Network Games

Piotr Sankowski
Uniwersytet Warszawski
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Outline

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Economic Games

Prisoner's dilemma:

- two members of criminal gang after being caught can choose to either say nothing, or to betray the other,
- if both say nothing, then both will stay 2 years in prison,
- if one betrays then he will get 1 year, but the other one will be imprisoned for 5 years,
- if both betray each other then both will stay in prison for 4 years.

Economic Games

It is profitable for both prisoners to betray the other.

To minimize the social cost both prisoners should remain silent.

Nevertheless from game theoretical perspective they will choose the solution with the highest cost and betray each other.

Economic Games

If there is no coordination, the players make decisions that optimize their own cost only, and the obtained solutions can be much worse then the optimal one.

- How bad can these solutions be?
- Can we propose a good stable solution to the players?
- Can be improve the mechanism of the game, e.g., by introducing taxes?
- ...?

Prisoner’s Dilemma and Reality

Consider two internet service providers (ISP), who need to forward their traffic.

In the case of outgoing transfer the ISP can choose where to route it, and this way influence the cost of other ISPs.

It is possible to construct Prisoner’s Dilemma this way.

Prisoner's Dilemma and Reality

However, in reality we mostly cope with the case where there are more than just two players.

In such cases similar dilemma is possible.

Examples:

- Pollution Game,
- Tragedy of the Commons.

Pollution Game

There are n countries playing this game.

Every country can choose to introduce pollution control.

Introducing pollution control costs 3 for every country.

Not introducing pollution control costs everyone 1.

In optimal solution of cost 3 every country controls pollution, but in the only stable solution of cost n no one is controlling pollution.

Tragedy of the Commons

In the above two examples our strategy does not depend on what others are doing.

We will show a game where our strategy depends on what others do.

During the exercise session we will discuss the Tragedy of the Commons game.

Coordination Games

Examples of the games that have many stable solutions.

Coordination game ”battle of the sexes”.

Anti-coordination game ”routing congestion game”.

Mixed Strategies

In all these examples there did exist pure (deterministic) stable states.

In the coin game, every players has a coin, which can be put head or tail up.

The 1st player wins where both coins are put in the same way.

The 2nd player wins when the coins are put differently.

Is stable state both players randomly with probability 1/2 choose one of the two strategies.

Simultaneous Move Games

The examples we have given so far belong to the class of *simultaneous move games*, i.e., games where players simultaneously choose their strategies.

Formally:

- the set of players contains n players $A = \{1, 2, \ldots, n\}$,
- the player i has his own set of strategies S_i,
- to play the player i chooses a strategy $s_i \in S_i$.

Simultaneous Move Games

We use $s = (s_1, \ldots, s_n)$ to denote the *vector of strategies* selected by players.

By $S = \times_i S_i$ we denote the set of all vectors of strategies.

The vector of strategies $s \in S$ determines the payoff of each player.

The results can be different for each player.

Simultaneous Move Games

We could define a transitive relation on the preference set of each player.

It is better to assign a numeric value to each outcome — depending on the situation this will be called gain or cost.

We denote these functions as $u_i : S \rightarrow \mathcal{R}$ for gain or $c_i : S \rightarrow \mathcal{R}$ for cost.

They can be used interchangeably because $u_i(s) = -c_i(s)$.

Simultaneous Move Games

In algorithmic game theory we need to define the way how the games are given to us.

One solution is to give a list of possible vectors of strategies and give their cost to each player.

Such form of a game is called *standard form* or *matrix form*.

Simultaneous Move Games

This definition is very useful when we have just two players.

However, in most interesting cases the games have exponential size when this ”natural” form is used.

In the pollution game we have 2^n possible strategy vectors.

Dominant Strategies

In the case of Prisoner's dilemma and pollution games every player has a single strategy that does no depend on what other players do.

When this is true we say that the game has a *dominat strategy*.

For a vector s, let s_i be the strategy chosen by player i, whereas by s_{-i} we denote the $(n-1)$ dimensional vector corresponding to the strategies of all other players.

Dominant Strategies

Let $u_i(s)$ be the gain of player i, we denote gain as $u_i(s_i, s_{-i})$ as well.

Formally the strategy vector $s \in S$ is *dominating* when for every player i and for every vector $s' \in S$ we have

$$u_i(s_i, s'_{-i}) \geq u_i(s'_i, s'_{-i}).$$

The dominating strategy does not need to be optimal.

We design auctions in such a way that they have dominat strategies.

Pure Nash Equilibrium

However, games very rarely have dominat strategies, so we want to introduce a weaker concept of stable solutions.

Nash Equilibrium is a stable solution, where each player maximize his own (selfish) goal.

These are solutions, where no player can gain anything by changing his strategy.

Pure Nash Equilibrium

The vector s is called *Nash equilibrium* if for every player i and for every alternative vector $s' \in S$ we have:

$$u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}).$$

The dominat strategy is a Nash equilibrium as well.

If it is strictly dominating then it is the unique Nash equilibrium.

Games can have many different equilibria.

Mixed Nash Equilibrium

The above equilibrium is called *pure*, because every player chooses deterministically his strategy.

Not every game has such equilibrium, e.g., the coin game.

In such case the players maximize their expected gain.

We assume players are not risk averse, i.e., that maximizing expectation is their goal.

Mixed Nash Equilibrium

Formally, every player chooses a probability distribution over his strategies.

This distribution is called *mixed strategy*.

We assume that players choose their strategies independently.

This gives a probability distribution over the set of possible outcomes s.

Mixed Nash Equilibrium

Nash in 1951 proved the following theorem.

Theorem 1 *Every game with finite number of players and finite number of strategies has a mixed Nash equilibria.*

Inefficiency of Equilibria

The goal of this lecture is to answer the question how bad a Nash equilibrium can be?

Equilibrium in Prisoner's Dilemma seem to be much worse than the case when prisoners do not betray.

This is the case with respect to the most reasonable social cost functions.

Inefficiency of Equilibria

The most widely used cost functions are:

- *utilitarian* – the sum of all costs,
- *egalitarian* – the maximum cost.

Nash equilibrium does not minimize any of the above two functions.

Introduction of these functions allows us to measure inefficiency of equilibrium and allow us to say how well equilibrium approximates the optimum.

Inefficiency of Equilibria

Price of anarchy (PoA) is the most important measure of this inefficiency.

PoA is defined as the ratio of the cost of the worst Nash equilibria to the cost of optimum solution.

$$PoA = \max_{Nash} \frac{c(Nash)}{c(Opt)}$$

PoA depends not only on the game, but on the cost function.

Inefficiency of Equilibria

The game that has many equilibria can have very high PoA even when just one of them is very inefficient.

In such case we can measure *the price of stability*, which is defined as a ratio of the cost of the best Nash equilibria to the cost of the optimum

$$PoS = \max_{Nash} \frac{c(Nash)}{c(Opt)}$$

This is the cost of the best stable solution we can propose to the players.

Nonatomic selfish flow/routing

Consider a *multicommodity flow network* given by a graph $G = (V, E)$ together with a set of source-sink pairs $(s_1, t_1), \ldots, (s_k, t_k)$.

Each such pair is called *commodity* and controlled by one player.

Let $\mathcal{P}_i$ be the set of $s_i - t_i$ paths in G.

We assume that $\mathcal{P}_i \neq \emptyset$ and denote by $\mathcal{P} = \bigcup_{i=1}^{k} \mathcal{P}_i$.

Nonatomic selfish flow

The paths chosen by the players are described by a *flow*,i.e., a vector indexed by elements of $\mathcal{P}$.

For a flow f and path $P \in \mathcal{P}_i$ we interpret f_P as the amount of the commodity i that is send from s_i to t_i on P.

Every player needs to send r_i units of the commodity.

Nonatomic selfish flow

The flow *satisfies* the vector r when every commodity is send in full

$$\forall_i \sum_{P \in \mathcal{P}_i} f_P = r_i.$$

We do not assign any capacities to the edges.

However, every edge e has a *cost function* $c_e : \mathcal{R}^+ \to \mathcal{R}^+$.

We assume that the cost are nonnegative, continuous and nondecreasing.

Nonatomic selfish flow

Let us now formalize the equilibrium concept for these *nonatomic* flow games.

Let us define the cost of a path P with respect to the flow f as the sum of cost of edges:

$$c_P(f) = \sum_{e \in P} c_e(f_e),$$

where

$$f_e = \sum_{P \in \mathcal{P} : e \in P} f_P,$$

denotes the flow on edge e.

Nonatomic selfish flow

Definition 2 *Let f be a flow that satisfies r. f is a nonatomic flow equilibrium if for every commodity i and every pair $P, P' \in \mathcal{P}_i$ of paths from s_i to t_i such that $f_P > 0$ we have:*

$$c_P(f) \leq c_{P'}(f).$$

In other words, all paths used in the equilibrium flow have the smallest possible cost (for give source-sink pair and given flow f).

Nonatomic selfish flow

We will consider the utilitarian cost defined as:

$$C(f) = \sum_{P \in \mathcal{P}} c_P(f) f_P.$$

After changing the order of summation we get:

$$C(f) = \sum_{e \in E} c_e(f_e) f_e.$$

We say that the flow f is *optimal* if is minimizes this cost.

Nonatomic selfish flow

The cost of anarchy is the ratio between the cost of the worst equilibrium flow to the cost of optimal flow:

$$KA = \max_{f \texttt{equlibirum}} \frac{c(f)}{c(f_{Opt})}$$

Pigou – $KA = \frac{4}{3}$.

Braess – $KA = \frac{4}{3}$.

Atomic Selfish Flow

The *atomic selfish flow* game is given by the graph $G = (V, E)$, the set source-sink pairs $(s_1, t_1), \ldots, (s_k, t_k)$, positive values r_i of the flows, and nonnegative, continuous, nondecreasing cost functions c_e for every edge e.

The difference between nonatomic and atomic games:

- in the nonatomic game the flow can be split in an arbitrary way and each infinitesimal amount of flow is controlled by a different player,
- in atomic games the flows represent the players and the flow needs to be send on a single path.

Atomic Selfish Flow

In the language of simultaneous games we have k players, one player for each pair $s_i - t_i$.

The set of strategies of player i is the set of paths $\mathcal{P}_i$, that can be used to send r_i units of flow.

Now the flow f is a vector indexed by both paths and players.

Let $f_P^{(i)}$ denote the flow that is send by player i on path P.

Atomic Selfish Flow

The flow f satisfies r when for each player i, $f_P^{(i)}$ is equal to r_i for exactly one path in $\mathcal{P}_i$ and is 0 for all other paths.

The cost of the flow $C(f)$ is defined in the same way as in the nonatomic games:

$$C(f) = \sum_{P \in \mathcal{P}} c_P(f) f_P = \sum_{e \in E} c_e(f_e) f_e.$$

Atomic Selfish Flow

Definition 3 *Let f be the flow satisfying r. f is an atomic equilibrium flow if for every player i and every pair $P, P' \in \mathcal{P}_i$ of paths from s_i to t_i such that $f_P^{(i)} > 0$ we have:*

$$c_P(f) \leq c_{P'}(f'),$$

where f' is identical to f with the exception that ${f'}_P^{(i)} = 0$ and ${f'}_{P'}^{(i)} = r_i$.

This definition corresponds to pure Nash equilibrium.

Atomic Selfish Flow

AAE example - on the blackboard.

This example shows that PoA is at least $\frac{5}{2}$.

There exist cases when pure Nash equilibrium does not exists, e.g., when general costs are allowed.

Nonatomic flows: existence

We will prove that nonatomic equilibrium flows allays exists, and all have equal cost.

In other words, we will show that $PoA = PoS$.

Theorem 4 *Let* (G, r, c) *be the nonatomic flow game, then:*

(a) *there exists an equilibrium flow in* (G, r, c)*,*

(b) *if* f *and* f' *are equilibria then* $c_e(f_e) = c_e(f'_e)$ *for every edge* e*.*

Nonatomic flows: existence

We are going to use the *potential function method*.

We will define a function over the state of the game, such that its minima will correspond to equilibrium flows.

Changes of the value of the function will correspond to changes of players cost when they change their strategy.

However, first we are going to characterize optimal flows.

Nonatomic flows: existence

Assume that $x \cdot c_e(x)$ is a convex differentiable function.

$x \cdot c_e(x)$ is the total contribution of e to the social cost.

The marginal cost function for edge e is defined as

$$c_e^*(x) = (x \cdot c_e(x))' = c_e(x) + x \cdot c_e'(x).$$

Let $c_P^*(f) = \sum_{e \in P} c_e^*(f)$ be the sum of marginal costs on path P.

Nonatomic flows: existence

Theorem 5 *f^* is the optimal flow for (G, r, c) iff for every $i \in \{1, \ldots, n\}$ and every pair $P, P' \in \mathcal{P}_i$ of paths such that $f^*_P > 0$ we have:*

$$c^*_P(f^*) \leq c^*_{P'}(f^*).$$

This is a consequence of convex flow properties.

We skip this proof as it is too "algorithmic".

Nonatomic flows: existence

From this theorem and the definition of equilibria flow we immediately obtain.

Corollary 6 *Let* (G, r, c) *be a nonatomic flow game such that for every edge the function* $x \cdot c_e(x)$ *is convex and differentiable. Then* f^* *is the optimal flow for* (G, r, c) *iff it is the equilibrium flow in* (G, r, c^*).

Pigou example.

Nonatomic flows: existence

In order to construct the potential function let us "flip" the above corollary and consider for which functions equilibria flows are optimal.

We need to find a function $h_e(x)$ that will play the role of $x \cdot c_e(x)$, i.e., $h'_e(x) = c_e(x)$.

We obtain $h_e(x) = \int_0^x c_e(y)dy$ for every edge e.

Nonatomic flows: existence

The function $\Phi(f)$ defined as:

$$\Phi(f) = \sum_{e \in E} \int_0^{f_e} c_e(x)dx$$

is the *potential function* for the game (G, r, c).

By applying Theorem 5 to $h_e(x)$ instead of $x \cdot c_e(x)$ we otain:

Lemma 7 *The flow f is the equilibrium flow in (G, r, c) iff when it minimizes the function $\Phi(f)$.*

Nonatomic flows: existence

Proof of Theorem 4.

The set of possible flows is a compact set.

The potential function is continuous.

From Weiersrass's theorem we know that it has a minimum.

From Lemma 7 we obtain that these minima are equilibria flows, what proves (a).

Nonatomic flows: existence

In order to prove (b) we first notice that $\Phi(f)$ is convex.

Let f and f' be two equilibria flows.

We know that both of them minimize the potential function.

Consider the flow of the form $\lambda f + (1-\lambda)f'$ for $\lambda \in [0,1]$.

Nonatomic flows: existence

From the convexity of $\Phi(f)$ we obtain:

$$\Phi(\lambda f + (1-\lambda)f') \leq \lambda\Phi(f) + (1-\lambda)\Phi(f').$$

The right side is equal to the minimum value of Φ, so this inequality needs to hold with equality.

This can be true only when every term $\int_0^x c_e(y)dy$ in the sum in Φ is linear.

This in turn means that c_e is constant between f and f'.

Atomic flows: existence

In general atomic games the equilibria flows do not need to exist.

It is possible to prove their existence in the case when every r_i is the same.

Theorem 8 *Let* (G, r, c) *be an atomic flow game such that* $r_i = 1$*, then there exist an equilibrium flow in* (G, r, c)*.*

During exercise session we will prove this for any r but linear cost functions.

Atomic flows: existence

Let us discretize the potential function:

$$\Phi_a(f) = \sum_{e \in E} \sum_{i=1}^{f_e} c_e(i).$$

The game is finite so the set of possible strategy vectors is finite as well. Hence $\Phi(a)$ has a minimum.

Let f be the global minimum of Φ_a.

Atomic flows: existence

We will prove that f is equilibrium flow.

Let us assume by contradiction that in f the player i can decrease his cost by changing his path from P to P'. Let f' be the new flow.

We have:

$$0 > c_{P'}(f') - c_P(f) =$$

$$= \sum_{e \in P'-P} c_e(f_e + 1) - \sum_{e \in P-P'} c_e(f_e) =$$

$$= \Phi_a(f') - \Phi_a(f),$$

what contradicts the minimality of f.

Nonatomic flow: PoA

We want to prove that PoA depends only on the cost function but not on:

- size of the network,
- structure of the network,
- number of commedities.

We will prove that there exists a uniform bound on PoA that depends only on the degree of nonlinearity of the cost function.

Intuitively the potential function is an approximation of the cost function.

Nonatomic flow: PoA

Theorem 9 *Let (G, r, c) be a nonatomic flow game and let $x \cdot c_e(x) \leq \gamma \cdot \int_0^x c_e(y)dy$ for every $e \in E$ and $x > 0$, then PoA is at most γ.*

Let f be the equilibrium and f^* be the optimal flow then:

$$C(f) \leq \gamma\Phi(f) \leq \gamma\Phi(f^*) \leq \gamma C(f^*),$$

because $\int_0^x c_e(y) \leq x \cdot c_e(x)$.

For polynomials of degree p we have $\gamma = p + 1$.

Nonatomic flow: PoA

Definition 10 *Let C be the set of cost functions, Pigou bound $\alpha(C)$ is defined as:*

$$\alpha(C) = \sup_{c \in C} \sup_{x,r \geq 0} \frac{r \cdot c(r)}{x \cdot c(x) + (r - x)c(r)}.$$

Theorem 11 *For a nonatomic game (G, r, c) with cost functions C:*

- *PoA is bounded by $\alpha(C)$,*
- *there exist examples where PoA is equal to $\alpha(C)$.*

We will prove the second part during the exercise session.

Atomic flow: PoA

In the atomic case we cannot use the same proof technique because the flows are nonunique.

For simplicity let us consider only linear cost functions, i.e., $c_e(x) = a_e x + b_e$ for $a_e, b_e \geq 0$.

We will prove:

Theorem 12 *If (G, r, c) is an atomic flow game with linear cost functions then PoA is bounded by* $(3 + \sqrt{5})/2 \approx 2.618$.

Atomic flow: PoA

The following lemma is a simple consequence of the definition of the equilibrium flow.

Lemma 13 *Let f be the equilibrium and f^* be the optimal flow, assume that player i is using the path P_i in f, and the path P_i^* in f^*, then:*

$$\sum_{e \in P_i} a_e f_e + b_e \leq \sum_{e \in P_i^*} a_e (f_e + r_i) + b_e.$$

Lemma 14 $C(f) \leq C(f^*) + \sum_{e \in E} a_e f_e f_e^*.$

Atomic flow: PoA

Let us multiply inequalities from Lemma 13 by r_i, then:

$$C(f) \leq \sum_{i=1}^{k} r_i \left(\sum_{e \in P_i^*} a_e(f_e + r_i) + b_e \right) \leq$$

$$\leq \sum_{i=1}^{k} r_i \left(\sum_{e \in P_i^*} a_e(f_e + f_e^*) + b_e \right) =$$

$$= \sum_{e \in E} [a_e(f_e + f_e^*) + b_e] f_e^* = C(f^*) + \sum_{e \in E} a_e f_e f_e^*.$$

Atomic flow: PoA

Theorem 15 *If* (G, r, c) *is an atomic flow game with linear cost functions then PoA is bounded by* $(3+\sqrt{5})/2 \approx 2.618$.

By applying the Cauchy-Schwarz inequality to vectors $\{\sqrt{a_e} f_e\}_{e \in E}$ and $\{\sqrt{a_e} f_e^*\}_{e \in E}$ we obtain:

$$\sum_{e \in E} a_e f_e f_e^* \leq \sqrt{\sum_{e \in E} a_e f_e^2} \cdot \sqrt{\sum_{e \in E} a_e (f_e^*)^2} \leq$$

$$\leq \sqrt{C(f)} \cdot \sqrt{C(f^*)}.$$

Atomic flow: PoA

By combining this inequality with the one in Lemma 14 and after dividing by $C(f^*)$ we obtain:

$$\frac{C(f)}{C(f^*)} - 1 \leq \sqrt{\frac{C(f)}{C(f^*)}}.$$

by squaring both sides and solving the inequality $x^2 - 3x + 1 \leq 0$ we get:

$$\frac{C(f)}{C(f^*)} \leq \frac{3+\sqrt{5}}{2} \approx 2.618.$$

For polynomials one obtain $\sim p^p$.