The Formal Theory of Monoidal Monads

Marek Zawadowski
Institute of Mathematics, University of Warsaw
ul. Banacha 2, 00-913 Warsaw, Poland
zawado@mimuw.edu.pl

December 2, 2010*

Abstract

We give a 3-categorical, purely formal argument explaining why on the category of Kleisli algebras for a lax monoidal monad, and dually on the category of Eilenberg-Moore algebras for an oplax monoidal monad, we always have a natural monoidal structures. The key observation is that the 2-category of lax monoidal monads in any 2-category D with finite products is isomorphic to the 2-category of monoidal objects with oplax morphisms in the 2-category of monads with lax morphisms in D. We explain at the end of the paper that a similar phenomenon occurs in many other situations.

1 Introduction

It is well known (cf. [Day] p. 30), that the category of Kleisli algebras for a monoidal monad carries a monoidal structure. Dually, the category of Eilenberg-Moore algebras for an opmonoidal monad carries a monoidal structure, as well. Theorem 7.2 of [Mo], considerably improved this result and then Theorem 2.9 of [McC] (cf. [Ch]) gives a still stronger formulation putting this result into 2-categorical context. Theorem 2.9 of [McC] says that the 2-category of monoidal categories, oplax morphisms, and monoidal natural transformations admits Eilenberg-Moore objects. The main goal of this paper is to put those considerations into 3-categorical context. We show that in fact any 2-category $\mathbf{Mon}_{op}(\mathcal{D})$ of monoidal objects, oplax 1-morphisms, and monoidal 2-cells constructed in any 2-category $\mathcal{D}$ with finite products and admitting Eilenberg-Moore objects, admits itself Eilenberg-Moore objects. As we are more interested in lax monoidal monads, we will be dealing with them and Kleisli objects. We will be also only pointing out what it implies in the dual case of oplax monoidal monads and Eilenberg-Moore objects. The proof of the main Theorem 4.1 is simple and purely formal based on the observation, Lemma 3.1, that the 2-categorical structures of monoidal objects and of monads commute, if taken with appropriate 1-cells. The name 'Formal Category Theory' for such kind of study was suggested by S. MacLane. It was first developed in [Gray] and later in many other places as in [St] for monads.

The author's main motivation for this paper is the study of structures like signatures, signatures with amalgamations, symmetric signatures, polynomial and analytic functors (cf. [Z]) and references there. Each of these structures carries a monoidal structure, giving rise to other algebraic structures via construction of the category of monoids. Many authors used such monoids as a tool to define the set of opetopes, the category of opetopes,

*This is the version of September 10, 2011 with corrections after the referee report.

and/or the category of opetopic sets see e.g. [BD], [HMP], [Le] [C] [KJBM], [Z], [SZ]. These notions are basic in the opetopic approach to the higher dimensional categories. Here we separate the case where the existence of monoidal structure on some of the structures mentioned above is simple for a very general reason.

Due to the fact that the symmetrization monad on multisorted signatures is not only monoidal, but it also has some additional properties, good monoidal structures exist not only on Kleisli categories like categories of signatures with amalgamation, and of polynomial functors, but also on the categories of Eilenberg-Moore algebras like categories of symmetric signatures and of analytic functors. The study of the monoidal monads with these additional properties is deferred to another paper.

In that sense this paper is meant to be a contribution to 2-category theory that will help to develop weak higher-dimensional category theory. Particulary, the goal is to understand better the relations between seemingly different approaches pursed by the mentioned authors.

The anonymous referee pointed out that this study can be also placed in a more general context involving monoidal objects in a monoidal bicategory and suggested references to such context, including [CLS].

The paper is organized as follows. For the sake of completeness, in Section 2, we describe in detail why the 2-categorical definition of the Kleisli objects (cf. [St]), gives all the data we expect and that it agrees with the usual Kleisli category when considered in 2-category of categories **Cat**. To appreciate the construction even more, we organize the data so constructed into various cells in 4-category **3CAT** of 3-categories, 3-functors, pseudo 3-natural transformations, pseudo 3-modifications, and perturbations. In particular, we show how real life situations may lead to perturbations. In Section 3, we spell the definition of a monoidal category in a 2-category with finite products of 0-cells. Moreover, we state key technical result (Lemma 3.1), explaining in what sense the monoidal and the monad structures commute. Using this fact, we prove, in Section 4, Theorem 4.1 concerning the existence of Kleisli objects in 2-categories of monoidal objects in 2-categories with finite products. We also present this result in an even more abstract form, Theorem 4.3, as a certain lifting property. In Section 5, we state these result in the dual case concerning oplax monoidal monads and Eilenberg-Moore objects. Finally, in Section 6, we show that such results also holds, if we replace monoidal objects by braided or symmetric monoidal objects or even by either monads or comonads, proviso we keep the 'laxness' of these structures opposite to the 'laxness' of the monads involved in the definition of either the Kleisli or the Eilenberg-Moore objects.

I would like to thank Stanisław Szawiel for the useful discussions.

2 The Kleisli and Eilenberg-Moore objects

The content of this section is well known, possibly with some minor exception. We spell the definitions in detail as we will be referring to them later.

In this section $\mathcal{D}$ is an arbitrary 2-category. Recall that a *monad* in $\mathcal{D}$ consists of an object $\mathcal{C}$ of $\mathcal{D}$, a 1-endocell $\mathcal{S} : \mathcal{C} \to \mathcal{C}$, two 2-cells $\eta : 1_{\mathcal{C}} \to \mathcal{S}$ and $\mu : \mathcal{S}^2 \to \mathcal{S}$ so that $\mu \circ \eta_{\mathcal{S}} = 1_{\mathcal{S}} = \mu \circ \mathcal{S}(\eta)$ and $\mu \circ (\mu_{\mathcal{S}}) = \mu \circ \mathcal{S}(\mu)$.

2.1 The Kleisli objects

An *oplax morphism of monads* is a pair $(F, \tau) : (\mathcal{C}, \mathcal{S}, \eta, \mu) \to (\mathcal{C}', \mathcal{S}', \eta', \mu')$ such that $F : \mathcal{C} \to \mathcal{C}'$ is a 1-cell and $\tau : F\mathcal{S} \to \mathcal{S}'F$ is a 2-cell so that the diagram

$$
\begin{array}{ccccc}
 & & F\mathcal{S} & \xleftarrow{F(\mu)} & F\mathcal{S}^2 \\
 & \overset{F(\eta)}{\nearrow} & & & \\
F & & \Big\downarrow \tau & & \Big\downarrow \mathcal{S}'(\tau)\circ\tau_{\mathcal{S}} \\
 & \underset{\eta'_F}{\searrow} & & & \\
 & & \mathcal{S}'F & \xleftarrow[\mu'_F]{} & \mathcal{S}'^2F
\end{array}
$$

commutes. The composition of two composable oplax morphisms of monads is given by $(F',\tau')\circ(F,\tau) = (F'\circ F, \tau'_F \circ F'(\tau))$. A *transformation* $\sigma : (F,\tau) \to (F',\tau')$ of two (parallel) oplax morphisms of monads is a 2-cell $\sigma : F \to F'$ making the square

$$
\begin{array}{ccc}
F\mathcal{S} & \xrightarrow{\sigma_{\mathcal{S}}} & F'\mathcal{S} \\
\tau\Big\downarrow & & \Big\downarrow\tau' \\
\mathcal{S}'F & \xrightarrow[S'(\sigma)]{} & \mathcal{S}'F'
\end{array}
$$

commute. This defines the 2-category $\mathbf{Mnd}_{op}(\mathcal{D})$ of monads in $\mathcal{D}$ with oplax morphisms and transformations of oplax morphisms. $\mathbf{Mnd}_{op}$ is a 3-endofunctor on the 3-category of 2-categories $2\mathbf{Cat}$. On 1- 2- 3-cells $\mathbf{Mnd}_{op}$ is defined in the obvious way. We have an embedding 2-functor $\iota_{op,\mathcal{D}} : \mathcal{D} \to \mathbf{Mnd}_{op}(\mathcal{D})$ sending an object $\mathcal{C}$ of $\mathcal{D}$ to the identity monad on $\mathcal{C}$. We often abbreviate $\iota_{op,\mathcal{D}}$ to ι_{op}. ι_{op} has always a right 2-adjoint $|-| = |-|_{\mathcal{D}}$ sending a monad to its underlying category. If ι_{op} has a left 2-adjoint $\mathcal{K} = \mathcal{K}_{\mathcal{D}}$ we say (cf. [St]), that $\mathcal{D}$ admits Kleisli objects.

$$
\mathbf{Mnd}_{op}(\mathcal{D}) \;\begin{array}{c} \xrightarrow{\mathcal{K}} \\ \xleftarrow[\iota_{op}]{} \\ \xrightarrow[|-|]{} \end{array}\; \mathcal{D}
$$

If $H : \mathcal{D} \to \mathcal{D}'$ is a 2-functor between two 2-categories that admit Kleisli objects, then we say that H *preserves Kleisli objects* if the canonical 2-natural transformation in the square

$$
\begin{array}{ccc}
\mathbf{Mnd}_{op}(\mathcal{D}) & \xrightarrow{\mathbf{Mnd}_{op}(H)} & \mathbf{Mnd}_{op}(\mathcal{D}') \\
\mathcal{K}_{\mathcal{D}}\Big\downarrow & & \Big\downarrow\mathcal{K}_{\mathcal{D}'} \\
\mathcal{D} & \xrightarrow[H]{} & \mathcal{D}'
\end{array}
$$

is a 2-natural isomorphism. If the forgetful 1-cell $|-| : \mathbf{Mnd}(\mathcal{D}) \to \mathcal{D}$ preserves Kleisli objects we say that $\mathbf{Mnd}(\mathcal{D})$ *has standard Kleisli objects.*

2.2 The Eilenberg-Moore objects

A *lax morphism of monads* is a pair $(F,\tau) : (\mathcal{C},\mathcal{S},\eta,\mu) \to (\mathcal{C}',\mathcal{S}',\eta',\mu')$ such that $F : \mathcal{C} \to \mathcal{C}'$ is a 1-cell and $\tau : \mathcal{S}'F \to F\mathcal{S}$ is a 2-cell so that the diagram

$$
\begin{array}{ccccc}
 & & \mathcal{S}'F & \xleftarrow{\mu'_F} & \mathcal{S}'^2F \\
 & \overset{\eta'_F}{\nearrow} & & & \\
F & & \Big\downarrow \tau & & \Big\downarrow \tau_{\mathcal{S}}\circ\mathcal{S}'(\tau) \\
 & \underset{F(\eta)}{\searrow} & & & \\
 & & F\mathcal{S} & \xleftarrow[F(\mu)]{} & F\mathcal{S}^2
\end{array}
$$

commutes. The composition of two composable lax morphisms of monads is given by $(F',\tau')\circ(F,\tau) = (F'\circ F, F'(\tau)\circ\tau'_F)$. A *transformation* $\sigma : (F,\tau) \to (F',\tau')$ of two (parallel) lax morphisms of monads is a 2-cell $\sigma : F \to F'$ making the square

$$\begin{array}{ccc} \mathcal{S}'F & \xrightarrow{S'(\sigma)} & \mathcal{S}'F' \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\tau'} \\ F\mathcal{S} & \xrightarrow[\sigma_{\mathcal{S}}]{} & F'\mathcal{S} \end{array}$$

commute. This defines the 2-category $\mathbf{Mnd}(\mathcal{D})$ of monads in $\mathcal{D}$ with lax morphisms and transformations of lax morphisms. **Mnd** is a 3-endofunctor on the 3-category of 2-categories $2\mathbf{Cat}$. We have an embedding 2-functor $\iota_{\mathcal{D}} : \mathcal{D} \to \mathbf{Mnd}(\mathcal{D})$ sending an object $\mathcal{C}$ of $\mathcal{D}$ to the identity monad on $\mathcal{C}$. We often abbreviate $\iota_{\mathcal{D}}$ to ι. It has always a left 2-adjoint $|-| = |-|_{\mathcal{D}}$ sending a monad to its underlying category. If ι has a right 2-adjoint $EM = EM_{\mathcal{D}}$ we say (cf. [St]), that $\mathcal{D}$ admits Eilenberg-Moore objects or EM objects.

$$\mathcal{D} \; \underset{EM}{\overset{|-|}{\leftleftarrows}} \!\!\xrightarrow[\iota]{}\; \mathbf{Mnd}(\mathcal{D})$$

The notions of preservation of EM objects and of standard EM objects are defined in the same way as in the case of Kleisli objects.

2.3 Some 3-categories and 3-functors

$2\mathbf{Cat}$ is the 3-category of 2-categories, i.e. with 2-categories as 0-cells, 2-functors as 1-cells, 2-natural transformations as 2-cells, and 2-modifications as 3-cells.

By a 2-category with finite products, we will always mean a 2-category with finite products of 0-cells. Let $2\mathbf{Cat}_\times$ be the sub-3-category of $2\mathbf{Cat}$ full on 2-transformations and 2-modifications, whose 0-cells are 2-categories with finite products, and 1-cells are 2-functors preserving finite products.

Let $2\mathbf{Cat}_k$ be the sub-3-category of $2\mathbf{Cat}$ full on 2-transformations and 2-modifications, whose 0-cells are 2-categories that admit Kleisli objects, and 1-cells are 2-functors preserving Kleisli objects.

Let $2\mathbf{Cat}_{em}$ be the sub-3-category of $2\mathbf{Cat}$ full on 2-transformations and 2-modifications, whose 0-cells are 2-categories that admit EM objects, and 1-cells are 2-functors preserving EM objects.

These properties can be combined together. For example $2\mathbf{Cat}_{kem\times}$ is the sub-3-category of $2\mathbf{Cat}$ full on 2-transformations and 2-modifications, that admit all the mentioned constructions. We require that Kleisli objects commute with finite products when they are both assumed to exist. Note that, as both EM objects and finite products are weighted limits they always commute.

As we already mentioned, we have 3-functors

$$\mathbf{Mnd}, \mathbf{Mnd}_{op} : 2\mathbf{Cat} \longrightarrow 2\mathbf{Cat}$$

and these 3-functors restrict to 3-functors on some sub-3-categories like $2\mathbf{Cat}_\times$, $2\mathbf{Cat}_{k\times}$ with the codomian restricted in a suitable way. Thus we also have for example

$$\mathbf{Mnd} : 2\mathbf{Cat}_\times \longrightarrow 2\mathbf{Cat}_\times \qquad \mathbf{Mnd}_{op} : 2\mathbf{Cat}_{k\times} \longrightarrow 2\mathbf{Cat}$$

To see this, note that in the 2-category $\mathcal{D}$ with finite products, the product of the monads $(C, \mathcal{S}, \eta, \mu)$ and $(C', \mathcal{S}', \eta', \mu')$ is, the monad $(C \times C', \mathcal{S} \times \mathcal{S}', (\eta, \eta'), (\mu, \mu'))$.

2.4 The 2-categorical description of the Kleisli objects

We describe below the above 3-categorical definition of the Kleisli objects in 2-categorical terms.

Thus we have 2-adjunctions $\mathcal{K} \dashv \iota_{op} \dashv |-|$. Let us fix a monad $(\mathcal{C}, \mathcal{S}, \eta, \mu)$ in $\mathcal{D}$. We will often abbreviate it to $\mathcal{S}$. The unit of the adjunction $\iota_{op} \dashv |-|$ on $\mathcal{C}$ is the identity $1_{\mathcal{C}} : \mathcal{C} \to |\iota_{op}(\mathcal{C})|$. The counit of this adjunction on $\mathcal{S}$ is $(1_{\mathcal{C}}, \eta) : \iota_{op}|\mathcal{S}| \to \mathcal{S}$.

The unit of the 2-adjunction $\mathcal{K} \dashv \iota_{op}$ on $\mathcal{S}$ is the morphism adjoint to $1_{\mathcal{K}(\mathcal{S})}$

$$\frac{\mathcal{K}(\mathcal{S}) \xrightarrow{\;1_{\mathcal{K}(\mathcal{S})}\;} \mathcal{K}(\mathcal{S})}{\mathcal{S} \xrightarrow[(F_{\mathcal{S}}, \kappa)]{} \iota_{op}\mathcal{K}(\mathcal{S}) = 1_{\mathcal{C}_{\mathcal{S}}}} \;\dashv$$

Thus $\mathcal{C}_{\mathcal{S}}$ is a 0-cell in $\mathcal{D}$, $F_{\mathcal{S}} : \mathcal{C} \to \mathcal{C}_{\mathcal{S}}$ is a 1-cell in $\mathcal{D}$, and $\kappa : F_{\mathcal{S}} \circ \mathcal{S} \to F_{\mathcal{S}}$ is a 2-cell in $\mathcal{D}$ so that in the diagram

$$F_{\mathcal{S}} \circ \mathcal{S}^2 \underset{F_{\mathcal{S}}(\mu)}{\overset{\kappa_{\mathcal{S}}}{\rightrightarrows}} F_{\mathcal{S}} \circ \mathcal{S} \underset{F_{\mathcal{S}}(\eta)}{\overset{\kappa}{\rightleftarrows}} F_{\mathcal{S}}$$

we have $\kappa \circ F_{\mathcal{S}} = 1_{F_{\mathcal{S}}}$ and $\kappa \circ F_{\mathcal{S}}(\mu) = \kappa \circ (\kappa_{F_{\mathcal{S}}})$. In such circumstances we say that $(F_{\mathcal{S}}, \kappa)$ *subcoequalizes* the monad $\mathcal{S}$. The counit of this adjunction on $\mathcal{C}$ is $1_{\mathcal{C}} : \mathcal{C} = \mathcal{K}\iota_{op}(\mathcal{C}) \to \mathcal{C}$.

One can check directly that $(\mathcal{S}, \mu) : \mathcal{S} \to \iota_{op}(|\mathcal{S}|) = 1_{\mathcal{C}}$ is an oplax morphism of monads. By adjunction

$$\frac{\mathcal{S} \xrightarrow{\;(\mathcal{S}, \mu)\;} \iota_{op}|\mathcal{S}|}{\mathcal{K}(\mathcal{S}) \xrightarrow[U_{\mathcal{S}}]{} |\mathcal{S}|} \;\dashv$$

we get the 1-cell $U_{\mathcal{S}}$. Using twice the adjunction $\mathcal{K} \dashv \iota_{op}$ we obtain

$$\begin{array}{c}
\mathcal{S} \xrightarrow{\;(F_{\mathcal{S}}, \kappa)\;} \iota_{op}\mathcal{K}(\mathcal{S}) \xrightarrow{\;\iota_{op}(U_{\mathcal{S}})\;} \iota_{op}|\mathcal{S}| \\
\hline
\mathcal{K}(\mathcal{S}) \xrightarrow{\;1_{\mathcal{K}(\mathcal{S})}\;} \mathcal{K}(\mathcal{S}) \xrightarrow{\;U_{\mathcal{S}}\;} |\mathcal{S}| \\
\hline
\mathcal{S} \xrightarrow{\;(\mathcal{S}, \mu)\;} \iota_{op}|\mathcal{S}|
\end{array}
\begin{array}{c} \\ \dashv \\ \\ \dashv \\ \\ \end{array}$$

and by the uniqueness of adjoints, we get $(\mathcal{S}, \mu) = \iota_{op}(U_{\mathcal{S}}) \circ (F_{\mathcal{S}}, \kappa) = (U_{\mathcal{S}} F_{\mathcal{S}}, U_{\mathcal{S}}(\kappa))$.

We shall explain that $F_{\mathcal{S}} \dashv U_{\mathcal{S}}$ in $\mathcal{D}$ as 1-cells in $\mathcal{D}$. The unit of the adjunction $F_{\mathcal{S}} \dashv U_{\mathcal{S}}$ in $\mathcal{D}$ is η. In order to define ε, the counit of this adjunction, we proceed as follows. First note that we have equalities of oplax morphism of monads from $\mathcal{S}$ to $\iota_{op}\mathcal{K}(\mathcal{S}) = 1_{\mathcal{C}_{\mathcal{S}}}$:

$$(F_{\mathcal{S}} \circ \mathcal{S}, F_{\mathcal{S}}(\mu)) = \iota_{op}(|F_{\mathcal{S}}, \kappa|) \circ (\mathcal{S}, \mu) = \iota_{op}(F_{\mathcal{S}}) \circ \iota_{op}(U_{\mathcal{S}}) \circ (F_{\mathcal{S}}, \kappa)$$

Note that the codomains of the morphisms are correct as $\iota_{op}|\iota_{op}\mathcal{K}(\mathcal{S})| = \iota_{op}\mathcal{K}(\mathcal{S})$. The above morphism is parallel to $(\mathcal{S}, \mu)$. Since $\kappa \circ F_{\mathcal{S}}(\mu) = \kappa \circ \kappa_{\mathcal{S}}$ it follows that

$$\kappa : (F_{\mathcal{S}} \circ \mathcal{S}, F_{\mathcal{S}}(\mu)) \to (F_{\mathcal{S}}, \kappa)$$

is a transformation of oplax morphisms of monads, i.e. a 2-cell in $\mathbf{Mnd}_{op}(\mathcal{D})$. The adjoint correspondence of the 2-cells below defines the counit ε:

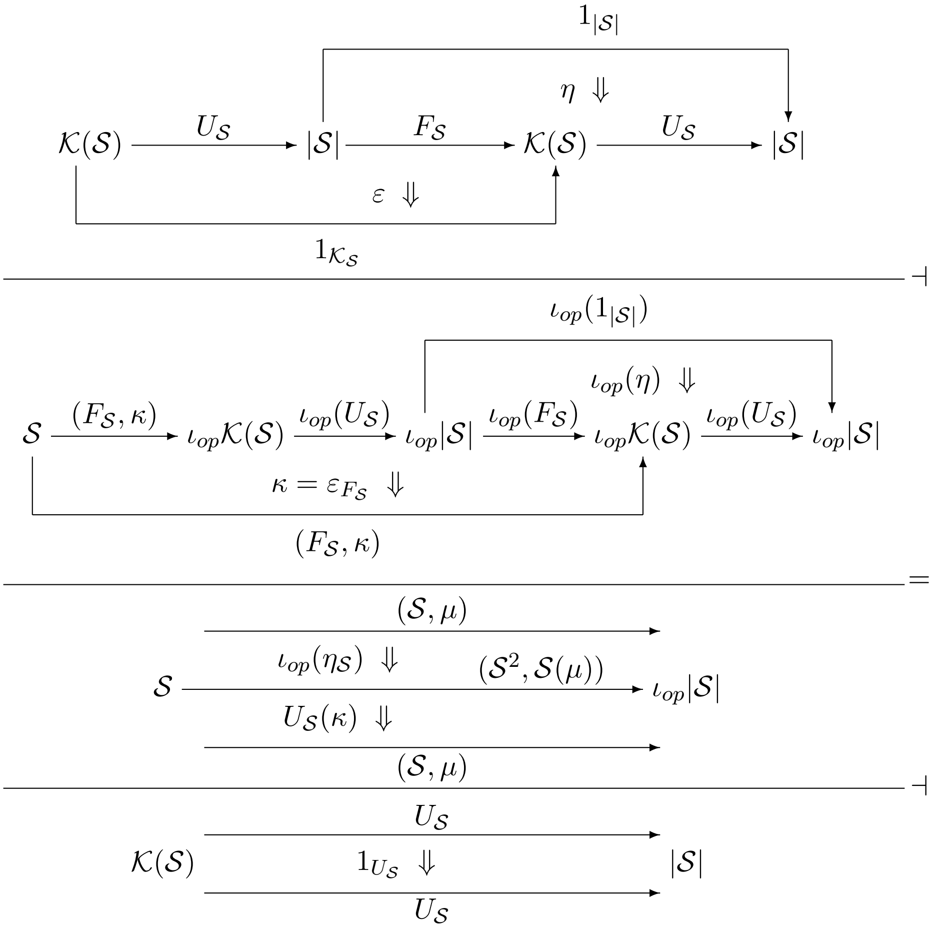
$$\begin{array}{c}
\mathcal{S} \xrightarrow{\;(F_{\mathcal{S}}, \kappa)\;} \iota_{op}\mathcal{K}(\mathcal{S}) \xrightarrow{\;\iota_{op}(U_{\mathcal{S}})\;} \iota_{op}|\mathcal{S}| \xrightarrow{\;\iota_{op}(F_{\mathcal{S}})\;} \iota_{op}\mathcal{K}(\mathcal{S}) \\
\kappa \Downarrow \\
\mathcal{S} \xrightarrow[\;(F_{\mathcal{S}}, \kappa)\;]{} \iota_{op}\mathcal{K}(\mathcal{S}) \\
\hline
\mathcal{K}(\mathcal{S}) \xrightarrow{\;U_{\mathcal{S}}\;} |\mathcal{S}| \xrightarrow{\;F_{\mathcal{S}}\;} \mathcal{K}(\mathcal{S}) \\
\varepsilon \Downarrow \\
\mathcal{K}(\mathcal{S}) \xrightarrow[\;1_{\mathcal{K}(\mathcal{S})}\;]{} \mathcal{K}(\mathcal{S})
\end{array}
\;\dashv$$

We note for the record that $\varepsilon_{F_{\mathcal{S}}} = \kappa$. Next, we verify the triangular equalities. We have

$$\varepsilon_{F_{\mathcal{S}}} \circ F_{\mathcal{S}}(\eta) = \kappa \circ F_{\mathcal{S}}(\eta) = 1_{F_{\mathcal{S}}}$$

The last equality follows from the fact that $(F_{\mathcal{S}}, \kappa) : \mathcal{S} \to 1_{\mathcal{C}_{\mathcal{S}}}$ is an oplax morphism of monads, i.e. $(F_{\mathcal{S}}, \kappa)$ subequalizes $\mathcal{S}$.

To see the other triangular equality, we consider the following correspondences of 2-cells



The first and the last are adjoint correspondences. In the middle, we have equality of 2-cells. The last 2-cell is $1_{U_{\mathcal{S}}}$ since before last is

$$U_{\mathcal{S}}(\varepsilon_{F_{\mathcal{S}}}) \circ \iota_{op}(\eta_{\mathcal{S}}) = U_{\mathcal{S}}(\kappa) \circ \eta_{\mathcal{S}} = \mu \circ \eta_{\mathcal{S}} = 1_{(S,\mu)}$$

This ends the 2-categorical explanation why $\mathcal{K}$ 'produces' the Kleisli object, if they exist. The categorical explanation will be given in Subsection 2.6.

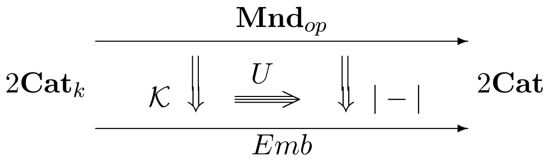
2.5 The 4-categorical perspective

We bring here some order to the data constructed above by describing it as some cells in the 4-category **3Cat** of (strict) 3-categories, 3-functors, pseudo-natural 3-transformations, pseudo 3-modifications, and perturbations.

We need some notation to be used only in the remainder of this subsection. For a monad $\mathcal{S} = (\mathcal{C}, \mathcal{S}, \eta, \mu)$ in a 2-category $\mathcal{D}$ the unit η (and all other constructs derived from the monad $\mathcal{S}$) will be denoted with a subscript $[\mathcal{D}, \mathcal{S}]$. Thus we write $\mathcal{C}_{[\mathcal{D},\mathcal{S}]}$ for $\mathcal{C}$, $\eta_{[\mathcal{D},\mathcal{S}]}$

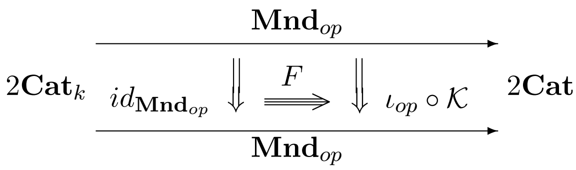
for the unit η, $\varepsilon_{[\mathcal{D},\mathcal{S}]}$ for the counit ε of the adjunction $F_{\mathcal{S}} \dashv U_{\mathcal{S}}$, i.e. $F_{[\mathcal{D},\mathcal{S}]} \dashv U_{[\mathcal{D},\mathcal{S}]}$, and so on.

We have a modification U:



The 3-functor $\mathbf{Mnd}_{op}$ is defined above, Emb is the obvious embedding 3-functor. $|-| : \mathbf{Mnd}_{op} \to Emb$ is a (strict) 3-transformation so that $|-|_{\mathcal{D}} : \mathbf{Mnd}_{op}(\mathcal{D}) \to \mathcal{D}$ is associating to a monad $\mathcal{S}$ in $\mathcal{D}$, its underlying category $|\mathcal{S}|_{\mathcal{D}} = \mathcal{C}_{[\mathcal{D},\mathcal{S}]}$. $\mathcal{K} : \mathbf{Mnd}_{op} \to Emb$ is a (pseudo) 3-transformation so that $\mathcal{K}_{\mathcal{D}} : \mathbf{Mnd}_{op}(\mathcal{D}) \to \mathcal{D}$ is associating to a monad $\mathcal{S}$ in $\mathcal{D}$ its Kleisli category $\mathcal{K}_{\mathcal{D}}(\mathcal{S})$. The component $U_{\mathcal{D}} : \mathcal{K}_{\mathcal{D}} \to |-|_{\mathcal{D}}$ of the modification $U : \mathcal{K} \to |-|$ at $\mathcal{D}$ is a 2-transformation of 2-functors such that at the monad S it is $U_{[\mathcal{D},\mathcal{S}]} : \mathcal{K}_{\mathcal{D}}(\mathcal{S}) \to |\mathcal{S}|_{\mathcal{D}}$, i.e. the forgetful 1-cell in $\mathcal{D}$ from the Kleisli object for $\mathcal{S}$ to the underlying category of $\mathcal{S}$.

We also have a modification F:



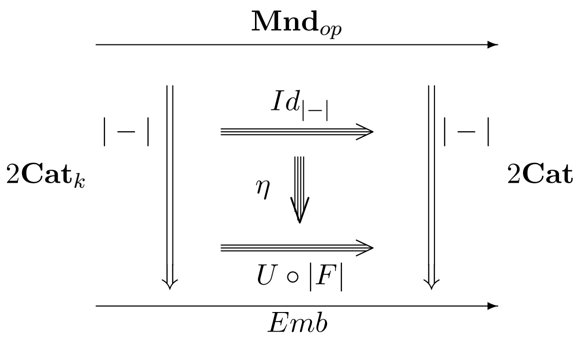
$\iota_{op} \circ \mathcal{K} : \mathbf{Mnd}_{op} \to \mathbf{Mnd}_{op}$ is a (pseudo) 3-transformation so that $\iota_{op,\mathcal{D}} \circ \mathcal{K}_{\mathcal{D}} : \mathbf{Mnd}_{op}(\mathcal{D}) \to \mathbf{Mnd}_{op}(\mathcal{D})$ is associating to a monad $\mathcal{S}$ in $\mathcal{D}$ the identity monad on $\mathcal{K}_{\mathcal{D}}(\mathcal{S})$, i.e. $\iota_{op,\mathcal{D}} \circ \mathcal{K}_{\mathcal{D}}(\mathcal{S}) = 1_{\mathcal{K}_{\mathcal{D}}(\mathcal{S})}$.

The component $F_{\mathcal{D}} : Id_{\mathbf{Mnd}_{op}(\mathcal{D})} \to \iota_{op,\mathcal{D}} \circ \mathcal{K}_{\mathcal{D}}$ of the modification $F : id_{\mathbf{Mnd}_{op}} \to \iota_{op} \circ \mathcal{K}$ at $\mathcal{D}$ is a 2-transformation of 2-functors such that at the monad S it is $(F_{[\mathcal{D},\mathcal{S}]}, \kappa_{[\mathcal{D},\mathcal{S}]}) : \mathcal{S} \to 1_{\mathcal{K}_{\mathcal{D}}(\mathcal{S})}$. In particular $F_{\mathcal{S}} = F_{[\mathcal{D},\mathcal{S}]} = |(F_{[\mathcal{D},\mathcal{S}]}, \kappa_{[\mathcal{D},\mathcal{S}]})|$ is the free Kleisli algebra 1-cell in $\mathcal{D}$ from the underlying category of $\mathcal{S}$ to the Kleisli object for $\mathcal{S}$.

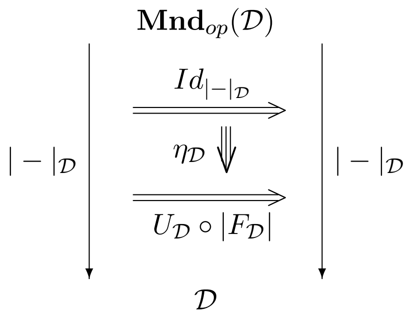
Now if we compose the 3-transformation $|-|$ with the 3-modification F we get a 3-modification

$$|F| : |-| \longrightarrow |-| \circ \iota_{op} \circ \mathcal{K} = \mathcal{K}$$

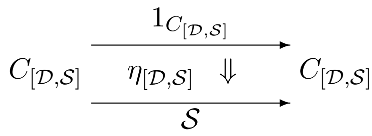
Thus we can compose the 3-modifications $|F|$ and U both ways. The perturbation η (i.e. a 4-cell in the 4-category $3\mathbf{Cat}$) from $Id_{|-|}$ to $U \circ |F|$ is described below. The following diagram



describes all the faces of η. The component of the above diagram at a 2-category (with Kleisli objects) $\mathcal{D}$ is

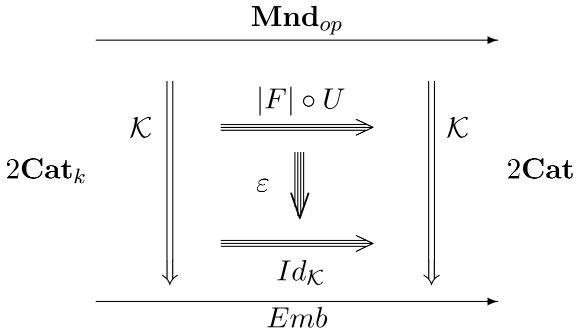


The component of the above diagram at a monad $\mathcal{S}$ in $\mathcal{D}$ is

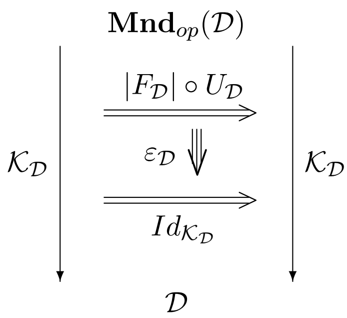


This means that η is the collection of all the units of all Kleisli adjunctions $F_{\mathcal{S}} \dashv U_{\mathcal{S}}$ of all the monads $\mathcal{S}$ in all the 2-categories $\mathcal{D}$ that admit Kleisli objects.

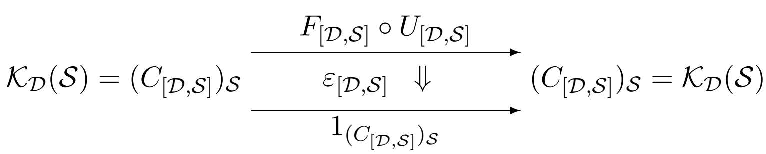
Similarly, ε, defined below, is a perturbation from $|F| \circ U$ to $Id_{\mathcal{K}}$.



The component of the above diagram at a 2-category (with Kleisli objects) $\mathcal{D}$ is



The component of the above diagram at a monad $\mathcal{S}$ in $\mathcal{D}$ is



This means that ε is the collection of all the counits of all Kleisli adjunctions $F_{\mathcal{S}} \dashv U_{\mathcal{S}}$ of all the monads $\mathcal{S}$ in all the 2-categories $\mathcal{D}$ that admit Kleisli objects. Needless to say that the perturbations η and ε satisfy the triangular equalities.

2.6 The categorical description of the Kleisli objects

If $\mathcal{D}$ is **Cat** the 2-category of categories, then the Kleisli objects coincide with the usual categories of Kleisli algebras. For a monad $(\mathcal{C}, \mathcal{S}, \eta, \mu)$ the category $\mathcal{C}_{\mathcal{S}}$ has the same objects as $\mathcal{C}$. A morphism in $f : A \to B$ in $\mathcal{C}_{\mathcal{S}}$ is a morphism in $f : A \to \mathcal{S}(A)$ with the usual identities, compositions, $U_{\mathcal{S}}$, and $F_{\mathcal{S}}$. The component at X in $\mathcal{C}$ of the natural transformation $\kappa : F_{\mathcal{S}} \circ \mathcal{S} \to F_{\mathcal{S}}$ is $1_{\mathcal{S}(X)}$.

If a 2-category $\mathcal{D}$ admits Kleisli objects we can ask whether the Kleisli 2-functor $\mathcal{K} : \mathbf{Mnd}_{op}(\mathcal{D}) \to \mathcal{D}$ preserves limits of a particular kind. We have

Lemma 2.1. *The Kleisli 2-functor $\mathcal{K} : \mathbf{Mnd}_{op}(\mathbf{Cat}) \to \mathbf{Cat}$ preserves products of 0-cells.*

Proof. We will sketch the construction for binary products. Let $(\mathcal{C}, \mathcal{S}, \eta, \mu)$ and $(\mathcal{C}', \mathcal{S}', \eta', \mu')$ in **Cat**. Then their product in $\mathbf{Mnd}_{op}(\mathbf{Cat})$ is $(\mathcal{C} \times \mathcal{C}', \mathcal{S} \times \mathcal{S}', (\eta, \eta'), (\mu, \mu'))$. One can easily verify that the unique morphism

$$H : (\mathcal{C} \times \mathcal{C}')_{\mathcal{S} \times \mathcal{S}'} \longrightarrow \mathcal{C}_{\mathcal{S}} \times \mathcal{C}'_{\mathcal{S}'}$$

such that $H \circ F_{\mathcal{S} \times \mathcal{S}'} = F_{\mathcal{S}} \times F_{\mathcal{S}'}$ and $H(\kappa^{\mathcal{S} \times \mathcal{S}'}) = (\kappa^{\mathcal{S}}, \kappa^{\mathcal{S}'})$ is an isomorphism. □

Remark. Note that, as the 2-functor $EM : \mathbf{Mnd}(\mathbf{Cat}) \to \mathbf{Cat}$ is a right 2-adjoint it preserves all limits.

3 Monoidal objects in 2-categories

Let $\mathcal{D}$ be a 2-category with finite products of 0-cells. In such a 2-category $\mathcal{D}$, we can talk about monoidal objects, (op)lax monoidal 1-cells, and monoidal 2-cells, as we talk about monoidal categories, (op)lax monoidal functors, and monoidal natural transformations in the 2-category **Cat**. A *monoidal object in $\mathcal{D}$* consists of a 0-cell $\mathcal{C}$, two 1-cells $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$, $I : 1 \to \mathcal{C}$, and three invertible 2-cells

$$\alpha : \otimes \circ (1 \times \otimes) \Rightarrow \otimes \circ (\otimes \times 1) \qquad \lambda : \otimes \circ \langle I, 1_{\mathcal{C}} \rangle \Rightarrow 1_{\mathcal{C}} \qquad \rho : \otimes \circ \langle 1_{\mathcal{C}}, I \rangle \Rightarrow 1_{\mathcal{C}}$$

making the pentagon

$$\otimes\langle \alpha, 1 \rangle \circ \alpha_{1_{\mathcal{C}} \times \otimes \times 1_{\mathcal{C}}} \circ \otimes\langle 1, \alpha \rangle = \alpha_{\otimes \times 1_{\mathcal{C}} \times 1_{\mathcal{C}}} \circ \alpha_{1_{\mathcal{C}} \times 1_{\mathcal{C}} \times \otimes}$$

and the triangle

$$\otimes\langle \varrho_{\pi_1}, 1_{\pi_2} \rangle \circ \alpha_{\langle \pi_1, I, \pi_2 \rangle} = \otimes\langle 1_{\pi_1}, \lambda_{\pi_2} \rangle$$

commute, where $\langle \pi_1, I, \pi_2 \rangle : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C} \times \mathcal{C} \times \mathcal{C}$ is the obvious morphism.

A *lax monoidal morphism of monoidal objects*

$$(F, \varphi, \bar{\varphi}) : (\mathcal{C}, \otimes, I, \alpha, \lambda, \varrho) \longrightarrow (\mathcal{C}', \otimes', I', \alpha', \lambda', \varrho')$$

consists of a 1-cell and two 2-cells

$$F : \mathcal{C} \to \mathcal{C}', \qquad \bar{\varphi} : I' \Rightarrow F \circ I, \qquad \varphi : \otimes' \circ (F \times F) \Rightarrow F \circ \otimes$$

such that the following three diagrams

$$\begin{CD}
\otimes' \circ (1 \times \otimes') \circ (F \times F \times F) @>{\alpha'_{F\times F\times F}}>> \otimes' \circ (\otimes' \times 1) \circ (F \times F \times F)\\
@V{\otimes'(1,\varphi)}VV @VV{\otimes'(\varphi,1)}V\\
\otimes' \circ (F \times F) \circ (1 \times \otimes) @. \otimes' \circ (F \times F) \circ (\otimes \times 1)\\
@V{\varphi_{(1\times\otimes)}}VV @VV{\varphi_{(\otimes\times 1)}}V\\
F \circ \otimes \circ (1 \times \otimes) @>>{F(\alpha)}> F \circ \otimes \circ (\otimes \times 1)
\end{CD}$$

$$\begin{CD}
\otimes' \circ \langle 1_{\mathcal{C}}, I' \rangle \circ F @>{\rho'_F}>> F\\
@V{\otimes'(1,\bar{\varphi})}VV @AA{F(\rho)}A\\
\otimes' \circ (F \times F) \circ \langle 1_{\mathcal{C}}, I \rangle @>>{\varphi_{\langle 1_{\mathcal{C}}, I \rangle}}> F \circ \otimes \circ \langle 1_{\mathcal{C}}, I \rangle
\end{CD}$$

and

$$\begin{CD}
\otimes' \circ \langle I', 1_{\mathcal{C}} \rangle \circ F @>{\lambda'_F}>> F\\
@V{\otimes'(\bar{\varphi},1)}VV @AA{F(\lambda)}A\\
\otimes' \circ (F \times F) \circ \langle I, 1_{\mathcal{C}} \rangle @>>{\varphi_{\langle I, 1_{\mathcal{C}} \rangle}}> F \circ \otimes \circ \langle I, 1_{\mathcal{C}} \rangle
\end{CD}$$

commute.

An *oplax monoidal morphism of monoidal objects*

$$(F, \varphi, \bar{\varphi}) : (\mathcal{C}, \otimes, I, \alpha, \lambda, \varrho) \longrightarrow (\mathcal{C}', \otimes', I', \alpha', \lambda', \varrho')$$

consists of a 1-cell and two 2-cells

$$F : \mathcal{C} \to \mathcal{C}', \qquad \bar{\varphi} : F \circ I \Rightarrow I', \qquad \varphi : F \circ \otimes \Rightarrow \otimes' \circ (F \times F)$$

(note the change of direction!) satisfying similar diagrams as those for lax monoidal morphism.

A *transformation of lax monoidal morphism*

$$\tau : (F, \varphi, \bar{\varphi}) \Rightarrow (F', \varphi', \bar{\varphi}')$$

is a 2-cell $\tau : F \to F'$ such that the diagrams

$$\begin{CD}
\otimes' \circ (F \times F) @>{\otimes'(\sigma,\sigma)}>> \otimes' \circ (F' \times F')\\
@V{\varphi}VV @AA{\varphi'}A\\
F \circ \otimes @>>{\sigma_{\otimes}}> F' \circ \otimes
\end{CD}
\qquad\qquad
\begin{array}{ccc}
 & \overset{\bar{\varphi}}{\nearrow} & F \circ I\\
I' & & \Big\downarrow \sigma_I\\
 & \underset{\bar{\varphi}'}{\searrow} & F' \circ I
\end{array}$$

commute. The transformations of oplax monoidal morphism are defined similarly.

Recall from 2.3 that $2\mathbf{Cat}_\times$ is the 3-category of 2-categories with finite products. We have 3-functors

$$\mathbf{Mon}, \mathbf{Mon}_{op} : 2\mathbf{Cat}_\times \longrightarrow 2\mathbf{Cat}_\times$$

$\mathbf{Mon}$ ($\mathbf{Mon}_{op}$) sends a 2-category $\mathcal{D}$ with finite products to the 2-category $\mathbf{Mon}(\mathcal{D})$ ($\mathbf{Mon}_{op}(\mathcal{D})$) of monoidal objects, (op)lax monoidal morphism, and their transformations. We also have 3-transformations

$$\mathcal{U} : \mathbf{Mon} \Rightarrow \mathbf{Id}, \qquad \mathcal{U}_{op} : \mathbf{Mon}_{op} \Rightarrow \mathbf{Id}$$

whose components are forgetful functors forgetting the monoidal structure. $\mathbf{Id}$ is the identity functor on $2\mathbf{Cat}_\times$.

The following theorem says that, in any 2-category $\mathcal{D}$ with finite products, monoidal monads are 'the same things' as monoidal categories in the 2-category of monads over $\mathcal{D}$. However there are subtleties concerning (op)laxness of 1-cells.

Lemma 3.1. *The following diagrams of* 3*-functors*

$$\begin{array}{ccc} 2\mathbf{Cat}_\times & \xrightarrow{\mathbf{Mon}} & 2\mathbf{Cat}_\times \\ {\scriptstyle \mathbf{Mnd}_{op}}\downarrow & & \downarrow{\scriptstyle \mathbf{Mnd}_{op}} \\ 2\mathbf{Cat}_\times & \xrightarrow[\mathbf{Mon}]{} & 2\mathbf{Cat}_\times \end{array} \qquad \begin{array}{ccc} 2\mathbf{Cat}_\times & \xrightarrow{\mathbf{Mon}_{op}} & 2\mathbf{Cat}_\times \\ {\scriptstyle \mathbf{Mnd}}\downarrow & & \downarrow{\scriptstyle \mathbf{Mnd}} \\ 2\mathbf{Cat}_\times & \xrightarrow[\mathbf{Mon}_{op}]{} & 2\mathbf{Cat}_\times \end{array}$$

commute up to natural 3*-isomorphisms* ξ *and* ξ'*, respectively. Moreover, these isomorphisms are compatible with* 3*-transformations* ι *and* $\mathcal{U}$ *in the sense that the diagrams of* 3*-transformations*

$$\begin{array}{ccccc} & & \mathbf{Mnd}_{op}\mathbf{Mon} & & \\ & \nearrow^{(\iota_{op})_{\mathbf{Mon}}} & & \searrow^{\mathbf{Mnd}_{op}(\mathcal{U})} & \\ \mathbf{Mon} & & \downarrow \xi & & \mathbf{Mnd}_{op} \\ & \searrow_{\mathbf{Mon}(\iota_{op})} & & \nearrow_{\mathcal{U}_{\mathbf{Mnd}_{op}}} & \\ & & \mathbf{Mon}\mathbf{Mnd}_{op} & & \end{array}$$

$$\begin{array}{ccccc} & & \mathbf{Mnd}\mathbf{Mon}_{op} & & \\ & \nearrow^{(\iota)_{\mathbf{Mon}_{op}}} & & \searrow^{\mathbf{Mnd}(\mathcal{U}_{op})} & \\ \mathbf{Mon}_{op} & & \downarrow \xi' & & \mathbf{Mnd} \\ & \searrow_{\mathbf{Mon}_{op}(\iota)} & & \nearrow_{(\mathcal{U}_{op})_{\mathbf{Mnd}}} & \\ & & \mathbf{Mon}_{op}\mathbf{Mnd} & & \end{array}$$

commute.

Proof. For any 2-category $\mathcal{D}$ with products the cells in the 2-categories $\mathbf{MonMnd}_{op}(\mathcal{D})$, $\mathbf{Mnd}_{op}\mathbf{Mon}(\mathcal{D})$, $\mathbf{Mon}_{op}\mathbf{Mnd}(\mathcal{D})$, $\mathbf{MndMon}_{op}(\mathcal{D})$ are tuples of cells from $\mathcal{D}$ satisfying certain (equational) coherence conditions. An easy but long verification shows, for example, that 0-cells of both $\mathbf{MonMnd}_{op}(\mathcal{D})$, $\mathbf{Mnd}_{op}\mathbf{Mon}(\mathcal{D})$ are tuples of cells that differ only by the cells order, but not the conditions they satisfy. Similarly for 1- and 2-cells. The morphism ξ' just permutes these tuples. One can easily check that this 'permutation isomorphism' is compatible with both ι and $\mathcal{U}$, as stated in the theorem.

More explicitly, we can identify 0-cells of both $\mathbf{MonMnd}_{op}(\mathcal{D})$ and $\mathbf{Mnd}_{op}\mathbf{Mon}(\mathcal{D})$ as 11-tuples

$$(\mathcal{C}, \otimes, I, \alpha, \lambda, \varrho, \mathcal{S}, \varphi, \bar{\varphi}, \eta, \mu)$$

satisfying certain conditions that we explain below.

In both cases $(\mathcal{C}, \mathcal{S}, \eta, \mu)$ is a monad. Moreover, in $\mathbf{MonMnd}_{op}(\mathcal{D})$

$$(\otimes, \varphi) : (\mathcal{C} \times \mathcal{C}, \mathcal{S} \times \mathcal{S}, (\eta, \eta), (\mu, \mu)) \longrightarrow (\mathcal{C}, \mathcal{S}, \eta, \mu)$$

$$(I, \bar{\varphi}) : (\mathcal{C}, 1, 1, 1) \longrightarrow (\mathcal{C}, \mathcal{S}, \eta, \mu)$$

is an oplax morphisms of monads. This condition is equivalent to the condition that

$$\eta : (1_{\mathcal{C}}, 1, 1) \longrightarrow (\mathcal{S}, \varphi, \bar{\varphi}), \qquad \mu : (\mathcal{S}^2, \mathcal{S}(\varphi) \circ \varphi_{\mathcal{S}\times\mathcal{S}}, \mathcal{S}(\bar{\varphi}) \circ \bar{\varphi}) \longrightarrow (\mathcal{S}, \varphi, \bar{\varphi})$$

are monoidal transformations of lax monoidal morphisms. The later condition is required for such tuple to be in $\mathbf{Mnd}_{op}\mathbf{Mon}(\mathcal{D})$. Finally, the conditions that

$$\alpha : (\otimes \circ (1 \times \otimes), \varphi_{1\times\otimes} \circ \otimes(1, \varphi)) \longrightarrow (\otimes \circ (\otimes \times 1), \varphi_{\otimes\times 1} \circ \otimes(\varphi, 1))$$

$$\lambda : (\otimes \circ \langle I, 1_{\mathcal{C}}\rangle, \varphi_{\langle I, 1_{\mathcal{C}}\rangle} \circ \otimes(\bar{\varphi}, 1_{\mathcal{S}})) \longrightarrow (1_{\mathcal{C}}, 1_{\mathcal{S}})$$

$$\varrho : (\otimes \circ \langle 1_{\mathcal{C}}, I\rangle, \varphi_{\langle 1_{\mathcal{C}}, I\rangle} \circ \otimes(1_{\mathcal{S}}, \bar{\varphi})) \longrightarrow (1_{\mathcal{C}}, 1_{\mathcal{S}})$$

are transformations of oplax morphisms of monads, required for the tuple to be in $\mathbf{MonMnd}_{op}(\mathcal{D})$ is equivalent to the condition that

$$(\mathcal{S}, \varphi, \bar{\varphi}) : (\mathcal{C}, \otimes, I, \alpha, \lambda, \varrho) \longrightarrow (\mathcal{C}, \otimes, I, \alpha, \lambda, \varrho)$$

is a lax monoidal morphisms. This is another condition required for the tuple to be in $\mathbf{Mnd}_{op}\mathbf{Mon}(\mathcal{D})$.

In that sense the conditions imposed on such 11-tuple to be either in $\mathbf{MonMnd}_{op}(\mathcal{D})$ or $\mathbf{Mnd}_{op}\mathbf{Mon}(\mathcal{D})$ are the same. The similar thing happen with $1-$ and 2-cells in those 2-categories. Thus they are isomorphic.

The remaining details are left for the reader. □

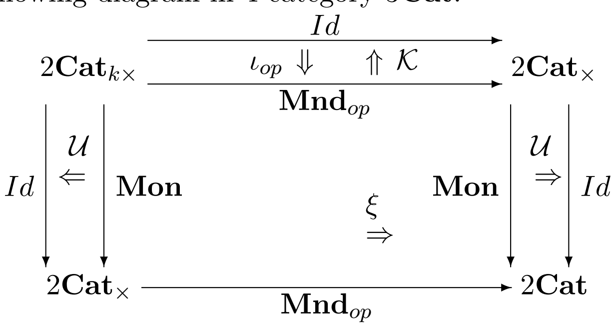
Remark. This fact is a fragment of a much wider phenomena, deserving a serious independent studies, that if we combine together two 'algebraic structures' then they cooperate well when one is taken with lax morphisms and the other with oplax morphisms like $\mathbf{Mnd}_{op}\mathbf{Mnd} \cong \mathbf{MndMnd}_{op}$, $\mathbf{Mon}_{op}\mathbf{Mon} \cong \mathbf{MonMon}_{op}$.

4 The Kleisli objects in 2-categories of monoidal objects

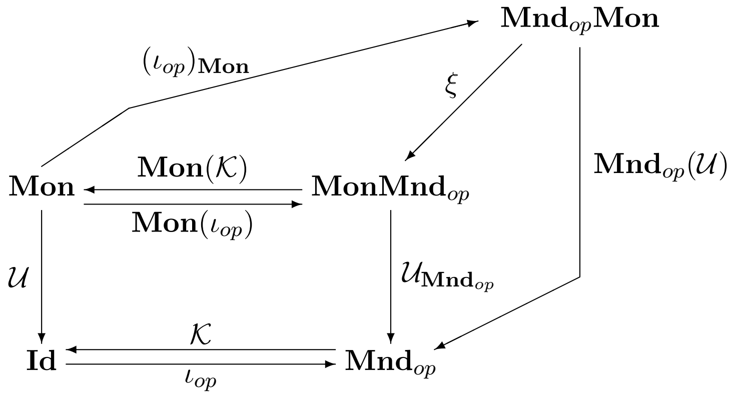
In this section we give a 3-categorical proof of

Theorem 4.1. *Let $\mathcal{D}$ be a 0-cell in $2\mathbf{Cat}_{k\times}$, i.e. a 2-category with finite products that admits Kleisli objects and such that Kleisli objects commute with finite products in $\mathcal{D}$. Then the 2-category $\mathbf{Mon}(\mathcal{D})$ admits Kleisli objects and they are standard.*

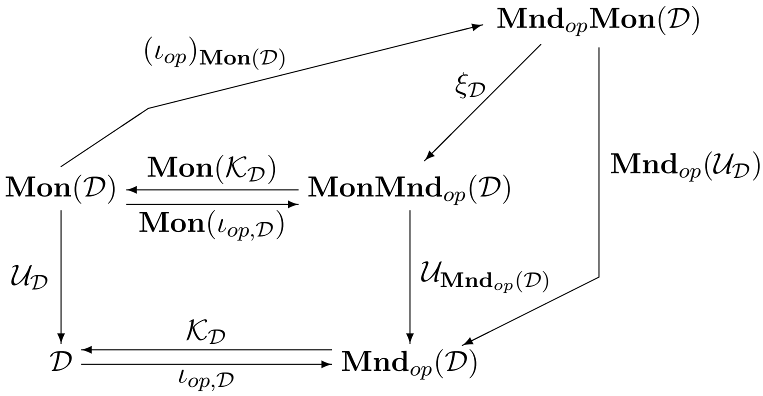
Proof. Consider the following diagram in 4-category $3\mathbf{Cat}$:



Using the above diagram we can form a diagram in the 3-category $3\mathbf{Cat}(2\mathbf{Cat}_{k\times}, 2\mathbf{Cat})$



By Lemma 3.1 and the fact that $\mathcal{U}$ is a 3-natural transformation we see that that the above diagram commutes. Evaluating this diagram at the 2-category $\mathcal{D}$ in $2\mathbf{Cat}_{k\times}$ we get a commuting diagram of 2-categories and 2-functors



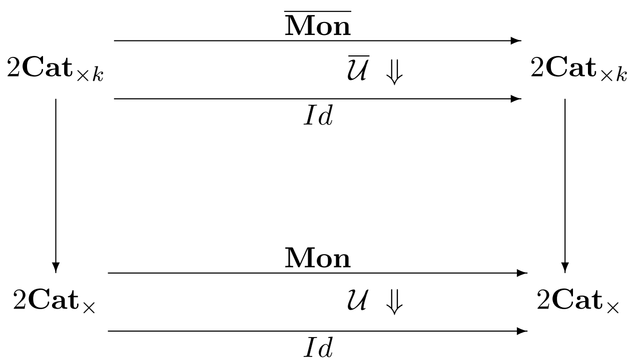
As $\mathcal{K}_{\mathcal{D}} \dashv \iota_{op,\mathcal{D}}$ and the 3-functor **Mon** preserves 2-adjunctions, we have $\mathbf{Mon}(\mathcal{K}_{\mathcal{D}}) \dashv \mathbf{Mon}(\iota_{op,\mathcal{D}})$. Since $\xi_{\mathcal{D}}$ is an isomorphism we get that $\mathbf{Mon}(\mathcal{K}_{\mathcal{D}}) \circ \xi_{\mathcal{D}} \dashv \iota_{op,\mathbf{Mon}(\mathcal{D})}$, i.e. $\mathbf{Mon}(\mathcal{D})$ indeed admits Kleisli objects and that they are standard. $\square$

By Lemma 2.1 we get from the above theorem

Corollary 4.2. *The* 2*-category* $\mathbf{Mon}(\mathbf{Cat})$ *admits standard Kleisli objects.*

Note that Theorem 4.1 can be rephrased in a slightly more general form as a lifting property.

Theorem 4.3. *The* 3*-functor* **Mon** *and the* 3*-transformation* $\mathcal{U}$ *can be lifted to the* 3*-functor* $\overline{\mathbf{Mon}}$ *and the* 3*-transformation* $\overline{\mathcal{U}}$ *with the* 0*-codomain* $2\mathbf{Cat}_{\times k}$ *so that the diagram of* 3*-categories,* 3*-functors, and* 3*-transformations commutes up to a canonical isomorphism*



Proof. From Theorem 4.1, we know that, if we apply the 3-functor **Mon** to a 2-category $\mathcal{D}$ that has not only finite products but also Kleisli objects, then we will get $\mathbf{Mon}(\mathcal{D})$ that also has finite products and Kleisli objects. We need also to verify that **Mon** applied to a 2-functor $F : \mathcal{D} \to \mathcal{D}'$ that preserves Kleisli objects also preserves Kleisli objects. This can be proved using a similar argument as the one in Theorem 4.1. We leave it for the reader. □

5 The EM objects in 2-categories of monoidal objects

The dual statement of Theorem 4.1 is

Theorem 5.1. *Let $\mathcal{D}$ be a 2-category with finite products admitting EM objects. Then the 2-category $\mathbf{Mon}_{op}(\mathcal{D})$ admits EM objects and they are standard.*

Putting $\mathcal{D}$ to be **Cat** in the above Theorem, we obtain a result by I. Moerdijk [Mo] in a sharper version of P. McCrudden [McC]

Corollary 5.2 (Moerdijk, McCrudden)**.** *The 2-category $\mathbf{Mon}_{op}(\mathbf{Cat})$ admits standard EM objects.*

6 Some other algebraic structures

If we replace the 3-functor **Mon** ($\mathbf{Mon}_{op}$) by the 3-functor **BMon** ($\mathbf{BMon}_{op}$) of braided monoidal objects with lax (oplax) monoidal morphisms and monoidal transformations or 3-functor **SMon** ($\mathbf{SMon}_{op}$) of symmetric monoidal objects with lax (oplax) monoidal morphisms and monoidal transformations or 3-functor **Cmd** ($\mathbf{Cmd}_{op}$) of comonads with lax (oplax) monoidal morphisms and transformations, or even 3-functor **Mnd** ($\mathbf{Mnd}_{op}$), we can repeat the whole reasoning again. In this way we obtain

Theorem 6.1. *Let $\mathcal{D}$ be a 2-category that admits Kleisli objects. Then the 2-categories $\mathbf{Mnd}(\mathcal{D})$ and $\mathbf{Cmd}(\mathcal{D})$ admit Kleisli object and they are standard.*

Moreover, if $\mathcal{D}$ has finite products that commute with Kleisli objects, then the 2-categories $\mathbf{BMon}(\mathcal{D})$, $\mathbf{SMon}(\mathcal{D})$ admit Kleisli objects and they are standard.

Theorem 6.2. *Let $\mathcal{D}$ be a 2-category that admits EM objects. Then the 2-category $\mathbf{Mnd}_{op}(\mathcal{D})$ and $\mathbf{Cmd}_{op}(\mathcal{D})$ admit EM objects and they are standard.*

Moreover, if $\mathcal{D}$ has finite products, then the 2-categories $\mathbf{BMon}_{op}(\mathcal{D})$, $\mathbf{SMon}_{op}(\mathcal{D})$ admit EM objects and they are standard.

Remarks.

1. The above facts suggest that the results of this paper can be still generalized. One way is to axiomatize the formal properties of the relation of 3-functors **Mon**, **BMon**, **SMon**, $\mathbf{Mnd}(\mathcal{D})$, and $\mathbf{Cmd}(\mathcal{D})$ with respect to the 3-functor $\mathbf{Mnd}_{op}$ and the relation of 3-functors $\mathbf{Mon}_{op}$, $\mathbf{BMon}_{op}$, $\mathbf{SMon}_{op}$, $\mathbf{Mnd}(\mathcal{D})$, and $\mathbf{Cmd}(\mathcal{D})$ with respect to the 3-functor **Mnd** and get this way still more abstract statement. This would be worth trying if some new natural examples were to be found, other than iterations of the 3-functors listed above.

2. The other more specific generalization would be to show that 'any' algebraic 2-categorical structure will do. The precise formulation what such algebraic structure should be is still to be found. It is possible that the 2-categories of pseudo-algebras with lax/oplax morphisms for pseudo-monads provide the right language to formulate this phenomenon in a more abstract form. The work of M. Hyland and his coworkers [Hy] might be also of a help.

References

[BD] J. Baez, J. Dolan, *Higher-dimensional algebra III: n-Categories and the algebra of opetopes.* Advances in Math. 135, 1998, pp 145-206.

[C] E. Cheng, *The category of opetopes and the category of opetopic sets*, Theory and Applications of Categories, Vol. 11, 2003, No. 16, pp 353-374.

[CLS] D. Chikhladze, S. Lack, R. Street. *Hopf monoidal comonads.* Theory and Applications of Categories, Vol. 24, 2010, No. 19, pp 554-563.

[Ch] D. Chikhladze. *A category of quantum categories* . Theory and Applications of Categories, Vol. 25, 2011, No. 1, pp 1-37.

[Day] B. Day, *On Closed category of Functors II.* Proceedings Sydney Category Theory Seminar 1972/1973, Lecture Notes in Math no 420, Springer-Verlag, 1974, pp 2054.

[Gray] J. Gray, *Formal Category Theory: Adjointness for* 2*-Categorires.* Lecture Notes in Math. 391, Springer-Verlag, 1974.

[Hy] M. Hyland, *Sorts of algebraic theories.* Talk at International Workshop on Categorical Logic August 28-29, 2010. Masaryk University, Brno, Czech Republic.

[KJBM] J. Kock, A. Joyal, M. Batanin, J.-F. Mascari, Polynomial functors and opetopes, Advances in Math., Vol. 224, 2010 pp 2690-2737.

[Le] T. Leinster, *Higher Operads, Higher Categories.* London Math. Soc. Lecture Note Series. Cambridge University Press, Cambridge, 2004. (math.CT/0305049)

[HMP] C. Hermida, M. Makkai, J. Power, *On weak higher dimensional categories I* Parts 1,2,3, Journal of Pure and Applied Algebra Vol. 153, 2000, pp 221-246, Vol. 157, 2001, pp 247-277, Vol. 166, 2002, pp 83-104.

[McC] P. McCrudden, *Opmonoidal monads.* Theory and Applications of Categories, Vol. 10, No. 19, 2002, pp 469485.

[Mo] I. Moerdijk, *Monads on tensor categories.* Journal of Pure and Applied Algebra 168, 2002, pp 189-208.

[SZ] S. Szawiel, M. Zawadowski, *The Web Monoid and Opetopic Sets.* arXiv:1011.2374v1 [math.CT], 2010, pp 1-63.

[St] R. Street, *The Formal Theory of Monads.* Journal of Pure and Applied Algebra 2, 1972, pp 149-168.

[Z] M. Zawadowski, *Lax Monoidal Fibrations*, in Models, Logics and Higher-Dimensional Categories: A Tribute to the Work of Mihaly Makkai, Proceedings of the CRM, accepted. (arXiv:0912.4464v2 [math.CT])