

BRA Working Group "GENTZEN"  
Group on TYPE RECONSTRUCTION  
List of Open Problems (Utrecht, March 1993)

1. Define the set  $T$  of *types*, ranged over by  $a, b$ , starting from a countable set  $C$  of type constants, in the following way:

$$C \subseteq T$$

$$a, b \in T \Rightarrow a \rightarrow b, a \wedge b \in T$$

Let  $E$  be a finite set of recursive equations between types, i.e.,

$$E = \{a_i = F_i(a_1, \dots, a_n) \mid 1 \leq i \leq n\}$$

where  $F_i(a_1, \dots, a_n)$  is a type containing the constants  $a_1, \dots, a_n$ .

The  $E$ -type assignment system consists of the following rules:

$$(ax) \frac{}{B \cup \{x : a\} \vdash_E x : a}$$

$$(\rightarrow I) \frac{B \cup \{x : a\} \vdash_E M : b}{B \vdash_E \lambda x. M : a \rightarrow b}$$

$$(\rightarrow E) \frac{B \vdash_E M : a \rightarrow b \quad B \vdash_E N : a}{B \vdash_E MN : b}$$

$$(\wedge I) \frac{B \vdash_E M : a \quad B \vdash_E M : b}{B \vdash_E M : a \wedge b}$$

$$(\wedge E) \frac{B \vdash_E M : a \wedge b}{B \vdash_E M : a(B \vdash_E M : b)}$$

$$(=) \frac{B \vdash_E M : a \quad a = b}{B \vdash_E M : b}$$

- Under what conditions an  $E$ -type assignment system has the (strong) normalization property (i.e.,  $B \vdash_E M : a \Rightarrow M$  is (strongly) normalizing)?

Example:

Both  $E = \{a_1 = a_2 \rightarrow a_1, a_2 = a_1 \rightarrow a_2\}$  and  $E' = \{a_1 = a_2 \rightarrow a_1 \wedge a_1 \rightarrow a_2, a_2 = a_1 \rightarrow a_2\}$  induce a strong normalizing system. Note that, while  $E$  is a set of monotone equations, the equations in  $E'$  are not monotone.

2. The inhabitation problem for the  $E$ -type assignment systems defined in the preceding point. Namely:

Given a type  $a$ , it is decidable if there is a closed term such that  $\emptyset \vdash_E M : a$ .

3. A recursive type  $\mu t. \sigma$  is *positively recursive* if and only if the type variable  $t$  occurs in the type  $\sigma$  in positive position. A type assignment system  $\vdash$  can be extended to positive recursion if its set of types is extended to contain positively recursive types and the set of rules is extended to contain the rules:

$$(unfold) \frac{B \vdash M : \mu t. \sigma}{B \vdash M : [\mu t. \sigma / t] \sigma}$$

$t_i$

$$(fold) \frac{B \vdash M : [\mu t. \sigma / t] \sigma}{B \vdash M : \mu t. \sigma}$$

It was proved by Urzyczyn that polymorphic type assignment system (PTA), when extended to positive recursion, is stronger than PTA itself. Namely, there are terms that cannot be typed in PTA while they can be typed in PTA plus positive recursion.

Is  $F_\omega$ , the higher order type assignment system, extended to positive recursion, stronger than  $F_\omega$ ?

4. Given a typed system, i.e., a type assignment system assigning types to typed terms, is there a combinatory version of it?

In particular, consider LF (Edinburgh Logical Framework) and the typed version of  $F_\omega$ .

5. Is there a typed version of the intersection type assignment system?
6. Consider the Curry type Assignment system, equipped with a decidable subtyping relation  $\leq$  on types, induced by an order relation on atomic types, and with the following rule:

$$\frac{B \vdash M : \sigma \quad \sigma \leq \tau}{B \vdash M : \tau}$$

. What is the complexity of the type inference?  
(partial results have been proved by Tiuryn).

7. Consider the following subtyping relation  $\leq$  on polymorphic types, defined by Mitchell:

$\forall t. \sigma \leq \forall u. [\tau / t] \sigma$  (if  $t$  is not free in  $\forall t. \sigma$ )

$\forall t. \sigma \rightarrow \tau \leq (\forall t. \sigma) \rightarrow \forall t. \tau$

$\sigma' \leq \sigma, \tau \leq \tau' \Rightarrow \sigma \rightarrow \tau \leq \sigma' \rightarrow \tau'$

$\sigma \leq \tau \leq \rho \Rightarrow \sigma \leq \rho$

$\sigma \leq \tau \Rightarrow \forall t. \sigma \leq \forall t. \tau$ .

The interest of this subtyping relation lies in the fact that the polymorphic type assignment system enriched with the rule:

$$\frac{B \vdash M : \sigma \quad \sigma \leq \tau}{B \vdash M : \tau}$$

has the property that typings are preserved by  $\eta$ -reduction.

Is this subtypings relation decidable?

8. Consider the typed system  $F_\leq$  introduced by Pierce in his thesis. Types in this system are built using as constructors  $\rightarrow$  and  $\wedge$ , and a subtyping relation  $\leq$  is defined on them. The original subtyping relation introduced by Pierce (after Ghelli) is undecidable. Recently Castagna proposed a decidable subtyping relation for this system.

What are the conditions on the  $\leq$  relation, in order to be decidable and to allow a type checking algorithm of tractable complexity?

9. Giannini and Ronchi defined a complete stratification of the polymorphic type assignment system, indexed by integers. Namely,  $\vdash_n$  ( $n \leq 0$ ) is such that  $B \vdash M : \sigma$  is a typing in the polymorphic type assignment system if and only if  $B \vdash_n M : \sigma$ , for some  $n$ . It turns out that  $\vdash_0$  is the Curry type assignment system, and  $\vdash_1$  is a system that can assign types to every normal form.

Find a characterization of  $\vdash_n$ , for more  $n$ .

The following problems were stated as open in the meeting of Utrecht, and have been solved:

1. The inhabitation problem for intersection type assignment system. Namely:  
Given a type  $\sigma$ , it is decidable if there is a closed term such that  $\emptyset \vdash M : \sigma$ ?  
(This problem was proved to be undecidable by Urzyczyn).
2. The type inference problem for polymorphic type assignment system. Namely:  
Given a term  $M$ , there are a basis  $B$  and a type  $\sigma$  such that  $B \vdash M : \sigma$ ?  
(This problem was proved to be undecidable by Wells).