The category of 3-computads is not cartesian closed

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Abstract

We show, using [CJ] and Eckmann-Hilton argument, that the category of 3-computads is not cartesian closed. As a corollary we get that neither the category of all computads nor the category of n-computads, for $n > 2$, do form locally cartesian closed categories, and hence elementary toposes.

1 Introduction

S.H. Schanuel (unpublished) made an observation, c.f. [CJ], that the category of 2-computads $\mathbf{Comp_2}$ is a presheaf category. We show below that neither the category of computads nor the categories n-computads, for $n > 2$, are locally cartesian closed. This is in contrast with a remark in [CJ] on page 453, and an explicit statement in [B] claiming that these categories are presheaves categories. Note that some interesting subcategories of computads, like many-to-one computads, do form presheaf categories, c.f. [HMP], [HMZ].

We thank the anonymous referee for comments that helped to clarify the exposition of the example. The diagrams for this paper were prepared with a help of *catmac* of Michael Barr.

2 Computads

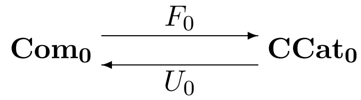
Computads were introduced by R.Street in [S], see also [B]. Recall that a computad is an ω-category that is levelwise free. Below we recall one of the definitions.

Let $\mathbf{nCat}$ be the category of n-categories and n-functors between them, $\omega\mathbf{Cat}$ be the category of ω-categories and ω-functors between them. We have the obvious truncation functors

$$tr_{n-1} : \mathbf{nCat} \longrightarrow \mathbf{(n-1)Cat}$$

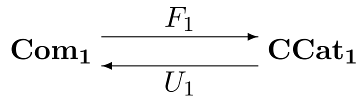
By $\mathbf{Comp_n}$ we denote the category of n-computads, a non-full subcategory of the category $\mathbf{nCat}$. By $\mathbf{CCat_n}$ we denote the non-full subcategory of $\mathbf{nCat}$, whose objects are 'computads up to the level $n-1$', i.e. an n-functor $f : A \to B$ is a morphism in $\mathbf{CCat_n}$ if and only if $tr_{n-1}(f) : tr_{n-1}(A) \to tr_{n-1}(B)$ is a morphism in $\mathbf{Comp_{n-1}}$. Clearly $\mathbf{CCat_n}$ is defined as soon as $\mathbf{Comp_{n-1}}$ is defined. The categories $\mathbf{Comp_n}$ and n-comma category $\mathbf{Com_n}$ are defined below.

The categories $\mathbf{Comp_0}$, $\mathbf{CCat_0}$ and $\mathbf{Com_0}$ are equal to Set, the category of sets. We have an adjunction

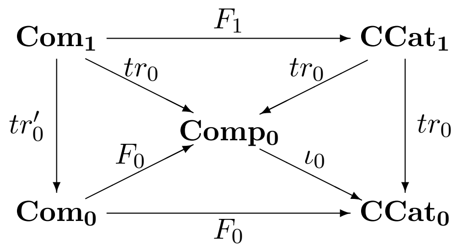


with both functors being the identity on Set, $F_0 \dashv U_0$. $\mathbf{Comp_0}$ is the image of $\mathbf{Com_0}$ under F_0.

$\mathbf{Com_1}$ is the category of graphs, i.e. an object of $\mathbf{Com_1}$ is a pair of sets and a pair of functions between them $\langle d, c : E \to V\rangle$. $\mathbf{CCat_1}$ is simply $\mathbf{Cat}$, the category of all small categories. The forgetful functor U_1 (forgetting compositions and identities) has a left adjoint F_1 'the free category (over a graph)' functor

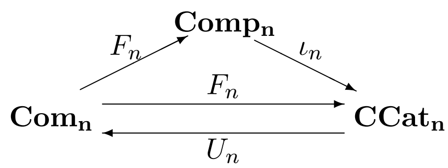


We have a diagram



where three triangles commute, moreover the left triangle and the outer square commute up to an isomorphism. tr_1 and tr'_1 are the obvious truncation morphisms. Then we define the category of 1-computads $\mathbf{Comp_1}$ as the essential (non-full) image of the functor F_1 in $\mathbf{CCat_1}$, i.e. 1-computads are the free categories over graphs and computad maps between them are functors sending indets (=indeterminates=generators) to indets.

Now suppose that we have an adjunction $U_n \dashv F_n$



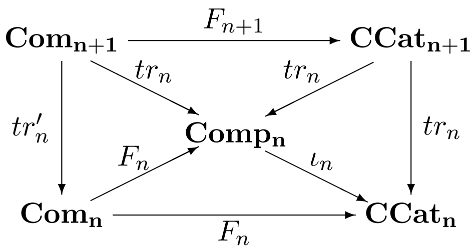
and $\mathbf{Comp_n}$ is defined as the the essential (non-full) image of the functor F_n in $\mathbf{CCat_n}$. We define the n-parallel pair functor

$$\Pi_n : \mathbf{Comp_n} \longrightarrow Set$$

such that

$$\Pi_n(A) = \{\langle a, b\rangle \mid a, b \in A_n,\ d(a) = d(b),\ c(a) = c(b)\}$$

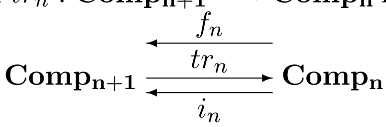
for any n-computad A. The $(n+1)$-comma category $\mathbf{Com_{n+1}}$ is the category $Set \downarrow \Pi_n$. Thus an object in $\mathbf{Com_{n+1}}$ is a pair $(A, \langle d, c\rangle : X \to \Pi_n(A)$, such that A is an n-computad X is a set of $(n+1)$-indets and $\langle d, c\rangle$ is a function associating n-domains and n-codomains. The forgetful functor $U_{n+1} : \mathbf{CCat_{n+1}} \longrightarrow \mathbf{Com_{n+1}}$ (forgetting compositions and identities at the level $n+1$) creates limits and satisfies the solution set condition. Thus it has a left adjoint F_{n+1}. We get a diagram



where three triangles commute, moreover the left triangle and the outer square commute up to an isomorphism. tr_n are the obvious truncation functors and tr'_n is a truncation functor that at the level n leaves the indets only. Then we define the category of $(n+1)$-computads $\mathbf{Comp_{n+1}}$ as the essential (non-full) image of the functor F_{n+1} in $\mathbf{CCat_{n+1}}$, i.e. $(n+1)$-computads are the free $(n+1)$-categories over $(n+1)$-comma categories and $(n+1)$-computad maps between them are $(n+1)$-functors sending indets to indets. The category of computads $\mathbf{Comp}$ is a (non-full) subcategory of the category of ω-categories and ω-functors $\omega\mathbf{Cat}$ such, that for each n, the truncation of objects and morphisms to $\mathbf{nCat}$ is in $\mathbf{Comp_n}$. As $F_n : \mathbf{Com_n} \to \mathbf{CCat_n}$ is faithful and full on isomorphisms, after restricting the codomain we get an equivalence of categories $F_n : \mathbf{Com_n} \to \mathbf{Comp_n}$.

Notation. If A is a computad then A_n denotes the set of n-cells of A and $|A|_n$ denotes the set of n-indets of A.

The truncation functor $tr_n : \mathbf{Comp_{n+1}} \longrightarrow \mathbf{Comp_n}$ has both adjoints $i_n \dashv tr_n \dashv f_n$



where

$$i_n(A) = F_{n+1}(A, \emptyset \to \Pi_n(A))$$

and

$$f_n(A) = F_{n+1}(A, id_{\Pi_n(A)} : \Pi_n(A) \to \Pi_n(A))$$

for A in $\mathbf{Comp_n}$. This shows that tr_n preserves limits and colimits. The colimits in $\mathbf{Comp_{n+1}}$ are calculated in $\mathbf{(n+1)Cat}$ but the limits in $\mathbf{Comp_{n+1}}$ are more involved. It is more convenient to describe them in $\mathbf{Com_{n+1}}$ and then apply the functor F_{n+1}. If $H : \mathcal{J} \to \mathbf{Com_{n+1}}$ is a functor and P is the limit of its truncation $tr_n \circ H$ to $\mathbf{Comp_n}$ then $Lim\, H$, the limit of H, truncated to $\mathbf{Comp_n}$ is P and the $(n+1)$-indets $|Lim\, H|_{n+1}$ of $Lim\, H$ are as follows

$$|Lim\, H|_{n+1} = \{\langle a_i\rangle_{i\in\mathcal{J}} |\ a_i \in |H(i)|_{n+1},\ \langle d(a_i)\rangle_{i\in\mathcal{J}},\ \langle c(a_i)\rangle_{i\in\mathcal{J}} \in P_n\}$$

The terminal object 1_n in $\mathbf{Comp_n}$ is quite complicated, for $n \geq 2$. However the $\mathbf{Com_2}$ part of 1_2 is still easy to describe. 1_2 has one 0-indet x and one 1-indet $\xi : x \to x$. Thus the 1-cells can be identified with finite (possibly empty) strings of of arrows:

$$x, \qquad x \xrightarrow{\xi} x \xrightarrow{\xi} x \ \cdots \ x \xrightarrow{\xi} x$$

or simply with elements of ω. The set $|1_2|_2$ of 2-indets in 1_2 contains exactly one indet for every pair of strings. The first element of such a pair is the domain of the indet and the second element of the pair is the codomain of the indet. Thus $|1_2|_2$ can be identified with the set $\omega \times \omega$. In particular $\langle 0, 0\rangle$ correspond to the only indet from id_x to id_x (id_x is the identity on x). The description of all 2-cells in 1_2 is more involved but we don't need it here.

3 The counterexample

Lemma 3.1 $\mathbf{Comp_3}$ *is not cartesian closed.*

Proof. As it was noted in Lemma 4.2 [CJ], the functor Π_2 factorizes as

$$\mathbf{Comp_2} \xrightarrow{\widehat{\Pi_2}} Set \downarrow \Pi_2(1_2) \xrightarrow{\Sigma} Set$$

where $\widehat{\Pi_2}(A) = \Pi_2(! : A \to 1_2)$, and $\Sigma(b : B \to \Pi_2(1_2)) = B$, for A in $\mathbf{Comp_2}$ and b in $Set \downarrow \Pi_2(1_2)$. Moreover, the category $Set \downarrow \Pi_2$, which is equivalent to $\mathbf{Comp_3}$, is also equivalent to $(Set \downarrow \Pi_2(1_2)) \downarrow \widehat{\Pi_2}$. Now, as $\mathbf{Comp_2}$ and $Set \downarrow \Pi_2(1_2)$ are cartesian closed categories with initial objects (in fact both categories are presheaf toposes) and $\widehat{\Pi_2}$ preserves the terminal object, by Theorem 4.1 of [CJ], $\mathbf{Comp_3}$ is a cartesian closed category if and only if $\widehat{\Pi_2}$ preserves binary products. We finish the proof by showing that $\widehat{\Pi_2}$ does not preserves the binary products.

Let A be a 2-computad with one 0-cell x, one 1-cell id_x the identity on x (no 1-indets). Moreover A has as 2-cells all cells generated by the two indeterminate 2-cells $a_1, a_2 : id_x \to id_x$. Thus, by Eckmann-Hilton argument, any 2-cell in A is of form $a_1^m \circ a_2^n$, for $m, n \in \omega$ (if $m = n = 0$ then $a_1^m \circ a_2^n = id_{id_x}$). Let B be a 2-computad isomorphic to A with indeterminate 2-cells b_1, b_2. Let x be the unique 0-cell in 1_2, c be the only indeterminate 2-cell in 1_2 that has id_x as its domain and codomain and C a subcomputad of 1_2 generated by c. The unique maps of 2-computads $! : A \to 1_2$ and $! : B \to 1_2$ sends a_i and b_i to c, for $i = 1, 2$. Thus they factor through C as $\alpha : A \to C$ and $\beta : B \to C$, respectively. The 2-computad C does not play a crucial role in the counterexample but it makes the explanations simpler.

Let us describe the product $A \times B$ in $\mathbf{Comp_2}$. The 0-cell and 1-cells are as in A, B and C. As there is only one 1-cell id_x in $A \times B$, the compatibility condition for domain and codomains of 2-indets is trivially satisfied, and the set 2-indets of $A \times B$ is just the product of 2-indets of A and B, i.e.

$$|A \times B|_2 = \{\langle a_i, b_j \rangle | \; i, j = 1, 2\}$$

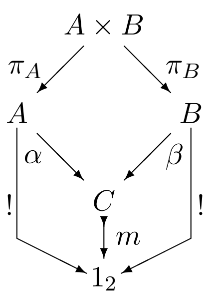
and the set of all 2-cells of $A \times B$ is

$$(A \times B)_2 = \{\langle a_1, b_1 \rangle^{n_1} \circ \langle a_1, b_2 \rangle^{n_2} \circ \langle a_2, b_1 \rangle^{n_3} \circ \langle a_2, b_2 \rangle^{n_2} | \; n_1, n_2, n_3, n_4 \in \omega\}$$

The projections

$$A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_B} B$$

are defined as the only 2-functors such that $\pi_A(a_i, b_j) = a_i$ and $\pi_A(a_i, b_j) = b_j$, for $i, j = 1, 2$. Thus we have a commuting square



(*)

As C is a subobject of the terminal object $A \times B$ is $A \times_C B$ and $A \times_{1_2} B$, i.e. both inner and outer squares in the above diagram are pullbacks.

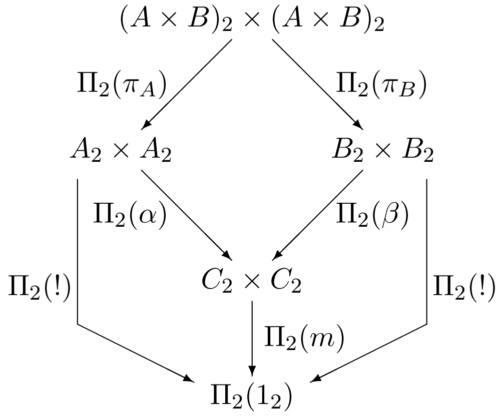
Since all the 2-cells in A, B, C and $A \times B$ are parallel we have

$$\Pi_2(A) = A_2 \times A_2, \quad \Pi_2(B) = B_2 \times B_2, \quad \Pi_2(C) = C_2 \times C_2,$$

and

$$\Pi_2(A \times B) = (A \times B)_2 \times (A \times B)_2$$

$\widehat{\Pi_2}$ preserves the product of A and B if in the diagram $(**)$ below, which is the application of Π_2 to the diagram $(*)$ above, the outer square is a pullback in *Set*



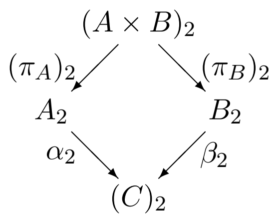
$(**)$

As $\Pi_2(m)$ is mono, the outer square in $(**)$ is a pullback in *Set* if and only if the inner square in $(**)$ is a pullback in *Set*. We have

$$\Pi_2(\pi_A) = (\pi_A)_2 \times (\pi_A)_2, \quad \Pi_2(\pi_B) = (\pi_B)_2 \times (\pi_B)_2,$$

$$\Pi_2(\alpha) = \alpha_2 \times \alpha_2, \quad \text{and} \quad \Pi_2(\beta) = \beta_2 \times \beta_2.$$

Hence the inner square in $(**)$ is a pullback if and only if the square $(***)$ below



$(***)$

is a pullback. But $(***)$ is not a pullback in *Set*. The two 2-cells

$$\langle a_1, b_1 \rangle \circ \langle a_2, b_2 \rangle, \quad \text{and} \quad \langle a_1, b_2 \rangle \circ \langle a_2, b_1 \rangle$$

in $A \times B$ are different since they are compositions of different indets. On the other hand

$$(\pi_A)_2((a_1, b_1) \circ (a_2, b_2)) = a_1 \circ a_2 = (\pi_A)_2((a_1, b_2) \circ (a_2, b_1))$$

and

$$(\pi_B)_2((a_1, b_1) \circ (a_2, b_2)) = b_1 \circ b_2 = b_2 \circ b_1 = (\pi_B)_2((a_1, b_2) \circ (a_2, b_1))$$

i.e. they agree on both projections and hence $(***)$ is not a pullback. Thus $\widehat{\Pi_2}$ does not preserve binary products, as required. □

Theorem 3.2 *The category of computads* **Comp** *and the categories of n-computads* $\mathbf{Comp_n}$*, for $n > 2$, are not locally cartesian closed.*

Proof. The slice categories **Comp** $\downarrow 1_3$, as well as $\mathbf{Comp_n} \downarrow 1_3$, for $n > 2$, are equivalent to $\mathbf{Comp_3}$, where 1_3 is the terminal object in $\mathbf{Comp_3}$ lifted (by adding suitable identities) to the category of appropriate computads. As, by Lemma 3.1, $\mathbf{Comp_n} \downarrow 1_3$ is not cartesian closed we get the theorem. $\Box$

Remark. In particular the categories mentioned in the above theorem are not presheaf (or even elementary) toposes.

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