n} \to \mathcal{P}$ a cell in $\overline{\mathcal{P}}_n$. Then*

$$!_n(f : S^{\sharp,n} \to \mathcal{P}) = S.$$

3. *Let $h : \mathcal{P} \to \mathcal{Q}$ be an object of* $\mathbf{Comma}_n^{m/1}$. *Then $\overline{h} : \overline{\mathcal{P}} \longrightarrow \overline{\mathcal{Q}}$ is a computad morphism.*

4. *Let $k \leq n$, S be an ordered face structure of dimension at most n, $f : S^* \longrightarrow \mathcal{P}$ a morphism in $\mathbf{Comp}_n^{m/1}$ and $y : Y \to S \in S^*_k$. We have that*

$$\overline{f}_k(y) = (f \circ y^*)^{\natural,k} (= f^{\natural,k} \circ y^{\sharp,k} : Y^{\sharp,k} \longrightarrow \mathcal{P}^{\natural,k}).$$

5. *Let S be an ordered face structure of dimension n, $\mathcal{P}$ many-to-one computad, $g, h : S^* \longrightarrow \mathcal{P}$ computad maps. Then*

$$g = h \qquad \text{iff} \qquad g_n(1_S) = h_n(1_S).$$

6. *Let S be an ordered face structure of dimension at most n, $\mathcal{P}$ be an object in $\mathbf{Comma}_n^{m/1}$. Then we have a bijective correspondence*

$$\frac{f : S^{\sharp,n} \longrightarrow \mathcal{P} \quad \in \mathbf{Comma}_n^{m/1}}{\overline{f} : S^* \longrightarrow \overline{\mathcal{P}} \quad \in \mathbf{Comp}_n^{m/1}}$$

such that, $\overline{f}_n(1_S) = f$, and for $g : S^* \longrightarrow \overline{\mathcal{P}}$ we have $g = \overline{g_n(1_S)}$.

7. *The map*

$$\kappa_n^{\mathcal{P}} : \coprod_S \mathbf{Comp}(S^*, \overline{\mathcal{P}}) \longrightarrow \overline{\mathcal{P}}_n$$

$$g : S^* \to \overline{\mathcal{P}} \quad \mapsto \quad g_n(1_S),$$

where coproduct is taken over all (up to iso) ordered face structures S of dimension at most n, is a bijection. In other words, any cell in $\overline{\mathcal{P}}$ has a unique description.

Proof. We prove all the statements simultaneously by induction on n. For $n = 0, 1$ all of them are easy.

Ad 1. We have to verify that $\overline{\mathcal{P}}$ satisfy the laws of ω-categories and that it is free in the appropriate sense. Laws ω-categories are left for the reader. We shall show that $\overline{\mathcal{P}}$ is free in the appropriate sense.

Let C be an ω-category, $g_{<n} : \mathcal{P}_{<n} \to C_{<n}$ an $(n-1)$-functor and $g_n : |\mathcal{P}|_n \to C_n$ a function so that the diagram

$$\begin{tikzcd}
{|\mathcal{P}|_n} \arrow[rr, "g_n"] \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] & & C_n \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] \\
\mathcal{P}_{n-1} \arrow[rr, "g_{n-1}"'] & & C_{n-1}
\end{tikzcd}$$

commutes serially. We shall define an n-functor $\overline{g} : \overline{\mathcal{P}} \to C$ extending g and g_n. For $x = (X, f) \in \overline{\mathcal{P}}_n$ we put

$$\overline{g}_n(x) = \begin{cases} 1_{g_{n-1} \circ f_{n-1}(x)} & \text{if } dim(X) < n, \\ g_n \circ f_n(m_X) & \text{if } dim(X) = n, \ X \text{ is principal}, \ X_n = \{m_X\} \\ \overline{g}_n(x^{\uparrow \check{a}});_k \overline{g}_n(x^{\downarrow \check{a}}) & \text{if } dim(X) = n, \ \check{a} \in Sd(S)_k \end{cases}$$

We need to check that $\overline{g}$ is well defined, unique one that extends g, preserves domains, codomains, compositions and identities.

All these calculations are similar, and they are very much like those in the proof of Proposition 13.1. We shall check, assuming that we already know that $\overline{g}$ is well defined, and preserves identities that compositions are preserved. So let $x = (X, f)$, $x_1 = (X_1, f_1)$, $x_2 = (X_2, f_2)$ be cells in $\overline{\mathcal{P}}_n$ such that $x = x_1 ;_k x_2$. Since $\overline{g}$ preserves identities, we can assume that $dim(X_1), dim(X_1) > k$. Let $l \in \omega$ be minimal such that $Sd(X)_l \neq \emptyset$. We have two cases:

Case 1. If $l < k$, then by Decomposition 3.2.a we have $\check{a} \in Sd(T_2)_l$, and then

$$\begin{array}{r} \overline{g}(x) = \\ \overline{g}(x^{\uparrow\check{a}});_l \overline{g}(x^{\downarrow\check{a}}) = \\ \overline{g}((x_1;_k x_2)^{\uparrow\check{a}});_l \overline{g}((x_1;_k x_2)^{\downarrow\check{a}}) = \\ \overline{g}(x_1^{\uparrow\check{a}};_k x_2^{\uparrow\check{a}});_l \overline{g}(x_1^{\downarrow\check{a}};_k x_2^{\downarrow\check{a}}) = \\ (\overline{g}(x_1^{\uparrow\check{a}});_k \overline{g}(x_2^{\uparrow\check{a}}));_l (\overline{g}(x_1^{\downarrow\check{a}});_k \overline{g}(x_2^{\downarrow\check{a}})) = \\ (\overline{g}(x_1^{\uparrow\check{a}});_l \overline{g}(x_1^{\downarrow\check{a}}));_k (\overline{g}(x_2^{\uparrow\check{a}});_l \overline{g}(x_2^{\downarrow\check{a}})) = \\ = \overline{g}(x_1);_k \overline{g}(x_2) \end{array}$$

Case 2. If $l = k$ then by Decomposition 3.2.a we have $\check{a} \in Sd(X_1)$ and

$$\begin{array}{r} \overline{g}(x) = \\ \overline{g}(x^{\uparrow\check{a}});_k \overline{g}(x^{\downarrow\check{a}}) = \\ \overline{g}(x_1^{\uparrow\check{a}});_k \overline{g}(x_1^{\downarrow\check{a}};_k x_2) = \\ \overline{g}(x_1^{\uparrow\check{a}});_k (\overline{g}(x_1^{\downarrow\check{a}});_k \overline{g}(x_2)) = \\ (\overline{g}(x_1^{\uparrow\check{a}});_k \overline{g}(x_1^{\downarrow\check{a}}));_k \overline{g}(x_2) = \\ = \overline{g}(x_1);_k \overline{g}(x_2) \end{array}$$

The remaining things are similar.

Ad 2. Let $! : \overline{\mathcal{P}} \longrightarrow \mathcal{T}$ be the unique computad map into the terminal object, S an ordered face structure such that $dim(S) = l \leq n$, $f : S^{\sharp,n} \longrightarrow \mathcal{P}$ a cell in $\overline{\mathcal{P}}_n$.

If $l < n$ then by induction we have $!_n(f) = S$. If $l = n$ and S is principal then we have, by induction

$$!_n(d(f) : (\mathbf{d}S)^{\sharp,n} \to \mathcal{P}) = \mathbf{d}S, \qquad !_n(c(f) : (\mathbf{c}S)^{\sharp,n} \to \mathcal{P}) = \mathbf{c}S.$$

As f is an indet in $\overline{\mathcal{P}}$, $!_n(f)$ is a principal ordered face structure. But the only (up to an iso) principal ordered face structure B such that

$$\mathbf{d}B = \mathbf{d}S, \qquad \mathbf{d}B = \mathbf{d}S$$

is S itself. Thus, in this case, $!_n(f) = S$.

Now assume that $l = n$, and S is not principal, and that for ordered face structures T of smaller size than S the statement holds. Let $a \in Sd(S)_k$. We have

$$!_n(f) = !_n(f^{\uparrow a};_k f^{\downarrow a}) = !_n(f^{\uparrow a});_k !_n(f^{\downarrow a}) = S^{\uparrow a};_k S^{\downarrow a} = S$$

where $f^{\uparrow a} = f \circ (\kappa^{\uparrow a})^{\sharp,n}$ and $f^{\downarrow a} = f \circ (\kappa^{\downarrow a})^{\sharp,n}$ and $\kappa^{\uparrow a}$ and $\kappa^{\downarrow a}$ are the monotone morphisms as in the following tensor square

$$\begin{tikzcd} S^{\uparrow a} \arrow[r, "\kappa^{\uparrow a}"] & S \\ \mathbf{c}^{(k)}S \arrow[u] \arrow[r] & S^{\downarrow a} \arrow[u, "\kappa^{\downarrow a}"'] \end{tikzcd}$$

Ad 3. The main thing is to show that $\overline{h}$ preserves compositions. This follows from the fact that the functor

$$(-)^{\sharp,n} : \mathbf{oFs} \longrightarrow \mathbf{Comma}_n^{m/1}$$

preserves special pushouts.

Ad 4. This is an immediate consequence of 3.

Ad 5. Let us fix ordered face structures S, Y, $dim(S) = n$, $\check{a} \in Sd(Y)$, and $f, g : S^* \longrightarrow \mathcal{P}$. Clearly, if $f = g$ then $f(1_S) = g(1_S)$. We shall prove the converse. As

$$\begin{tikzcd}
(Y^{\downarrow\check{a}})^{\sharp,k} \arrow[rr] & & Y^{\sharp,k} \\
(\mathbf{c}^{(l)}Y^{\downarrow\check{a}})^{\sharp,k} \arrow[u] \arrow[rr] & & (Y^{\uparrow\check{a}})^{\sharp,k} \arrow[u]
\end{tikzcd}$$

is a pushout in $\mathbf{Comma}^{m/1}_k$ we have that for any $y : Y \to S \in S^*_k$

$$f^{\natural,k} \circ y^{\sharp,k} = g^{\natural,k} \circ y^{\sharp,k} \text{ iff } f^{\natural,k} \circ (y^{\downarrow\check{a}})^{\sharp,k} = g^{\natural,k} \circ (y^{\downarrow\check{a}})^{\sharp,k} \text{ and } f^{\natural,k} \circ (y^{\uparrow\check{a}})^{\sharp,k} = g^{\natural,k} \circ (y^{\uparrow\check{a}})^{\sharp,k}$$

From this observation it is easy to see that if for some $y : Y \to S \in S^*$ we have $f(y) \neq g(y)$ then we can assume that this Y is principal. On the other hand, from the above observation, the fact that both f and g are ω-functors and that $f(1_S) = g(1_S)$ we can deduce that for any $y : Y \to S \in S^*$ with Y principal we have $f(y) = g(y)$. This together shows 5.

Ad 6. we shall use 5. Fix an ordered face structure S of dimension n and a many-to-one computad $\mathcal{P}$. For $f : S^{\sharp,n} \to \mathcal{P}^{\natural,n}$ in $\mathbf{Comma}^{m/1}_n$ we have

$$\overline{f}_n(1_S) = (f \circ (1_S)^{\sharp,n})^{\natural,n} = f \circ (1_S)^{\sharp,k} = f.$$

On the other hand, for a computad map $g : S^* \to P$ we have

$$\overline{g_n(1_S)}(1_S) = (g_n(1_S) \circ (1_S)^{\sharp,n})^{\natural,n} =$$

$$= (g^{\natural,n} \circ (1_S)^{\sharp,n} \circ (1_S)^{\sharp,n})^{\natural,n} = (g^{\natural,n} \circ (1_S)^{\sharp,n})^{\natural,n} = g(1_S).$$

Thus by 5. we have $\overline{g_n(1_S)} = g$.

Ad 7. It follows immediately from 6. $\Box$

The following Proposition says a bit more about descriptions than point 7. of the previous one.

Proposition 15.2 *Let $\mathcal{P}$ be a many-to-one computad, $n \in \omega$, and $a \in \mathcal{P}_n$. Let $T_a = !^{\mathcal{P}}_n(a)$ (where $!^{\mathcal{P}} : \mathcal{P} \longrightarrow \mathcal{T}$ is the unique morphism into the terminal many-to-one computad). Then there is a unique computad map $\tau_a : T^*_a \longrightarrow \mathcal{P}$ such that $(\tau_a)_n(1_{T_a}) = a$. Moreover, we have:*

1. *for any $a \in \mathcal{P}$ we have*

$$\tau_{da} = d(\tau_a) = \tau_{da} = \tau_a \circ (\mathbf{d}_{T_a})^*, \qquad \tau_{c(a)} = c(\tau_a) = \tau_{c(a)} = \tau_a \circ (\mathbf{c}_{T_a})^*,$$

$$\tau_{1_a} = \tau_a$$

2. *for any $a, b \in \mathcal{P}$ such that $c^{(k)}(a) = d^{(k)}(b)$ we have*

$$\tau_{a;_k b} = [\tau_a, \tau_b] : T^*_a ;_k T^*_b \longrightarrow \mathcal{P},$$

3. *for any ordered face structure S, for any computad map $f : S^* \longrightarrow \mathcal{P}$,*

$$\overline{\tau_{f_n(1_S)}} = f.$$

4. *for any ordered face structure S, any ω-functor $f : S^* \longrightarrow \mathcal{P}$ can be essentially uniquely factorized as*

$$
\begin{tikzcd}
S^* \arrow[rr, "f"] \arrow[dr, "f^{in}"'] & & \mathcal{P} \\
& T^*_{f(1_S)} \arrow[ur, "\tau_{f(1_S)}"'] &
\end{tikzcd}
$$

where f^{in} is an inner map (i.e. $f^{in}(1_S) = 1_{T_{f(1_S)}}$) and $(\tau_{f(1_S)}, T_{f(1_S)})$ is the description of the cell $f(1_S)$.

Proof. Using the above description of the many-to-one computad $\mathcal{P}$ we have that $a : (T_a)^{\sharp,n} \longrightarrow \mathcal{P}^{\natural,n}$. We put $\tau_a = \overline{a}$. By Proposition 15.1 point 6, we have that $(\tau_a)_n(1_{T_a}) = \overline{a}_n(1_{T_a}) = a$, as required. The uniqueness of (T_a, τ_a) follows from Proposition 15.1 point 5. The remaining part is left for the reader. $\Box$

16 Appendix

A definition of the many-to-one computads and the comma categories

The notion of a computad was introduced by Ross Street. We repeat this definition for a subcategory $\mathbf{Comp}^{m/1}$ of the category of all computads $\mathbf{Comp}$ that have indeterminates of a special shape, namely their codomains are again indeterminates. We use this opportunity to introduce the notation used in the paper. In order to define $\mathbf{Comp}^{m/1}$ we define three sequences of categories $\mathbf{Comp}^{m/1}_n$, $\mathbf{Comma}^{m/1}_n$, and $\mathbf{Comma}_n$.

1. For $n = 0$, the categories $\mathbf{Comp}^{m/1}_n$, $\mathbf{Comma}^{m/1}_n$, and $\mathbf{Comma}_n$ are just Set, and the functor $\overline{(-)}^n : \mathbf{Comma}^{m/1}_n \longrightarrow \mathbf{Comp}^{m/1}_n$ is the identity.

2. For $n = 1$, the categories $\mathbf{Comma}^{m/1}_n$ and $\mathbf{Comma}_n$ are the category of graphs (i.e. 1-graphs) and $\mathbf{Comp}^{m/1}_n$ is the category of free ω-categories over graphs with morphisms being the functors sending indets (=indeterminates=generators) to indets.

3. Let $n \geq 1$. We define the following functor

$$\varpi^{m/1}_n : \mathbf{Comp}^{m/1}_n \longrightarrow Set$$

such that

$$\varpi^{m/1}_n(\mathcal{P}) = \{(a,b) :\ a \in \mathcal{P}_n,\ b \in |\mathcal{P}|_n,\ d(a) = d(b),\ c(a) = c(b)\}$$

i.e. $\varpi^{m/1}_n(\mathcal{P})$ consists of those parallel pairs (a,b) of n-cells of $\mathcal{P}$ such that b is an indet. On morphisms ϖ_n is defined in the obvious way. We define $\mathbf{Comma}^{m/1}_{n+1}$ to be equal to the comma category $Set \downarrow \varpi^{m/1}_n$. So we have a diagram

$$\begin{tikzcd} & \mathbf{Comma}^{m/1}_{n+1} \arrow[dl, "(-)_{\leq n}"'] \arrow[dr, "|-|_{n+1}"] & \\ \mathbf{Comp}^{m/1}_n \arrow[rr, "\varpi^{m/1}_n"'] \arrow[rr, phantom, shift left=4, "\mu \Downarrow"] & & Set \end{tikzcd}$$

4. For $n \geq 1$, we can define also a functor

$$\varpi_n : n\mathbf{Cat} \longrightarrow Set$$

such that

$$\varpi_n(C) = \{(a,b) : a, b \in C_n,\ d(a) = d(b),\ c(a) = c(b)\}$$

i.e. $\varpi_n(C)$ consists of all parallel pairs (a,b) of n-cells of the n-category C. We define $\mathbf{Comma}_{n+1}$ to be equal to the comma category $Set \downarrow \varpi_n$. We often denote objects of $\mathbf{Comma}_{n+1}$ as quadruples $C = (|C|_{n+1}, C_{\leq n}, d, c)$, where $C_{\leq n}$ is an n-category, $|C|_{n+1}$ is a set and $(d,c) : |C|_{n+1} \longrightarrow \varpi_n(C_{\leq n})$ is a function. Clearly, the category $\mathbf{Comma}^{m/1}_{n+1}$ is a full subcategory of $\mathbf{Comma}_{n+1}$, moreover we have a forgetful functor

$$\mathcal{U}_{n+1} : (n+1)\mathbf{Cat} \longrightarrow \mathbf{Comma}_{n+1}$$

such that for an $(n+1)$-category A

$$\mathcal{U}_{n+1}(A) = (A_{n+1}, A_{\leq n}, d, c)$$

i.e. $\mathcal{U}_{n+1}$ forgets the structure of compositions and identities at the top level. This functor has a left adjoint

$$\mathcal{F}_{n+1} : \mathbf{Comma}_{n+1} \longrightarrow (n+1)\mathbf{Cat}$$

The category $\mathcal{F}_{n+1}(|B|_{n+1}, B, d, c)$ is said to be a *free extension* of the n-category B by the indets $|B|_{n+1}$. The category of many-to-one $(n+1)$-computads $\mathbf{Comp}^{m/1}_{n+1}$ is a subcategory of $(n+1)\mathbf{Cat}$ whose objects are free extensions of objects from $\mathbf{Comma}^{m/1}_{n+1}$. The morphisms in $\mathbf{Comp}^{m/1}_{n+1}$ are $(n+1)$-functors that sends indets to indets. Thus the functor $\mathcal{F}_{n+1}$ restricts to an equivalence of categories

$$\mathcal{F}^{m/1}_{n+1} : \mathbf{Comma}^{m/1}_{n+1} \longrightarrow \mathbf{Comp}^{m/1}_{n+1},$$

its essential inverse will be denoted by

$$\| - \|_{n+1} : \mathbf{Comp}^{m/1}_{n+1} \longrightarrow \mathbf{Comma}^{m/1}_{n+1}.$$

Thus for an $(n+1)$-computad $\mathcal{P}$ we have $\|\mathcal{P}\|_{n+1} = (|\mathcal{P}|_{n+1}, \mathcal{P}_{\leq n}, d, c)$.

5. The category $\mathbf{Comp}^{m/1}$ is the category of such ω-categories $\mathcal{P}$, that for every $n \in \omega$, $\mathcal{P}_{\leq n}$ is a many-to-one n-computad, and whose morphisms are ω-functors sending indets to indets.

   For $n \in \omega$, we have functors

   $$| - |_n : \mathbf{Comp}^{m/1} \longrightarrow Set$$

   associating to computads their n-indets, i.e.

   $$f : A \longrightarrow B \mapsto |f|_n : |A|_n \longrightarrow |B|_n,$$

   they all preserve colimits. Moreover we have a functor

   $$| - | : \mathbf{Comp}^{m/1} \longrightarrow Set$$

   associating to computads all their indets, i.e.

   $$f : A \longrightarrow B \mapsto |f| : |A| \longrightarrow |B|,$$

   where

   $$|A| = \coprod_{n\in\omega} |A|_n.$$

   It also preserves colimits and moreover it is is faithful.

6. We have a truncation functor

   $$(-)_{\leq n} : \omega Cat \longrightarrow n\mathbf{Cat}$$

   such that

   $$f : A \longrightarrow B \mapsto f_{\leq k} : A_{\leq k} \longrightarrow B_{\leq k}$$

   with $k \in \omega$, it preserves limits and colimits.

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