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NECESSARY AND SUFFICIENT CONDITIONS FOR THE STRONG LAW OF LARGE NUMBERS FOR *U*-STATISTICS

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Under some mild regularity on the normalizing sequence, we obtain necessary and sufficient conditions for the strong law of large numbers for (symmetrized) U-statistics. We also obtain necessary and sufficient conditions for the a.s. convergence of series of an analogous form.

1. Introduction. The general question addressed in this paper is that of necessary and sufficient conditions for

$$\frac{1}{\gamma_n}\sum_{\mathbf{i}\in I_n}\varepsilon_{\mathbf{i}}h(X_{\mathbf{i}})\to 0 \quad \text{a.s.},$$

where $I_n=\{\mathbf{i}=(i_i,i_2,\dots,i_d): 1\le i_1<i_2<\cdots<i_d\le n\}$, $\{X_j\}_{j=1}^{\infty}$ is a sequence of i.i.d. r.v.'s, $X_{\mathbf{i}}=(X_{i_1},\dots,X_{i_d})$. Without loss of generality, we may assume that h is symmetric in its arguments.

Further, as in [2] and in [12], it is also important to consider the question of the almost sure convergence to zero of

$$\frac{1}{\gamma_n}\max_{\mathbf{i}\in I_n}|h(X_{\mathbf{i}})|.$$

In fact, it is through the study of this problem that one is able to complete the characterization for the original question.

Without the symmetrization by Rademachers, Hoeffding [5] in 1961 proved that for general d and $\gamma_n=\binom{n}{d}$, mean zero is sufficient for the normalized sum above to go to zero almost surely. And, under a pth moment, one has the a.s. convergence to zero with $\gamma_n=n^{d/p}$ ([10] when $0<p<1$, in the product case with mean zero [11] for $1\le p<2$ and in the case of general degenerate h [4] for $1<p<2$).

It is somewhat surprising that it took until the 1990's to see that Hoeffding's sufficient condition was not necessary [4]. In the particular case in which $d=2$, $h(x,y)=xy$ and the variables are symmetric, necessary and sufficient conditions were given in [2] in 1995. This was later extended to $d\ge 3$ by Zhang [12]. Very recently Zhang [13] obtained "computable" necessary and sufficient conditions in the case $d=2$ and, in general, found equivalent conditions in terms of a law of large numbers for modified maxima. Other related work is that of [8] in which the different indices go to infinity at their own

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pace and [3] in which the variables in different coordinates can be based on different distributions.

In this paper we obtain necessary and sufficient conditions for strong laws for "maxima" for general d. This likely would have enabled one to complete Zhang's program. However, we also found a more classical way of handling the reduction of the case of sums to the case of maxima.

The organization of the paper is as follows. In Section 2 we introduce the necessary notation and give the basic lemmas. Now the form of our main theorem is inductive. The reason we present the result in this form is that the conditions in the case $d > 2$ are quite involved. Because of the format of our theorem we first present in Section 3 the case that the function h is the product of the coordinates. As mentioned earlier, this case received quite a bit of attention, culminating in Zhang's paper [12]. In the first part of Section 3 we show how the methods developed in this paper allow one to give a relatively simple, and perhaps transparent, proof of Zhang's result. We then prove the main result, namely, the necessary and sufficient conditions for the strong law for symmetric U-statistics. Again, because of our inductive format, in order to clearly bring out the main ideas of our proof, we also give a simple proof of Zhang's result for the case $d = 2$.

Finally, in Section 4 we consider the question of convergence of multidimensional random series $\sum_{\mathbf{i}\in Z_+^d} h_{\mathbf{i}}(\widetilde{X}_{\mathbf{i}})$. We obtain necessary and sufficient conditions for a.s. convergence in the case of nonnegative or symmetrized kernels. This generalizes the results of [6] (case $d = 2$ and $h_{i,\,j}(x, y) = a_{i,\,j}xy$).

2. Preliminaries and basic lemmas. Let us first introduce the multiindex notation we will use in the paper:

- $\mathbf{i} = (i_i, i_2, \ldots, i_d)$, the multiindex of size d;
- $X_{\mathbf{i}} = (X_{i_i}, X_{i_2}, \ldots, X_{i_d})$, where X_j is a sequence of i.i.d. random variables with values in some space E and the common law μ;
- $\widetilde{X}_{\mathbf{i}} = (X^{(1)}_{i_i}, X^{(2)}_{i_2}, \ldots, X^{(d)}_{i_d})$, where $(X^{(k)}_j)$, $k = 1, \ldots, d$, are independent copies of (X_j);
- $\varepsilon_{\mathbf{i}} = \varepsilon_{i_1}\varepsilon_{i_2}\cdots\varepsilon_{i_d}$, where (ε_i) is a Rademacher sequence (i.e., a sequence of i.i.d. symmetric random variables taking on values ± 1) independent of other random variables;
- $\tilde{\varepsilon}_{\mathbf{i}} = \varepsilon^{(1)}_{i_1}\varepsilon^{(2)}_{i_2}\cdots\varepsilon^{(d)}_{i_d}$, where $(\varepsilon^{(j)}_i)$ is a doubly indexed Rademacher sequence independent of other random variables;
- $\mu_k = \otimes_{i=1}^k \mu$, the product measure on E^k;
- for $I \subset \{1, 2, \ldots, d\}$, by E_I and E'_I we will denote expectation with respect to $(X^k_i)_{k\in I}$ and $(X^k_i)_{k\notin I}$, respectively;

- in the undecoupled case $E_I h(X_{\mathbf{i}})$ [resp. $E_I' h(X_{\mathbf{i}})$] will denote expectation with respect to $(X_{i_k})_{k\in I}$ [resp. $(X_{i_k})_{k\notin I}$];
- $\mathbf{i}_I = (i_k)_{k\in I}$ and $I' = \{1, 2, \dots, d\}\setminus I$ for $I \subset \{1, 2, \dots, d\}$;
- $I_n = \{\mathbf{i} = (i_i, i_2, \dots, i_d) : 1 \le i_1 < i_2 < \dots < i_d \le n\}$;
- $C_n = \{\mathbf{i} = (i_i, i_2, \dots, i_d) : 1 \le i_1, i_2, \dots, i_d \le n\}$;
- for $I \subset \{1, 2, \dots, d\}$ we put $\sum_{\mathbf{i}_I} a_{\mathbf{i}} = \sum_{\mathbf{j}\in C_n : \mathbf{j}_{I'} = \mathbf{i}_{I'}} a_{\mathbf{j}}$;
- $A^{I,x} = A^{x_I} = \{z \in E^{I'} : \exists a \in A, a_I = x_I, a_{I'} = z\}$ for $A \subset E^d, I \subset \{1, \dots, d\}$.

The results in this section were motivated by the difficulty in computing quantities such as

$$P\Bigl(\max_{i,\,j\le n} h(X_i, Y_j) > t\Bigr),$$

where $\{X_i\}$ are independent random variables and $\{Y_i\}$ is an independent copy, and h is, say, symmetric in its arguments.

In the one-dimensional case, namely, $P(\max_{i\le n} \xi_i > t)$, where $\{\xi_i\}$ are independent r.v.'s, we have the simple inequality

$$\frac{1}{2}\min\Bigl(\sum_i P(|\xi_i| > t), 1\Bigr) \le P\Bigl(\max_i |\xi_i| > t\Bigr) \le \min\Bigl(\sum_i P(|\xi_i| > t), 1\Bigr). \tag{1}$$

If this type of inequality held for any dimension, the proofs and results would look much the same as in dimension 1. Here we give an example to see the difference between the cases $d = 1$ and $d > 1$.

Consider the set in the unit square given by

$$A = \bigl\{(x, y) \in [0, 1]^2 : x < a, y < b \text{ or } x < b, y < a\bigr\}$$

and assume that the X_i, Y_j are i.i.d. uniformly distributed on $[0, 1]$. By (1) it easily follows that

$$P\Bigl(\max_{1\le i,j\le n} I_A(X_i, Y_j) > 0\Bigr) \sim \min(na, 1)\min(nb, 1),$$

which is equivalent to $\sum_{i,\,j=1}^{n} P(I_A(X_i, Y_j) > 0) \sim n^2 ab$ if and only if both a and b are of order $O(1/n)$.

LEMMA 1. *Suppose that the nonnegative functions $f_{\mathbf{i}}(x_{\mathbf{i}})$ satisfy the following conditions*:

$$f_{\mathbf{i}}(\widetilde{X}_{\mathbf{i}}) \le 1 \text{ a.s. for all } \mathbf{i} \in C_n \tag{2}$$

and

$$E_I \sum_{\mathbf{i}_I} f_{\mathbf{i}}(\widetilde{X}_{\mathbf{i}}) \le 1 \quad \text{a.s. for any } I \subset \{1, 2, \dots, d\}, 0 < \mathrm{Card}(I) < d. \tag{3}$$

Let $\tilde{m}_1 = E\sum_{\mathbf{i}\in C_n} f_{\mathbf{i}}(\widetilde{X}_{\mathbf{i}})$. *Then*

$$E\Bigg(\sum_{\mathbf{i}\in C_n} f_{\mathbf{i}}(\widetilde{X}_{\mathbf{i}})\Bigg)^2 \le \tilde{m}_1^2 + (2^d - 1)\tilde{m}_1 \tag{4}$$

and

$$P\Bigg(\sum_{\mathbf{i}\in C_n} f_{\mathbf{i}}(\widetilde{X}_{\mathbf{i}}) \ge \frac{1}{2}\tilde{m}_1\Bigg) \ge 2^{-d-2}\min(\tilde{m}_1, 1). \tag{5}$$

PROOF. Let $S(d)$ denote the family of nonempty subsets of $\{1,\dots,d\}$ and for a fixed $I \in S(d)$ and $\mathbf{i}$ let

$$\widetilde{J}(\mathbf{i}, I) = \{\mathbf{j}\in C_n : \mathbf{j}_I = \mathbf{i}_I \text{ and } j_k \ne i_k \text{ for all } k \notin I\}.$$

Then we have, by (2) and (3),

$$\begin{aligned} E\Bigg(\sum_{\mathbf{i}\in C_n} f_{\mathbf{i}}(\widetilde{X}_{\mathbf{i}})\Bigg)^2 &\le \Bigg(E\sum_{\mathbf{i}\in C_n} f_{\mathbf{i}}(\widetilde{X}_{\mathbf{i}})\Bigg)^2 + \sum_{I\in S(d)}\sum_{\mathbf{i}\in C_n} E_I E_I' f_{\mathbf{i}}(\widetilde{X}_{\mathbf{i}}) E_I' \sum_{j\in\widetilde{J}(\mathbf{i},I)} f_{\mathbf{j}}(\widetilde{X}_{\mathbf{j}}) \\ &\le \tilde{m}_1^2 + \sum_{I\in S(d)}\sum_{\mathbf{i}\in C_n} E_I E_I' f_{\mathbf{i}}(\widetilde{X}_{\mathbf{i}}) = \tilde{m}_1^2 + \big(2^d - 1\big)\tilde{m}_1. \end{aligned}$$

Inequality (5) follows by (4) and the Paley–Zygmund inequality (cf. [7], Lemma 0.2.1). $\square$

The next lemma is an undecoupled version of Lemma 1, the proof of which is similar to that of Lemma 1 and is omitted.

LEMMA 2. *Suppose that the nonnegative functions* $f_{\mathbf{i}}(x_{\mathbf{i}})$ *satisfy the following conditions:*

$$f_{\mathbf{i}}(X_{\mathbf{i}}) \le 1 \quad \text{a.s. for all } \mathbf{i}\in I_n$$

and

$$E_I' \sum_{j\in J(\mathbf{i}, I)} f_{\mathbf{j}}(X_{\mathbf{j}}) \le 1 \quad \text{a.s. for all } \mathbf{i} \text{ and } I \subset \{1, 2, \dots, d\}, 0 < \operatorname{Card}(I) < d,$$

where

$$J(\mathbf{i}, I) = \big\{\mathbf{j}\in I_n : \{k : \exists_l i_k = j_l\} = I\big\}.$$

Let $m_1 = E\sum_{\mathbf{i}\in I_n} f_{\mathbf{i}}(X_{\mathbf{i}})$. *Then*

$$E\Bigg(\sum_{\mathbf{i}\in I_n} f_{\mathbf{i}}(X_{\mathbf{i}})\Bigg)^2 \le m_1^2 + (2^d - 1)m_1 \tag{6}$$

and

$$P\Bigg(\sum_{\mathbf{i}\in I_n} f_{\mathbf{i}}(X_{\mathbf{i}}) \ge \frac{1}{2}m_1\Bigg) \ge 2^{-d-2}\min(m_1, 1). \tag{7}$$

In the rest of this paper we will refer to the next corollary as the "section lemma."

COROLLARY 1. *If the set $A \subset E^d$ satisfies the condition*

$$n^{d-l}\mu_{d-l}(A^{I,X_I}) \le 1$$
$$\text{a.s. for all } I \subset \{1,\dots,d\} \text{ with } 0 < \mathrm{Card}(I) = l < d,$$

then

$$P\big(\exists_{\mathbf{i}\in C_n} \widetilde{X}_{\mathbf{i}} \in A\big) \ge 2^{-d-2}\min\big(n^d\mu_d(A),1\big)$$

and, for $n \ge d$,

$$P\big(\exists_{\mathbf{i}\in I_n} X_{\mathbf{i}} \in\big) \ge 2^{-d-2}d^{-d}\min\big(n^d\mu_d(A),1\big).$$

PROOF. The first inequality follows immediately by Lemma 1 applied to $f_{\mathbf{i}} = I_A$. To prove the second inequality, we use Lemma 2 and notice that

$$\min\left(\binom{n}{d}\mu_d(A),1\right) \ge d^{-d}\min\big(n^d\mu_d(A),1\big). \qquad \square$$

3. Strong laws of large numbers. We will assume in this section that the sequence γ_n satisfies the following regularity conditions:

(8) $$\gamma_n \text{ is nondecreasing,}$$

(9) $$\gamma_{2n} \le C\gamma_n \quad \text{for any } n,$$

(10) $$\sum_{k\ge l}\frac{2^{dk}}{\gamma_{2^k}^2} \le C\frac{2^{dl}}{\gamma_{2^l}^2} \quad \text{for any } l = 1,2,\dots.$$

As mentioned in the Introduction, we first give a proof of Zhang's result [12] for the product case, that is, $h(x) = \prod_{i=1}^d x_i$ for $x \in R^d$. To state the SLLN in this case, we need to define numbers c_n by the formula

$$c_n = \min\left\{c > 0 : nE\left(\frac{X^2}{c^2}\wedge 1\right) \le 1\right\}.$$

THEOREM 1. *Assume that $h(x) = \prod_{i=1}^d x_i$ and that the r.v.'s X_i are symmetric. Then, under the regularity assumptions* (8)–(10), *the following are equivalent:*

(11) $$\frac{1}{\gamma_n}\sum_{\mathbf{i}\in I_n} h(X_{\mathbf{i}}) = \frac{1}{\gamma_n}\sum_{\mathbf{i}\in I_n}\prod_{r=1}^d X_{i_r} \to 0 \quad \text{a.s.,}$$

(12) $$\sum_{k=1}^\infty 2^{kl}P\left(\prod_{r=1}^l X_r^2 > \frac{\gamma_{2^k}^2}{c_{2^k}^{2(d-l)}}, \min_{r\le l} X_r^2 > c_{2^k}^2\right) < \infty \quad \text{for all } 1 \le l \le d.$$

PROOF. We give only the proof of the necessity of the conditions (12). The sufficiency can be proved as in Theorem 2. Let

$$T_n^{(r)} = \sum_{i_r=1}^{n} X_{i_r}^{(r)^2}$$

and

$$T_n^{(r)}(c) = \sum_{i_r=1}^{n} X_{i_r}^{(r)^2} \wedge c^2.$$

STEP 1. We first reduce to the sum of squares; that is, we will show that condition (11) implies

$$\gamma_n^{-2} \sum_{\mathbf{i}\in I_n} \prod_{r=1}^{d} X_{i_r}^2 \to 0 \quad \text{a.s.} \tag{13}$$

By the symmetry of X we have that $\gamma_n^{-1} \sum_{\mathbf{i}\in I_n} \prod_{r=1}^{d} \varepsilon_{i_r} X_{i_r} \to 0$ a.s. Thus, for a.a. sequences (X_i), the Walsh sums (i.e., the linear combinations of products of d Rademachers) converge to 0 a.s. Hence, they converge in probability. This implies (by a result of Bonami about hypercontractivity of Walshes [1]) that for a.a. sequences (X_i), $\gamma_n^{-2} \sum_{\mathbf{i}\in I_n} \prod_{r=1}^{d} X_{i_r}^2 \to 0$ and (13) is proved.

STEP 2. We now go to a dyadic subsequence and then decouple. By the Borel–Cantelli lemma, condition (13) implies that

$$\forall_{\varepsilon>0} \sum_{k=1}^{\infty} P\left(\sum_{\mathbf{i}\in I_{2^{k-1}}} \prod_{r=1}^{d} X_{i_r}^2 \geq \varepsilon \gamma_{2^k}^2 \right) < \infty.$$

Now let us notice that $I_{2^k} \supseteq \{\mathbf{i} \in I_{2^k} : (r-1)2^{k-l} < i_r \leq r2^{k-l} \text{ for } r = 1, \ldots, d\}$ if l is such that $2^l \geq d$. Moreover, the random variables in these blocks are independent of the other blocks. Thus, we obtain

$$\forall_{\varepsilon>0} \sum_{k=l+1}^{\infty} P\left(\sum_{\mathbf{i}\in C_{2^{k-l-1}}} \prod_{r=1}^{d} \left(X_{i_r}^{(r)} \right)^2 \geq \varepsilon \gamma_{2^k}^2 \right) < \infty.$$

Hence, using the regularity assumption (9),

$$\forall_{\varepsilon>0} \sum_{k=1}^{\infty} P\left(\prod_{r=1}^{d} T_{2^k}^{(r)} \geq \varepsilon \gamma_{2^k}^2 \right) < \infty. \tag{14}$$

STEP 3. At this point we use the one-dimensional case of Lemma 1. We apply it to

$$c_n^{-2} T_n^{(r)}(c_n) = \sum_{j=1}^{n} \frac{\left(X_j^{(r)} \right)^2}{c_n^2} \wedge 1$$

and notice that $Ec_n^{-2}T_n^{(r)}(c_n) = 1$ by the definition of c_n. We get that

$$P\left(T_n^{(r)}(c_n) \geq \tfrac{1}{2}c_n^2\right) = P\left(c_n^{-2}T_n^{(r)}(c_n) \geq \tfrac{1}{2}Ec_n^{-2}T_n^{(r)}(c_n)\right) \geq \tfrac{1}{8}.$$

Hence,

$$P\left(\prod_{r=l+1}^{d} T_n^{(r)} \geq \frac{c_n^{2(d-l)}}{2^{d-l}}\right) \geq P\left(\prod_{r=l+1}^{d} T_n^{(r)}(c_n) \geq \frac{c_n^{2(d-l)}}{2^{d-l}}\right) \geq \left(\frac{1}{8}\right)^{d-l}$$

and

$$P\left(\prod_{r=1}^{d} T_n^{(r)} \geq 2^{l-d}\gamma_{2^k}^2\right) \geq \left(\frac{1}{8}\right)^{d-l} P\left(\prod_{r=1}^{l} T_n^{(r)} \geq \frac{\gamma_{2^k}^2}{c_{2^k}^{2(d-l)}}\right).$$

Thus, condition (14) yields

$$\sum_{k=1}^{\infty} P\left(\max_{i_1,\dots,i_l \leq 2^k} \prod_{r=1}^{l} (X_{i_r}^{(r)})^2 > \frac{\gamma_{2_k}^2}{c_{2^k}^{2(d-l)}}\right) < \infty. \tag{15}$$

Now, here is the main point.

STEP 4. At this point, we need to replace the max inside the probability with 2^{kl} outside the probability. To do this, we use the section lemma (Corollary 1).

To get small sections, there are a variety of choices. To obtain Zhang's result, we reduce the probabilities even further by intersecting the sets in the following manner:

$$\sum_{k=1}^{\infty} P\left(\max_{i_1,\dots,i_l \leq 2^k} \prod_{r=1}^{l} \left(X_{i_r}^{(r)}\right)^2 I_{\{(X_{i_r}^{(r)})^2 > c_{2^k}^2\}} > \frac{\gamma_{2_k}^2}{c_{2^k}^{2(d-l)}}\right) < \infty.$$

To see why we have small sections, just note that

$$P(X^2 > c_{2^k}^2) \leq \frac{E(X^2 \wedge c_{2^k}^2)}{c_{2^k}^2} = \frac{1}{2^k}.$$

Now we just use the section lemma to get

$$\sum_{k=1}^{\infty} 2^{kl} P\left(\prod_{r=1}^{l} X_r^2 I_{\{X_r^2 > c_{2^k}^2\}} > \frac{\gamma_{2_k}^2}{c_{2^k}^{2(d-l)}}\right) < \infty.$$

Or, equivalently,

$$\sum_{k=1}^{\infty} 2^{kl} P\left(\prod_{r=1}^{l} X_r^2 > \frac{\gamma_{2^k}^2}{c_{2^k}^{2(d-l)}}, \min_{1\leq r \leq l} X_r^2 > c_{2^k}^2\right) < \infty,$$

which yields (12). □

In Theorem 2 we reduce the SLLN for symmetric or nonnegative kernels to an SLLN for "modified maxima." To see what this means, consider the case $d = 2$. Then

$$A_{k,2} = \Big\{(x, y) \in E^2 : h^2(x, y) \le \gamma_{2^k}^2, 2^k E_Y h^2 I_{\{h^2 \le \gamma_{2^k}\}}(x, Y) \le \gamma_{2^k}^2,$$
$$2^k E_X h^2 I_{\{h^2 \le \gamma_{2^k}\}}(X, y) \le \gamma_{2^k}^2\Big\}.$$

So that

$$\{\exists \mathbf{i} \in C_{2^k}, \tilde{X}_{\mathbf{i}} \notin A_{k,2}\} = \Big\{\max_{\mathbf{i} \in C_{2^k}} \varphi(\widetilde{X}_{\mathbf{i}}) > \gamma_{2^k}^2\Big\},$$

where

$$\varphi(x, y) = h^2(x, y) \vee 2^k E_Y h^2 I_{\{h^2 \le \gamma_{2^k}\}}(x, Y) \vee 2^k E_X h^2 I_{\{h^2 \le \gamma_{2^k}\}}(X, y).$$

In [13] Zhang, using different methods, also reduced the probem to "modified maxima." We continue in Theorem 3 to find necessary and sufficient conditions for the SLLN for the maximum, which, hence, could also be used to complete Zhang's program.

For a measurable function h on E^d, which is symmetric with respect to permutations of the variables, we define, for $k = 1, 2, \ldots$,

$$A_{k,1} = \{x \in E^d : h^2(x) \le \gamma_{2^k}^2\}$$

and, for $l = 1, \ldots, d-1$,

$$A_{k,l+1} = \Big\{x \in A_{k,l} : 2^{kl} E_I h^2 I_{A_{k,l}}(x) \le \gamma_{2^k}^2$$
$$\text{for all } I \subset \{1, 2, \ldots d\}, \operatorname{Card}(I) = l\Big\}.$$

THEOREM 2. *Suppose that assumptions* (8)–(10) *are satisfied and the sets* $A_{k,l}$ *are defined as above. Then the following conditions are equivalent:*

$$\frac{1}{\gamma_n} \sum_{\mathbf{i} \in I_n} \varepsilon_{\mathbf{i}} h(X_{\mathbf{i}}) \to 0 \quad a.s., \tag{16}$$

$$\frac{1}{\gamma_n} \sum_{\mathbf{i} \in C_n} \tilde{\varepsilon}_{\mathbf{i}} h(\widetilde{X}_{\mathbf{i}}) \to 0 \quad a.s., \tag{17}$$

$$\frac{1}{\gamma_n^2} \sum_{\mathbf{i} \in I_n} h^2(X_{\mathbf{i}}) \to 0 \quad a.s., \tag{18}$$

$$\frac{1}{\gamma_n^2} \sum_{\mathbf{i} \in C_n} h^2(\widetilde{X}_{\mathbf{i}}) \to 0 \quad a.s., \tag{19}$$

$$\sum_{k=1}^{\infty} P(\exists_{\mathbf{i} \in I_{2^k}} X_{\mathbf{i}} \notin A_{k,d}) < \infty, \tag{20}$$

$$\sum_{k=1}^{\infty} P(\exists_{\mathbf{i} \in C_{2^k}} \widetilde{X}_{\mathbf{i}} \notin A_{k,d}) < \infty. \tag{21}$$

PROOF. (16) $\Rightarrow$ (18) and (17) $\Rightarrow$ (19). Proofs of these implications are the same as in Proposition 4.7 in [2] (see also Step 1 in the proof of Theorem 1).

(18) $\Rightarrow$ (19). Let l be such that $2^l \geq d$. By the regularity of γ_n (8), (9) and the Borel–Cantelli lemma, (18) and (19) are equivalent, respectively, to

$$\sum_{k=1}^{\infty} P\biggl(\sum_{\mathbf{i}\in I_{2^k}} h^2(X_{\mathbf{i}}) \geq \varepsilon\gamma_{2^k}^2\biggr) < \infty \quad \text{for all } \varepsilon > 0 \tag{22}$$

and

$$\sum_{k=l+1}^{\infty} P\biggl(\sum_{\mathbf{i}\in C_{2^{k-l}}} h^2(\widetilde{X}_{\mathbf{i}}) \geq \varepsilon\gamma_{2^k}^2\biggr) < \infty \quad \text{for all } \varepsilon > 0. \tag{23}$$

Let

$$D_k = \{\mathbf{i} : (m-1)2^{k-l} < i_m \leq m2^{k-l} \text{ for } m = 1, \dots, d\}.$$

Then for $k \geq l$ we get

$$P\biggl(\sum_{\mathbf{i}\in I_{2^k}} h^2(X_{\mathbf{i}}) \geq \varepsilon\gamma_{2^k}^2\biggr) \geq P\biggl(\sum_{\mathbf{i}\in D_k} h^2(X_{\mathbf{i}}) \geq \varepsilon\gamma_{2^k}^2\biggr) = P\biggl(\sum_{\mathbf{i}\in C_{2^{k-l}}} h^2(\widetilde{X}_{\mathbf{i}}) \geq \varepsilon\gamma_{2^k}^2\biggr)$$

and (22) implies (23).

(18) $\Rightarrow$ (20). We will prove by induction that, for $l \leq d$,

$$\sum_{k=1}^{\infty} P(\exists_{\mathbf{i}\in I_{2^k}} X_{\mathbf{i}} \notin A_{k,l}) < \infty. \tag{24}$$

For $l = 1$ (24) is $\sum_{k=1}^{\infty} P(\exists_{\mathbf{i}\in I_{2^k}} h^2(X_{\mathbf{i}}) > \gamma_{2^k}^2) < \infty$ and follows easily by the Borel–Cantelli lemma. Assume that (24) holds for $l \leq d-1$. To show it for $l+1$, it is enough to prove that, for any I with $\mathrm{Card}(I) = l$,

$$\sum_{k=1}^{\infty} P\Bigl(\exists_{\mathbf{i}_{I'}\in I_{2^k}} 2^{kl} E_I h^2 I_{A_{k,l}}(X_{\mathbf{i}}) > \gamma_{2^k}^2\Bigr) < \infty. \tag{25}$$

By the symmetry of the kernel h, we may and will assume that $I = \{1, \dots, l\}$. From (18) it follows that

$$\frac{1}{\gamma_{2^k}^2} \sum_{\mathbf{i}\in I_{2^k}} h^2 I_{A_{k,l}}(X_{\mathbf{i}}) \to 0 \quad \text{a.s.}$$

By the regularity of γ_{2^k}, (9) and the Borel–Cantelli lemma, we get that

$$\sum_{k=1}^{\infty} P\biggl(\sum_{\mathbf{i}\in I_{2^{k+1}}} h^2 I_{A_{k,l}}(X_{\mathbf{i}}) \geq \frac{1}{2}\gamma_{2^k}^2\biggr) < \infty.$$

But

$$P_I\left(\sum_{\mathbf{i}\in I_{2^{k+1}}} h^2 I_{A_{k,l}}(X_{\mathbf{i}}) \geq \tfrac{1}{2}\gamma_{2^k}^2\right) \geq P_I\left(\max_{\mathbf{i}_{I'}\in J_{2^k}} \sum_{\mathbf{i}_I\in I_{2^k}} h^2 I_{A_{k,l}}(X_{\mathbf{i}}) \geq \tfrac{1}{2}\gamma_{2^k}^2\right)$$

$$\geq \max_{\mathbf{i}_{I'}\in J_{2^k}} P_I\left(\sum_{\mathbf{i}_I\in I_{2^k}} h^2 I_{A_{k,l}}(X_{\mathbf{i}}) \geq \tfrac{1}{2}\gamma_{2^k}^2\right),$$

where

$$J_{2^k} = \{(i_1,\dots,i_{d-l}) : 2^k < i_1 < i_2 < \cdots < i_{d-l} \leq 2^{k+1}\}.$$

Let us notice that by the definition of $A_{k,l}$ we have, for any $J \subset I$ with $\mathrm{Card}(J) = m < l$,

$$2^{km} E_J h^2 I_{A_{k,l}}(X_{\mathbf{i}}) \leq \gamma_{2^k}^2.$$

Therefore, by Lemma 2 we get that

$$\max_{\mathbf{i}_{I'}\in J_{2^k}} P_I\left(\sum_{\mathbf{i}_I\in I_{2^k}} h^2 I_{A_{k,l}}(X_{\mathbf{i}}) \geq \tfrac{1}{2}\gamma_{2^k}^2\right) \geq 2^{-l-2}$$

if $\max_{\mathbf{i}_{I'}\in J_{2^k}} E_I \sum_{\mathbf{i}_I\in I_{2^k}} h^2 I_{A_{k,l}}(X_{\mathbf{i}}) > \gamma_{2^k}^2$. Hence,

$$P\left(\sum_{\mathbf{i}\in I_{2^k}} h^2 I_{A_{k,l}}(X_{\mathbf{i}}) \geq \tfrac{1}{2}\gamma_{2^k}^2\right) \geq 2^{-l-2} P\left(\exists_{i_{I'}\in J_{2^k}} 2^{kl} E_I h^2 I_{A_{k,l}}(\widetilde{X}_{\mathbf{i}}) > \gamma_{2^k}^2\right)$$

and (25) follows.

(19) $\Rightarrow$ (21). This is the same as above, except we use Lemma 1 instead of Lemma 2.

(20) $\Rightarrow$ (16). By the regularity assumptions (8), (9) and the Borel–Cantelli lemma, it is enough to prove that, for any $t > 0$,

$$\sum_{k=1}^{\infty} P\left(\frac{1}{\gamma_{2^k}} \max_{n\leq 2^k}\left|\sum_{\mathbf{i}\in I_n} \varepsilon_{\mathbf{i}} h(X_{\mathbf{i}})\right| \geq t\right) < \infty.$$

By our assumption (20) it is enough to show that

$$\sum_{k=1}^{\infty} P\left(\frac{1}{\gamma_{2^k}} \max_{n\leq 2^k}\left|\sum_{\mathbf{i}\in I_n} \varepsilon_{\mathbf{i}} h I_{A_{k,d}}(X_{\mathbf{i}})\right| \geq t\right) < \infty.$$

Since $d_n = \sum_{\mathbf{i}\in I_n} \varepsilon_{\mathbf{i}} h I_{A_{k,d}}(X_{\mathbf{i}})$ is a martingale, by Doob's maximal inequality we get

$$P\left(\frac{1}{\gamma_{2^k}} \max_{n\leq 2^k}\left|\sum_{\mathbf{i}\in I_n} \varepsilon_{\mathbf{i}} h I_{A_{k,d}}(X_{\mathbf{i}})\right| \geq t\right) \leq \frac{1}{t^2\gamma_{2^k}^2} E\left(\sum_{\mathbf{i}\in I_{2^k}} \varepsilon_{\mathbf{i}} h I_{A_{k,d}}(X_{\mathbf{i}})\right)^2$$

$$\leq \frac{2^{dk}}{t^2\gamma_{2^k}^2} E h^2 I_{A_{k,d}}(\widetilde{X}).$$

Thus, it is enough to show that

$$\sum_{k=1}^{\infty} \frac{2^{dk}}{\gamma_{2^k}^2} Eh^2 I_{A_{k,d}}(\widetilde{X}) < \infty. \tag{26}$$

Let $\tau = \inf\{k : \widetilde{X} \in A_{k,d}\}$. Then (if we additionally define $A_{0,d} = \emptyset$) $\widetilde{X} \in A_{\tau,d} \backslash A_{\tau-1,d}$ so by (10) we get

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{2^{dk}}{\gamma_{2^k}^2} Eh^2 I_{A_{k,d}}(\widetilde{X}) &\le E \sum_{k=\tau}^{\infty} \frac{2^{dk}}{\gamma_{2^k}^2} h^2(\widetilde{X}) \\
&\le CE \frac{2^{d\tau}}{\gamma_{2^\tau}^2} h^2(\widetilde{X}) \\
&\le C \sum_{k=1}^{\infty} E \frac{2^{dk}}{\gamma_{2^k}^2} h^2 I_{A_{k,d}\backslash A_{k-1,d}}(\widetilde{X}).
\end{aligned}$$

Let us notice that by the definition of $A_{k,d}$ we have $h^2(\widetilde{X}) I_{A_{k,d}\backslash A_{k-1,d}}(\widetilde{X}) \le \gamma_{2^k}^2$ and $E_I 2^{kl} h^2 I_{A_{k,d}\backslash A_{k-1,d}}(\widetilde{X}) \le \gamma_{2^k}^2$ for any $I \subset \{1, \dots, d\}$ with $0 < \mathrm{Card}(I) = l < d$. Thus, by Lemma 2,

$$\begin{aligned}
&2^{-d-2} \min\left(\binom{2^{k-1}}{d} \frac{1}{\gamma_{2^k}^2} Eh^2 I_{A_{k,d}\backslash A_{k-1,d}}(\widetilde{X}), 1 \right) \\
&\quad \le P\left(\sum_{\mathbf{i} \in I_{2^{k-1}}} h^2 I_{A_{k,d}\backslash A_{k-1,d}}(X_{\mathbf{i}}) > 0 \right) \\
&\quad \le P(\exists_{\mathbf{i} \in I_{2^{k-1}}} X_{\mathbf{i}} \in A_{k,d} \backslash A_{k-1,d}) \\
&\quad \le P(\exists_{\mathbf{i} \in I_{2^{k-1}}} X_{\mathbf{i}} \notin A_{k-1,d}).
\end{aligned}$$

So condition (20) implies that

$$\sum_{k=1}^{\infty} \min\left(\frac{2^{dk}}{\gamma_{2^k}^2} Eh^2 I_{A_{k,d}\backslash A_{k-1,d}}(\widetilde{X}), 1 \right) < \infty$$

and (26) easily follows.

(21) $\Rightarrow$ (16) and (21) $\Rightarrow$ (17). In the same way as above we show that (21) implies (26) and that (26) implies (17). $\square$

The next theorem will show how to deal with condition (20). Suppose that the sets A_k are given and let us define the sets $C_{k,l}$ and $B_{k,I}$ for $I \subset \{1, \dots, d\}$ with $\mathrm{Card}(I) = l$ by induction over $d - l$:

$$\begin{aligned}
C_{k,d} &= A_k, \\
B_{k,I} &= \left\{ x_I \in E^l : 2^{k(d-l)} \mu_{d-l}(C_{k,l+1}^{x_I}) \ge 1 \text{ for } \mathrm{Card}(I) = l \right\}, \\
C_{k,l} &= \left\{ x \in C_{k,l+1} : x_I \notin B_{k,I} \text{ for all I with } \mathrm{Card}(I) = l \right\}.
\end{aligned}$$

THEOREM 3.

$$\sum_{k=1}^{\infty} P(\exists_{\mathbf{i}\in I_{2^k}} X_{\mathbf{i}} \in A_k) < \infty \tag{27}$$

if and only if the following conditions are satisfied:

$$\forall_{l=1,\dots,d-1}\forall_{I\subset\{1,\dots,d\},\,\mathrm{Card}(I)=l} \sum_{k=1}^{\infty} P\Big(\exists_{\mathbf{j}\in I^l_{2^k}} X_{\mathbf{j}} \in B_{k,I}\Big) < \infty, \tag{28}$$

$$\sum_{k=1}^{\infty} 2^{kd}\mu_d(C_{k,1}) < \infty. \tag{29}$$

PROOF. Let us notice that (29) immediately implies that

$$\sum_{k=1}^{\infty} P\Big(\exists_{\mathbf{i}\in I_{2^k}} X_{\mathbf{i}} \in C_{k,1}\Big) < \infty.$$

Since, by the definition of the sets $C_{k,l}$,

$$\{\exists_{\mathbf{i}\in I_{2^k}} X_{\mathbf{i}} \in A_k\} \subset \{\exists_{\mathbf{i}\in I_{2^k}} X_{\mathbf{i}} \in C_{k,1}\} \cup \bigcup_{l=1}^{d-1} \bigcup_{I\subset\{1,\dots,d\},\ \mathrm{Card}(I)=l} \{\exists_{\mathbf{i}_I\in I^l_{2^k}} X_{\mathbf{i}_I} \in B_{k,I}\},$$

(28) and (29) imply (27).

To prove the second implication, let us first notice that by the definition of $C_{k,l}$ we have

$$2^{k(d-m)}\mu_{d-m}(C^{x_I}_{k,l}) < 1 \quad \text{for any } I \text{ with } \mathrm{Card}(I) = m \geq l. \tag{30}$$

Hence, by Corollary 1,

$$P\Big(\exists_{\mathbf{i}\in I_{2^k}} X_{\mathbf{i}} \in A_k\Big) \geq P\Big(\exists_{\mathbf{i}\in I_{2^k}} X_{\mathbf{i}} \in C_{k,1}\Big) \geq c_d 2^{kd}\mu_d(C_{k,1})$$

so (27) implies (29).

By Corollary 1 and (30) we also get that for any $I \subset \{1,\dots,m\}$ with $\mathrm{Card}(I) = l = 1,\dots,d-1$ we have, for $J = I^c$ and any $x_I \in E^l$,

$$P\Big(\exists_{\mathbf{i}_J\in I^{d-l}_{2^k}}\ X_{\mathbf{i}_J} \in C^{x_I}_{k,l+1}\Big) \geq c_{d-l}2^{k(d-l)}\mu_{d-l}(C^{x_I}_{k,l+1}).$$

Thus,

$$P\Big(\exists_{\mathbf{i}\in I_{2^k}} X_{\mathbf{i}} \in A_k\Big) \geq P\Big(\exists_{\mathbf{i}\in I_{2^k}} X_{\mathbf{i}} \in C_{k,l+1}\Big) \geq c_{d-l}P\Big(\exists_{\mathbf{i}_I\in I^l_{2^k}} X_{\mathbf{i}_I} \in B_{k,I}\Big)$$

and (27) implies (28). $\square$

3.1. *Two-dimensional case.* In the two-dimensional case let us define, for $k = 1, 2, \dots,$

$$f_k(x) = 2^k E_Y\big(h^2(x, Y) \wedge \gamma_{2^k}^2\big). \tag{31}$$

THEOREM 4. *In the case of $d = 2$ each of the equivalent conditions* (16)–(21) *is equivalent to the following condition:*

$$\sum_{k=1}^{\infty} 2^k P\big(f_k(X) \geq \gamma_{2^k}^2\big) < \infty \tag{32a}$$

and

$$\sum_{k=1}^{\infty} 2^{2k} P\Big(h^2(X, Y) \geq \gamma_{2^k}^2, f_k(X) < \gamma_{2^k}^2, f_k(Y) < \gamma_{2^k}^2\Big) < \infty. \tag{32b}$$

PROOF. Again, we concentrate on the necessity, since the sufficiency can be proved as in Theorem 2. To obtain (32a), first reduce to the decoupled sum of squares as in Theorem 2 (19). One then has

$$\begin{aligned} &P\left(\sum_{i,\, j \leq 2^k} h^2(X_i, Y_j) \wedge \gamma_{2^k}^2 > \tfrac{1}{2}\gamma_{2^k}^2\right) \\ &\qquad \geq E_Y \max_{j \leq 2^k} P_X\left(\sum_{i \leq 2^k} h^2(X_i, Y_j) \wedge \gamma_{2^k}^2 > \tfrac{1}{2}\gamma_{2^k}^2\right). \end{aligned}$$

Applying Lemma 1 (the case $d = 1$) to the probability appearing in the last expectation, we see that

$$P_X\left(\sum_{i \leq 2^k} h^2(X_i, Y_j) \wedge \gamma_{2^k}^2 > \tfrac{1}{2}\gamma_{2^k}^2\right) \geq \tfrac{1}{8} I_{\{2^k E_X(h^2 \wedge \gamma_{2^k}^2) > \gamma_{2^k}^2\}}.$$

Hence,

$$\begin{aligned} &E_Y \max_{j \leq 2^k} P_X\left(\sum_{i \leq 2^k} h^2(X_i, Y_j) \wedge \gamma_{2^k}^2 > \tfrac{1}{2}\gamma_{2^k}^2\right) \\ &\qquad \geq \tfrac{1}{8} P_Y\left(\max_{i \leq 2^k} 2^k E_X(h^2 \wedge \gamma_{2^k}^2) > \gamma_{2^k}^2\right) \\ &\qquad \geq \tfrac{1}{16} \min\Big(1, 2^k P_Y\big(2^k E_X(h^2 \wedge \gamma_{2^k}^2) > \gamma_{2^k}^2\big)\Big), \end{aligned}$$

which implies (32a). But we also have

$$\begin{aligned} &P\left(\sum_{i,\, j \leq 2^k} h^2(X_i,\, Y_j) \wedge \gamma_{2^k}^2 \geq \gamma_{2^k}^2\right) \\ &\qquad \geq P\left(\max_{i,\, j \leq 2^k} h^2(X_i, Y_j) \wedge \gamma_{2^k}^2 I_{\{f_k(X_i), f_k(Y_j) \leq \gamma_{2^k}^2\}} \geq \gamma_{2^k}^2\right). \end{aligned}$$

Now, using the section lemma (Corollary 1), we have that the last quantity is greater than or equal to

$$2^{-4}\min\bigl(1, 2^{2k}P\bigl(h^2 \wedge \gamma_{2^k}^2 \geq \gamma_{2^k}^2, f_k(X), f_k(Y) < \gamma_{2^k}^2\bigr)\bigr).$$

And this implies (32b). □

4. Convergence of series. In this section we will present the multidimensional generalizations of the symmetric case of Kolmogorov's three series theorem, which states that for independent random variables X_i the following conditions are equivalent:

$$\sum_{i=1}^{\infty} \varepsilon_i X_i \text{ is a.s. convergent,}$$

$$\sum_{i=1}^{\infty} X_i^2 < \infty \quad \text{a.s.}$$

and

$$\sum_{i=1}^{\infty} E(X_i^2 \wedge 1) < \infty.$$

Let us first consider the two-dimensional case and define

$$c_i(x_i) = \sum_{j=1}^{\infty} E_Y\Bigl(h_{i,j}^2(x_i, Y_j) \wedge 1\Bigr),$$

$$d_j(y_j) = \sum_{i=1}^{\infty} E_X\Bigl(h_{i,j}^2(X_i, y_j) \wedge 1\Bigr).$$

THEOREM 5. *Suppose that the functions c_i, d_j are defined as above. Then the following conditions are equivalent:*

$$\lim_{n\to\infty} \sum_{i,j=1}^{n} \varepsilon_i^{(1)} \varepsilon_j^{(2)} h_{i,j}(X_i, Y_j) \textit{ is a.s. convergent}, \tag{33}$$

$$\sum_{i,j=1}^{\infty} h_{i,j}^2(X_i, Y_j) < \infty \quad \textit{a.s.} \tag{34}$$

and

$$c_i(X_i) < \infty \quad \textit{a.s. for all } i \quad \textit{and} \quad d_j(Y_j) < \infty \quad \textit{a.s. for all } j, \tag{35a}$$

$$\sum_{i=1}^{\infty} P\bigl(c_i(X_i) > 1\bigr) < \infty \quad \textit{and} \quad \sum_{j=1}^{\infty} P\bigl(d_j(Y_j) > 1\bigr) < \infty, \tag{35b}$$

$$\sum_{i,j=1}^{\infty} E\Bigl(h_{i,j}^2(X_i, Y_j) \wedge 1\Bigr) I_{\{c_i(X_i)\leq 1, d_j(Y_j)\leq 1\}} < \infty. \tag{35c}$$

PROOF. (33) $\Leftrightarrow$ (34). Let us first notice that (33) and (34) are equivalent, respectively, to the following two conditions:

$$\forall_{\varepsilon>0}\exists_n P\left(\sup_{k\geq n}\left|\sum_{n\leq i\vee j\leq k}\varepsilon_i^{(1)}\varepsilon_j^{(2)}h_{i,j}(X_i,Y_j)\right|>\varepsilon\right)<\varepsilon \tag{36}$$

and

$$\forall_{\varepsilon>0}\exists_n P\left(\sum_{n\leq i\vee j}h_{i,j}^2(X_i,Y_j)>\varepsilon\right)<\varepsilon. \tag{37}$$

By the hypercontractivity of the Walshes (i.e., for sums of products of Rademacher r.v.'s [1] or [7], Section 3.4.) and the Paley–Zygmund inequality, we have

$$P\left(\left(\sum_{n\leq i\vee j}\varepsilon_i^{(1)}\varepsilon_j^{(2)}h_{i,j}(X_i,Y_j)\right)^2\geq t\sum_{n\leq i\vee j}h_{i,j}^2(X_i,Y_j)\right)\geq\frac{(1-t)^2}{81}.$$

Hence, (36) implies (37). On the other hand, since $d_k=\sum_{n\leq i\vee j\leq k}\varepsilon_i^{(1)}\varepsilon_j^{(2)}\times h(X_i,Y_j)$ is a martingale, we get by Doob's inequality

$$P\left(\sup_{k\geq n}\left|\sum_{n\leq i\vee j\leq k}\varepsilon_i^{(1)}\varepsilon_j^{(2)}h_{i,j}(X_i,Y_j)\right|\geq t\left(\sum_{n\leq i\vee j}h_{i,j}^2(X_i,Y_j)\right)^{1/2}\right)\leq t^{-2}$$

and (37) implies (36).

(35) $\Rightarrow$ (34). By condition (35a) we get that $\sum_{j=1}^{\infty}h_{i,j}^2(X_i,Y_j)<\infty$ a.s. for any i and $\sum_{i=1}^{\infty}h_{i,j}^2(X_i,Y_j)<\infty$ a.s. for any j. Hence, by condition (35b) it is enough to prove that

$$Z=\sum_{i,j=1}^{\infty}\left(h_{i,j}^2(X_i,Y_j)\wedge 1\right)I_{\{c_i(X_i)\leq 1,d_j(Y_j)\leq 1\}}<\infty\quad\text{a.s.} \tag{38}$$

However, by Chebyshev's inequality

$$P(Z\geq t)\leq t^{-2}\sum_{i,j=1}^{\infty}E\left(h_{i,j}^2(X_i,Y_j)\wedge 1\right)I_{\{c_i(X_i)\leq 1,d_j(Y_j)\leq 1\}}$$

and (38) follows by (35c).

(34) $\Rightarrow$ (35). The condition $c_i(X_i)<\infty$ a.s. is equivalent to $\sum_{j=1}^{\infty}h_{i,j}^2(X_i,Y_j)<\infty$ a.s. Thus (35a) immediately follows by (34).

To prove condition (35b), let us notice that for sufficiently large n we have

$$P\left(\sum_{i=n,j=1}^{\infty}h_{i,j}^2(X_i,Y_j)\geq\tfrac{1}{2}\right)\leq 2^{-4}.$$

Let us notice that by Lemma 1 (case $d=1$) we have, for any $k\geq n$,

$$P_Y\left(\sum_{i=n,j=1}^{\infty}h_{i,j}^2(X_i,Y_j)\geq\tfrac{1}{2}c_k(X_k)\right)\geq P_Y\left(\sum_{j=1}^{\infty}h_{k,j}^2(X_k,Y_j)\wedge 1\geq\tfrac{1}{2}c_k(X_k)\right)$$
$$\geq 2^{-3}\min\bigl(c_k(X_k),1\bigr).$$

Thus,

$$P\left(\sum_{i=n,\, j=1}^{\infty} h_{i,\,j}^2(X_i, Y_j) \geq \frac{1}{2}\right) \geq 2^{-3} P\left(\max_{i \geq n} c_i(X_i) > 1\right),$$

so $P(\max_{i\geq n} c_i(X_i) > 1) \leq 1/2$, which implies that $\sum_{i=1}^{\infty} P(c_i(X_i) > 1) < \infty$. In an analogous way we prove that $\sum_{j=1}^{\infty} P(d_j(Y_j) > 1) < \infty$.

Finally, let

$$m = \sum_{i,\, j=1}^{\infty} E\Big(h_{i,j}^2(X_i, Y_j) \wedge 1\Big) I_{\{c_i(X_i) \leq 1,\ d_j(Y_j) \leq 1\}}.$$

We have

$$\begin{aligned} &E_X \sum_{i=1}^{\infty} h_{i,\,j}^2(X_i, Y_j) \wedge 1 I_{\{c_i(X_i) \leq 1,\ d_j(Y_j) \leq 1\}} \\ &\quad \leq \left(E_X \sum_{i=1}^{\infty} h_{i,\,j}^2(X_i, Y_j) \wedge 1\right) I_{\{d_j(Y_j) \leq 1\}} \leq 1 \end{aligned}$$

and by a similar argument

$$E_Y \sum_{j=1}^{\infty} (h_{i,\,j}^2(X_i, Y_j) \wedge 1) I_{\{c_i(X_i) \leq 1, d_j(Y_j) \leq 1\}} \leq 1.$$

Hence, by Lemma 1 we get

$$P\left(\sum_{i,\, j=1}^{\infty} (h_{i,j}^2(X_i, Y_j) \wedge 1) I_{\{c_i(X_i) \leq 1, d_j(Y_j)\}} \geq \frac{1}{2} m\right) \geq 2^{-4} \min(m, 1),$$

which implies that $m < \infty$. $\square$

Before formulating the result in the d-dimensional case, we will need a few more definitions. Let us define in this case $A_{0,\,\mathbf{i}} = E^d$ and then inductively, for $l = 1, \ldots, d-1$, $I \subset \{1, 2, \ldots, d\}$ with $\mathrm{Card}(I) = l$,

$$\begin{aligned} c_{\mathbf{i}_I}(x_{\mathbf{i}_I}) &= \sum_{\mathbf{i}_{I'}} E_I'\Big(h_{(\mathbf{i}_I, \mathbf{i}_{I'})}^2 I_{A_{l-1, (\mathbf{i}_I, \mathbf{i}_{I'})}}(x_{\mathbf{i}_I}, \widetilde{X}_{\mathbf{i}_{I'}}) \wedge 1\Big), \\ A_{l,\,\mathbf{i}} &= \big\{x_{\mathbf{i}} \in A_{l-1,\,\mathbf{i}} : c_{\mathbf{i}_I}(x_{\mathbf{i}_I}) \leq 1 \text{ for all } I \text{ with } \mathrm{Card}(I) = l\big\}. \end{aligned}$$

THEOREM 6. *Suppose that c_{i_I} and $A_{l,\,\mathbf{i}}$ are defined as above. Then the following conditions are equivalent:*

$$\sum_{\mathbf{i} \in Z_+^d} \varepsilon_{\mathbf{i}} h_{\mathbf{i}}(\widetilde{X}_{\mathbf{i}}) \text{ is a.s. convergent,} \tag{39}$$

$$\sum_{\mathbf{i} \in Z_+^d} h_{\mathbf{i}}^2(\widetilde{X}_{\mathbf{i}}) < \infty \quad \text{a.s.} \tag{40}$$

and

$$\sum_{\mathbf{i}_I \in Z_+^{d-1}} h_{\mathbf{i}}^2(\widetilde{X}_{\mathbf{i}}) < \infty \quad \text{a.s. for all } I \text{ with } \operatorname{Card}(I) = d-1, \tag{41a}$$

$$\sum_{\mathbf{i}_I \in Z_+^{l}} I_{\{c_{i_I}(\widetilde{X}_{i_I}) > 1\}} < \infty \quad \text{a.s. for all } I \text{ with} \tag{41b}$$

$$l = \operatorname{Card}(I) = 1, 2 \dots, d-1,$$

$$\sum_{\mathbf{i} \in Z_+^d} E\Big(h_{\mathbf{i}}^2(\widetilde{X}_{\mathbf{i}}) \wedge 1\Big) I_{A_{d-1,\mathbf{i}}}(\widetilde{X}_{\mathbf{i}}) < \infty. \tag{41c}$$

PROOF. As above. □

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