Royen's proof of the Gaussian correlation inequality

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Abstract

We present in detail Thomas Royen's proof of the Gaussian correlation inequality which states that $\mu(K \cap L) \geq \mu(K)\mu(L)$ for any centered Gaussian measure μ on $\mathbb{R}^d$ and symmetric convex sets K, L in $\mathbb{R}^d$.

1 Introduction

The aim of this note is to present in a self contained way the beautiful proof of the Gaussian correlation inequality, due to Thomas Royen [7]. Although the method is rather simple and elementary, we found the original paper not too easy to follow. One of the reasons behind it is that in [7] the correlation inequality was established for more general class of probability measures. Moreover, the author assumed that the reader is familiar with properties of certain distributions and may justify some calculations by herself/himself. We decided to reorganize a bit Royen's proof, restrict it only to the Gaussian case and add some missing details. We hope that this way a wider readership may appreciate the remarkable result of Royen.

The statement of the Gaussian correlation inequality is as follows.

Theorem 1. *For any closed symmetric sets K, L in $\mathbb{R}^d$ and any centered Gaussian measure μ on $\mathbb{R}^d$ we have*

$$\mu(K \cap L) \geq \mu(K)\mu(L). \tag{1}$$

For $d = 2$ the result was proved by Pitt [5]. In the case when one of the sets K, L is a symmetric strip (which corresponds to $\min\{n_1, n_2\} = 1$ in Theorem 2 below) inequality (1) was established independently by Khatri [3] and Šidák [9]. Hargé [2] generalized the Khatri-Šidak result to the case when one of the sets is a symmetric ellipsoid. Some other partial results may be found in papers of Borell [1] and Schechtman, Schlumprecht and Zinn [8].

Up to our best knowledge Thomas Royen was the first to present a complete proof of the Gaussian correlation inequality. Some other recent attempts may be found in [4] and [6], however both papers are very long and difficult to check. The first version of [4], placed on the arxiv before Royen's paper, contained a fundamental mistake (Lemma 6.3 there was wrong).

Since any symmetric closed set is a countable intersection of symmetric strips, it is enough to show (1) in the case when

$$K = \{x \in \mathbb{R}^d\colon\ \forall_{1\le i\le n_1}\ |\langle x, v_i\rangle| \le t_i\} \quad \text{and} \quad L = \{x \in \mathbb{R}^d\colon\ \forall_{n_1+1\le i\le n_1+n_2}\ |\langle x, v_i\rangle| \le t_i\},$$

where v_i are vectors in $\mathbb{R}^d$ and t_i nonnegative numbers. If we set $n = n_1+n_2$, $X_i := \langle v_i, G\rangle$, where G is the Gaussian random vector distributed according to μ, we obtain the following equivalent form of Theorem 1.

Theorem 2. *Let $n = n_1 + n_2$ and X be an n-dimensional centered Gaussian vector. Then for any $t_1, \ldots, t_n > 0$,*

$$\begin{aligned}\mathbb{P}(&|X_1| \le t_1, \ldots, |X_n| \le t_n)\\ &\ge \mathbb{P}(|X_1| \le t_1, \ldots, |X_{n_1}| \le t_{n_1})\mathbb{P}(|X_{n_1+1}| \le t_{n_1+1}, \ldots, |X_n| \le t_n).\end{aligned}$$

Remark 3. i) The standard approximation argument shows that the Gaussian correlation inequality holds for centered Gaussian measures on separable Banach spaces.
ii) Theorem 1 has the following functional form:

$$\int_{\mathbb{R}^d} fg d\mu \ge \int_{\mathbb{R}^d} f d\mu \int_{\mathbb{R}^d} g d\mu$$

for any centered Gaussian measure μ on $\mathbb{R}^d$ and even functions $f, g\colon \mathbb{R}^d \to [0, \infty)$ such that sets $\{f \ge t\}$ and $\{g \ge t\}$ are convex for all $t > 0$.
iii) Thomas Royen established Theorem 2 for a more general class of random vectors X such that $X^2 = (X_1^2, \ldots, X_n^2)$ has an n-variate gamma distribution with appropriately chosen parameters (see [7] for details).

Notation. By $\mathcal{N}(0, C)$ we denote the centered Gaussian measure with the covariance matrix C. We write $M_{n\times m}$ for a set of $n \times m$ matrices and $|A|$ for the determinant of a square matrix A. For a matrix $A = (a_{ij})_{i,j\le n}$ and $J \subset [n] := \{1, \ldots, n\}$ by A_J we denote the square matrix $(a_{ij})_{i,j\in J}$ and by $|J|$ the cardinality of J.

2 Proof of Theorem 2

Without loss of generality we may and will assume that the covariance matrix C of X is nondegenerate (i.e. positive-definite). We may write C as

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix},$$

where C_{ij} is the $n_i \times n_j$ matrix. Let

$$C(\tau) := \begin{pmatrix} C_{11} & \tau C_{12} \\ \tau C_{21} & C_{22} \end{pmatrix}, \quad 0 \le \tau \le 1.$$

Set $Z_i(\tau) := \frac{1}{2}X_i(\tau)^2$, $1 \le i \le n$, where $X(\tau) \sim \mathcal{N}(0, C(\tau))$.

We may restate the assertion as

$$\mathbb{P}(Z_1(1) \le s_1, \ldots, Z_n(1) \le s_n) \ge \mathbb{P}(Z_1(0) \le s_1, \ldots, Z_n(0) \le s_n),$$

where $s_i = \frac{1}{2}t_i^2$. Therefore it is enough to show that the function

$$\tau \mapsto \mathbb{P}(Z_1(\tau) \le s_1, \ldots, Z_n(\tau) \le s_n) \text{ is nondecreasing on } [0,1].$$

Let $f(x,\tau)$ denote the density of the random vector $Z(\tau)$ and $K = [0,s_1] \times \cdots \times [0,s_n]$. We have

$$\frac{\partial}{\partial\tau}\mathbb{P}(Z_1(\tau) \le s_1, \ldots, Z_n(\tau) \le s_n) = \frac{\partial}{\partial\tau}\int_K f(x,\tau)dx = \int_K \frac{\partial}{\partial\tau}f(x,\tau)dx,$$

where the last equation follows by Lemma 6 applied to $\lambda_1 = \ldots = \lambda_n = 0$. Therefore it is enough to show that $\int_K \frac{\partial}{\partial\tau}f(x,\tau) \ge 0$.

To this end we will compute the Laplace transform of $\frac{\partial}{\partial\tau}f(x,\tau)$. By Lemma 6, applied to $K = [0,\infty)^n$, we have for any $\lambda_1 \ldots, \lambda_n \ge 0$,

$$\int_{[0,\infty)^n} e^{-\sum_{i=1}^n \lambda_i x_i}\frac{\partial}{\partial\tau}f(x,\tau)dx = \frac{\partial}{\partial\tau}\int_{[0,\infty)^n} e^{-\sum_{i=1}^n \lambda_i x_i}f(x,\tau)dx.$$

However by Lemma 4 we have

$$\int_{[0,\infty)^n} e^{-\sum_{i=1}^n \lambda_i x_i}f(x,\tau)dx = \mathbb{E}\exp\left(-\frac{1}{2}\sum_{i=1}^n \lambda_i X_i^2(\tau)\right) = |I + \Lambda C(\tau)|^{-1/2},$$

where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$.

Formula (2) below yields

$$|I + \Lambda C(\tau)| = 1 + \sum_{\emptyset \ne J \subset [n]} |(\Lambda C(\tau))_J| = 1 + \sum_{\emptyset \ne J \subset [n]} |C(\tau)_J| \prod_{j \in J}\lambda_j.$$

Fix $\emptyset \ne J \subset [n]$. Then $J = J_1 \cup J_2$, where $J_1 := [n_1] \cap J$, $J_2 := J \setminus [n_1]$ and $C(\tau)_J = \begin{pmatrix} C_{J_1} & \tau C_{J_1 J_2} \\ \tau C_{J_2 J_1} & C_{J_2} \end{pmatrix}$. If $J_1 = \emptyset$ or $J_2 = \emptyset$ then $C(\tau)_J = C_J$, otherwise by (3) we get

$$\begin{aligned}|C(\tau)_J| &= |C_{J_1}||C_{J_2}|\left|I_{|J_1|} - \tau^2 C_{J_1}^{-1/2}C_{J_1J_2}C_{J_2}^{-1}C_{J_2J_1}C_{J_1}^{-1/2}\right| \\ &= |C_{J_1}||C_{J_2}|\prod_{i=1}^{|J_1|}(1 - \tau^2\mu_{J_1,J_2}(i)),\end{aligned}$$

where $\mu_{J_1,J_2}(i)$, $1 \le i \le |J_1|$ denote the eigenvalues of $C_{J_1}^{-1/2} C_{J_1 J_2} C_{J_2}^{-1} C_{J_2 J_1} C_{J_1}^{-1/2}$ (by (4) they belong to $[0,1]$). Thus for any $\emptyset \neq J \subset [n]$ and $\tau \in [0,1]$ we have

$$a_J(\tau) := -\frac{\partial}{\partial \tau}|C(\tau)_J| \ge 0.$$

Therefore

$$\begin{aligned}\frac{\partial}{\partial \tau}|I+\Lambda C(\tau)|^{-1/2} &= -\frac{1}{2}|I+\Lambda C(\tau)|^{-3/2}\sum_{\emptyset\neq J\subset[n]}\frac{\partial}{\partial\tau}|C(\tau)_J||\Lambda_J| \\ &= \frac{1}{2}|I+\Lambda C(\tau)|^{-3/2}\sum_{\emptyset\neq J\subset[n]} a_J(\tau)\prod_{j\in J}\lambda_j.\end{aligned}$$

We have thus shown that

$$\int_{[0,\infty)^n} e^{-\sum_{i=1}^n \lambda_i x_i}\frac{\partial}{\partial\tau} f(x,\tau)dx = \sum_{\emptyset\neq J\subset[n]}\frac{1}{2}a_J(\tau)|I+\Lambda C(\tau)|^{-3/2}\prod_{j\in J}\lambda_j.$$

Let $h_\tau := h_{3,C(\tau)}$ be the density function on $(0,\infty)^n$ defined by (5). By Lemmas 8 and 7 iii) we know that

$$|I+\Lambda C(\tau)|^{-3/2}\prod_{j\in J}\lambda_j = \int_{(0,\infty)^n} e^{-\sum_{i=1}^n \lambda_i x_i}\frac{\partial^{|J|}}{\partial x_J}h_\tau.$$

This shows that

$$\frac{\partial}{\partial\tau}f(x,\tau) = \sum_{\emptyset\neq J\subset[n]}\frac{1}{2}a_J(\tau)\frac{\partial^{|J|}}{\partial x_J}h_\tau(x).$$

Finally recall that $a_J(\tau)\ge 0$ and observe that by Lemma 7 ii),

$$\lim_{x_i\to 0+}\frac{\partial^{|I|}}{\partial x_I}h_\tau(x) = 0 \quad \text{ for } i \notin I \subset [n],$$

thus

$$\int_K \frac{\partial^{|J|}}{\partial x_J}h_\tau(x)dx = \int_{\prod_{j\in J^c}[0,s_j]} h_\tau(s_J, x_{J^c})dx_{J^c} \ge 0,$$

where $J^c = [n]\setminus J$ and $y = (s_J, x_{J^c})$ if $y_i = s_i$ for $i \in J$ and $y_i = x_i$ for $i \in J^c$. $\square$

3 Auxiliary Lemmas

Lemma 4. *Let X be an n dimensional centered Gaussian vector with the covariance matrix C. Then for any $\lambda_1,\dots,\lambda_n \ge 0$ we have*

$$\mathbb{E}\exp\left(-\sum_{i=1}^n \lambda_i X_i^2\right) = |I_n + 2\Lambda C|^{-1/2},$$

where $\Lambda := \mathrm{diag}(\lambda_1,\dots,\lambda_n)$.

Proof. Let A be a symmetric positive-definite matrix. Then $A = UDU^T$ for some $U \in O(n)$ and $D = \mathrm{diag}(d_1, d_2, \dots, d_n)$. Hence

$$\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle)dx = \int_{\mathbb{R}^n} \exp(-\langle Dx, x\rangle)dx = \prod_{k=1}^{n} \sqrt{\frac{\pi}{d_k}} = \pi^{n/2}|D|^{-1/2} = \pi^{n/2}|A|^{-1/2}.$$

Therefore for a canonical Gaussian vector $Y \sim \mathcal{N}(0, I_n)$ and a symmetric matrix B such that $2B < I_n$ we have

$$\begin{aligned}\mathbb{E}\exp(\langle BY, Y\rangle) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \exp\left(-\left\langle \left(\frac{1}{2}I_n - B\right) x, x \right\rangle\right) dx = 2^{-n/2} \left|\frac{1}{2}I_n - B\right|^{-1/2} \\ &= |I_n - 2B|^{-1/2}.\end{aligned}$$

We may represent $X \sim \mathcal{N}(0, C)$ as $X \sim AY$ with $Y \sim \mathcal{N}(0, I_n)$ and $C = AA^T$. Thus

$$\begin{aligned}\mathbb{E}\exp\left(-\sum_{i=1}^{n} \lambda_i X_i^2\right) &= \mathbb{E}\exp(-\langle \Lambda X, X\rangle) = \mathbb{E}\exp(-\langle \Lambda AY, AY\rangle) = \mathbb{E}\exp(-\langle A^T\Lambda AY, Y\rangle) \\ &= |I_n + 2A^T\Lambda A|^{-1/2} = |I_n + 2\Lambda C|^{-1/2},\end{aligned}$$

where to get the last equality we used the fact that $|I_n + A_1A_2| = |I_n + A_2A_1|$ for $A_1, A_2 \in M_{n\times n}$. $\square$

Lemma 5. *i) For any matrix $A \in M_{n\times n}$,*

$$|I_n + A| = 1 + \sum_{\emptyset \neq J \subset [n]} |A_J|. \tag{2}$$

ii) Suppose that $n = n_1 + n_2$ and $A \in M_{n\times n}$ is symmetric and positive-definite with a block representation $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, where $A_{ij} \in M_{n_i \times n_j}$. Then

$$|A| = |A_{11}||A_{22}| \left| I_{n_1} - A_{11}^{-1/2} A_{12} A_{22}^{-1} A_{21} A_{11}^{-1/2} \right|. \tag{3}$$

Moreover,

$$0 \leq A_{11}^{-1/2} A_{12} A_{22}^{-1} A_{21} A_{11}^{-1/2} \leq I_{n_1}. \tag{4}$$

Proof. i) This formula may be verified in several ways – e.g. by induction on n or by using the Leibniz formula for the determinant.

ii) We have

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11}^{1/2} & 0 \\ 0 & A_{22}^{1/2} \end{pmatrix} \begin{pmatrix} I_{n_1} & A_{11}^{-1/2} A_{12} A_{22}^{-1/2} \\ A_{22}^{-1/2} A_{21} A_{11}^{-1/2} & I_{n_2} \end{pmatrix} \begin{pmatrix} A_{11}^{1/2} & 0 \\ 0 & A_{22}^{1/2} \end{pmatrix}$$

and

$$\left|\begin{pmatrix} I_{n_1} & A_{11}^{-1/2}A_{12}A_{22}^{-1/2} \\ A_{22}^{-1/2}A_{21}A_{11}^{-1/2} & I_{n_2} \end{pmatrix}\right| = \left|\begin{pmatrix} I_{n_1} - A_{11}^{-1/2}A_{12}A_{22}^{-1}A_{21}A_{11}^{-1/2} & 0 \\ A_{22}^{-1/2}A_{21}A_{11}^{-1/2} & I_{n_2} \end{pmatrix}\right|$$
$$= \left|I_{n_1} - A_{11}^{-1/2}A_{12}A_{22}^{-1}A_{21}A_{11}^{-1/2}\right|.$$

To show the last part of the statement notice that $A_{11}^{-1/2}A_{12}A_{22}^{-1}A_{21}A_{11}^{-1/2} = B^TB \geq 0$, where $B := A_{22}^{-1/2}A_{21}A_{11}^{-1/2}$. Since A is positive-definite, for any $t \in \mathbb{R}$, $x \in \mathbb{R}^{n_1}$ and $y \in \mathbb{R}^{n_2}$ we have $t^2\langle A_{11}x, x\rangle + 2t\langle A_{21}x, y\rangle + \langle A_{22}y, y\rangle \geq 0$. This implies $\langle A_{21}x, y\rangle^2 \leq \langle A_{11}x, x\rangle\langle A_{22}y, y\rangle$. Replacing x by $A_{11}^{-1/2}x$ and y by $A_{22}^{-1/2}y$ we get $\langle Bx, y\rangle^2 \leq |x|^2|y|^2$. Choosing $y = Bx$ we get $\langle B^TBx, x\rangle \leq |x|^2$, i.e. $B^TB \leq I_{n_1}$. □

Lemma 6. *Let $f(x,\tau)$ be the density of the random vector $Z(\tau)$ defined above. Then for any Borel set K in $[0,\infty)^n$ and any $\lambda_1, \ldots, \lambda_n \geq 0$,*

$$\int_K e^{-\sum_{i=1}^n \lambda_i x_i} \frac{\partial}{\partial \tau} f(x,\tau)dx = \frac{\partial}{\partial \tau} \int_K e^{-\sum_{i=1}^n \lambda_i x_i} f(x,\tau)dx.$$

Proof. The matrix C is nondegenerate, therefore matrices C_{11} and C_{22} are nondegerate and $C(\tau)$ is nondegenerate for any $\tau \in [0,1]$. Random vector $X(\tau) \sim \mathcal{N}(0, C(\tau))$ has the density $|C(\tau)|^{-1/2}(2\pi)^{-n/2}\exp(-\frac{1}{2}\langle C(\tau)^{-1}x, x\rangle)$. Standard calculation shows that $Z(\tau)$ has the density

$$f(x,\tau) = |C(\tau)|^{-1/2}(4\pi)^{-n/2}\frac{1}{\sqrt{x_1 \cdots x_n}} \sum_{\varepsilon \in \{-1,1\}^n} e^{-\langle C(\tau)^{-1}\varepsilon\sqrt{x}, \varepsilon\sqrt{x}\rangle}\mathbf{1}_{(0,\infty)^n}(x),$$

where for $\varepsilon \in \{-1,1\}^n$ and $x \in (0,\infty)^n$ we set $\varepsilon\sqrt{x} := (\varepsilon_i\sqrt{x_i})_i$.

The function $\tau \mapsto |C(\tau)|^{-1/2}$ is smooth on $[0,1]$, in particular

$$\sup_{\tau \in [0,1]} |C(\tau)|^{-1/2} + \sup_{\tau \in [0,1]} \frac{\partial}{\partial \tau}|C(\tau)|^{-1/2} =: M < \infty.$$

Since $C(\tau) = \tau C(1) + (1-\tau)C(0)$ we have $\frac{\partial}{\partial \tau}C(\tau) = C(1) - C(0)$ and

$$\frac{\partial}{\partial \tau} e^{-\langle C(\tau)^{-1}\varepsilon\sqrt{x}, \varepsilon\sqrt{x}\rangle} = -\langle C(\tau)^{-1}(C(1) - C(0))C(\tau)^{-1}\varepsilon\sqrt{x}, \varepsilon\sqrt{x}\rangle e^{-\langle C(\tau)^{-1}\varepsilon\sqrt{x}, \varepsilon\sqrt{x}\rangle}.$$

The continuity of the function $\tau \mapsto C(\tau)$ gives

$$\langle C(\tau)^{-1}\varepsilon\sqrt{x}, \varepsilon\sqrt{x}\rangle \geq a\langle \varepsilon\sqrt{x}, \varepsilon\sqrt{x}\rangle = a\sum_{i=1}^n |x_i|$$

and

$$|\langle C(\tau)^{-1}(C(1)-C(0))C(\tau)^{-1}\varepsilon\sqrt{x},\varepsilon\sqrt{x}\rangle| \le b\langle\varepsilon\sqrt{x},\varepsilon\sqrt{x}\rangle = b\sum_{i=1}^n |x_i|$$

for some $a>0$, $b<\infty$. Hence for $x\in(0,\infty)^n$

$$\sup_{\tau\in[0,1]}\left|\frac{\partial}{\partial\tau}f(x,\tau)\right| \le g(x) := M\pi^{-n/2}\frac{1}{\sqrt{x_1\cdots x_n}}\left(1+b\sum_{i=1}^n |x_i|\right)e^{-a\sum_{i=1}^n |x_i|}.$$

Since $g(x)\in L_1((0,\infty)^n)$ and $e^{-\sum_{i=1}^n \lambda_i x_i}\le 1$ the statement easily follows by the Lebesgue dominated convergence theorem. □

Let for $\alpha>0$,

$$g_\alpha(x,y) := e^{-x-y}\sum_{k=0}^\infty \frac{x^{k+\alpha-1}}{\Gamma(k+\alpha)}\frac{y^k}{k!} \quad x>0, y\ge 0.$$

For $\mu,\alpha_1,\dots,\alpha_n>0$ and a random vector $Y=(Y_1,\dots,Y_n)$ such that $\mathbb{P}(Y_i\ge 0)=1$ we set

$$h_{\alpha_1,\dots,\alpha_n,\mu,Y}(x_1,\dots,x_n) := \mathbb{E}\left[\prod_{i=1}^n \frac{1}{\mu}g_{\alpha_i}\left(\frac{x_i}{\mu},Y_i\right)\right], \quad x_1,\dots,x_n>0.$$

Lemma 7. *Let $\mu>0$ and Y be a random n-dimensional vector with nonnegative coordinates. For $\alpha=(\alpha_1,\dots,\alpha_n)\in(0,\infty)^n$ set $h_\alpha := h_{\alpha_1,\dots,\alpha_n,\mu,Y}$.*
i) For any $\alpha\in(0,\infty)^n$, $h_\alpha\ge 0$ and $\int_{(0,\infty)^n} h_\alpha(x)dx=1$.
ii) If $\alpha\in(0,\infty)^n$ and $\alpha_i>1$ then $\lim_{x_i\to 0+}h_\alpha(x)=0$, $\frac{\partial}{\partial x_i}h_\alpha(x)$ exists and

$$\frac{\partial}{\partial x_i}h_\alpha(x) = h_{\alpha-e_i}-h_\alpha.$$

iii) If $\alpha\in(1,\infty)^n$ then for any $J\subset[n]$, $\frac{\partial^{|J|}}{\partial x_J}h_\alpha(x)$ exists and belongs to $L_1((0,\infty)^n)$. Moreover for $\lambda_1,\dots,\lambda_n\ge 0$,

$$\int_{(0,\infty)^n} e^{-\sum_{i=1}^n\lambda_i x_i}\frac{\partial^{|J|}}{\partial x_J}h_\alpha(x)dx = \prod_{i\in J}\lambda_i\int_{(0,\infty)^n} e^{-\sum_{i=1}^n\lambda_i x_i}h_\alpha(x)dx.$$

Proof. i) Obviously $h_\alpha\in[0,\infty]$. We have for any $y\ge 0$ and $\alpha>0$,

$$\int_0^\infty \frac{1}{\mu}g_\alpha\left(\frac{x}{\mu},y\right)dx = \int_0^\infty g_\alpha(x,y)dx = 1.$$

Hence by the Fubini theorem,

$$\int_{(0,\infty)^n} h_\alpha(x)dx = \mathbb{E}\prod_{i=1}^k\int_0^\infty \frac{1}{\mu}g_{\alpha_i}\left(\frac{x_i}{\mu},Y_i\right)dx_i = 1.$$

ii) It is well known that $\Gamma(x)$ is decreasing on $(0, x_0]$ and increasing on $[x_0, \infty)$, where $1 < x_0 < 2$ and $\Gamma(x_0) > 1/2$. Therefore for $k = 1, \ldots$ and $\alpha > 0$, $\Gamma(k+\alpha) \geq \frac{1}{2}\Gamma(k) = \frac{1}{2}(k-1)!$ and

$$g_\alpha(x,y) \leq e^{-x} \sum_{k=0}^{\infty} \frac{x^{k+\alpha-1}}{\Gamma(k+\alpha)} \leq 2\left(x^{\alpha-1}e^{-x} + x^\alpha \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!}e^{-x}\right) = 2x^{\alpha-1}(e^{-x}+x).$$

This implies that for $\alpha > 0$ and $0 < a < b < \infty$, $g_\alpha(x,y) \leq C(\alpha, a, b) < \infty$ for $x \in (a,b)$ and $y \geq 0$. Moreover,

$$h_\alpha(x) \leq \left(\frac{2}{\mu}\right)^n \prod_{i=1}^{n} \left(\frac{x_i}{\mu}\right)^{\alpha_i - 1} \left(1 + \frac{x_i}{\mu}\right).$$

In particular $\lim_{x_i \to 0+} h_\alpha(x) = 0$ if $\alpha_i > 1$. Observe that for $\alpha > 1$, $\frac{\partial}{\partial x} g_\alpha = g_{\alpha-1} - g_\alpha$. Standard application of the Lebegue dominated convergence theorem concludes the proof of part ii).

iii) By ii) we get

$$\frac{\partial^{|J|}}{\partial x_J} h_\alpha = \sum_{K \subset J} (-1)^{|J|-|K|} h_{\alpha - \sum_{i \in K} e_i} \in L_1((0,\infty)^n).$$

Moreover $\lim_{x_j \to 0+} \frac{\partial^{|J|}}{\partial x_J} h_\alpha(x) = 0$ for $j \notin J$. We finish the proof by induction on $|J|$ using integration by parts. $\square$

Let C be a positive-definite symmetric $n \times n$ matrix. Then there exists $\mu > 0$ such that $B := C - \mu I_n$ is positive-definite. Let $X^{(l)} := (X_i^{(l)})_{i \leq n}$ be independent Gaussian vectors $\mathcal{N}(0, \frac{1}{2\mu}B)$,

$$Y_i = \sum_{l=1}^{k} (X_i^{(l)})^2 \quad 1 \leq i \leq n$$

and

$$h_{k,C} := h_{\frac{k}{2}, \ldots, \frac{k}{2}, \mu, Y}. \tag{5}$$

Lemma 8. *For any $\lambda_1, \ldots, \lambda_n \geq 0$ we have*

$$\int_{(0,\infty)^n} e^{-\sum_{i=1}^n \lambda_i x_i} h_{k,C}(x) = |I_n + \Lambda C|^{-\frac{k}{2}},$$

where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$.

Proof. We have for any $\alpha, \mu > 0$ and $\lambda, y \geq 0$

$$\int_0^\infty \frac{1}{\mu} e^{-\lambda x} g_\alpha\left(\frac{x}{\mu}, y\right) dx = e^{-y} \sum_{k=0}^\infty \frac{y^k}{k!\Gamma(k+\alpha)} \int_0^\infty e^{-(\lambda+\frac{1}{\mu})x} \frac{x^{k+\alpha-1}}{\mu^{k+\alpha}} dx$$
$$= e^{-y} \sum_{k=0}^\infty \frac{y^k}{k!(1+\mu\lambda)^{k+\alpha}} = (1+\mu\lambda)^{-\alpha} e^{-\frac{\mu\lambda}{1+\mu\lambda} y}.$$

By the Fubini theorem we have

$$\int_{(0,\infty)^n} e^{-\sum_{i=1}^n \lambda_i x_i} h_{k,C}(x) dx = \mathbb{E} \prod_{i=1}^n \int_0^\infty e^{-\lambda_i x_i} \frac{1}{\mu} g_{k/2}\left(\frac{x_i}{\mu}, Y_i\right) dx_i$$
$$= |I_n + \mu\Lambda|^{-\frac{k}{2}} \mathbb{E} e^{-\sum_{i=1}^n \frac{\mu\lambda_i}{1+\mu\lambda_i} Y_i}.$$

Therefore by Lemma 4 we have

$$\int_{(0,\infty)^n} e^{-\sum_{i=1}^n \lambda_i x_i} h_{k,C}(x) dx = |I_n + \mu\Lambda|^{-\frac{k}{2}} \left| I + 2\mu\Lambda(I+\mu\Lambda)^{-1} \frac{1}{2\mu} B \right|^{-\frac{k}{2}} = |I_n + \Lambda C|^{-\frac{k}{2}}.$$

□

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