

# Forced structure and connectivity in digraphs

## PhD defense

Amadeus Reinald

*before the jury:*

Alantha Newman, DR, *CNRS, G-SCOP, Univ. Grenoble-Alpes* - Reviewer

Jean-Sébastien Sereni, DR, *CNRS, ICube, Univ. Strasbourg* - Reviewer

Marthe Bonamy, DR, *CNRS, LABRI, Univ. Bordeaux* - Examiner

Alexandre Pinlou, PR, *LIRMM, Univ. Montpellier* - Examiner

Pierre Charbit, MCF, *IRIF, Univ. Paris-Cité* - Guest

Stéphane Bessy, PR, *LIRMM, Univ. Montpellier* - Supervisor

Daniel Gonçalves, CR, *CNRS, LIRMM, Univ. Montpellier* - Supervisor

September 12th, 2025





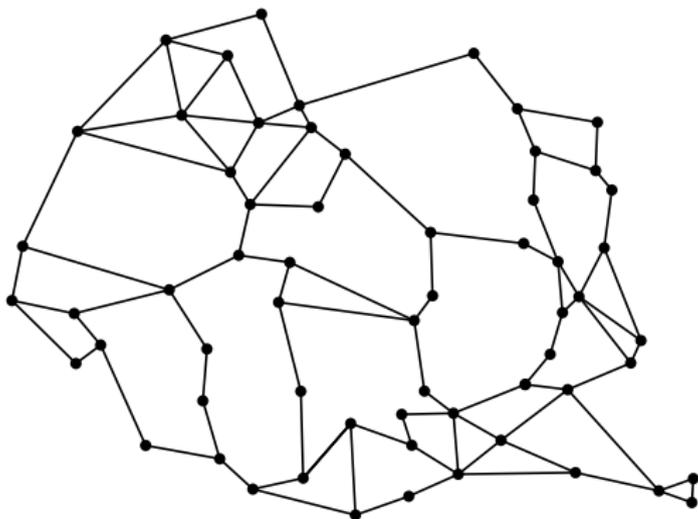






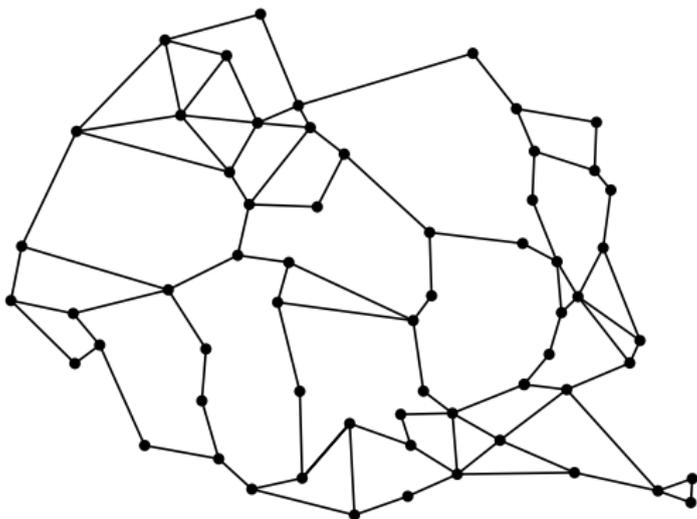
# Graphs and oriented graphs

Graph  $G = (V, E)$ : Vertices  $V$ , Edges  $E$ : pairs in  $V$



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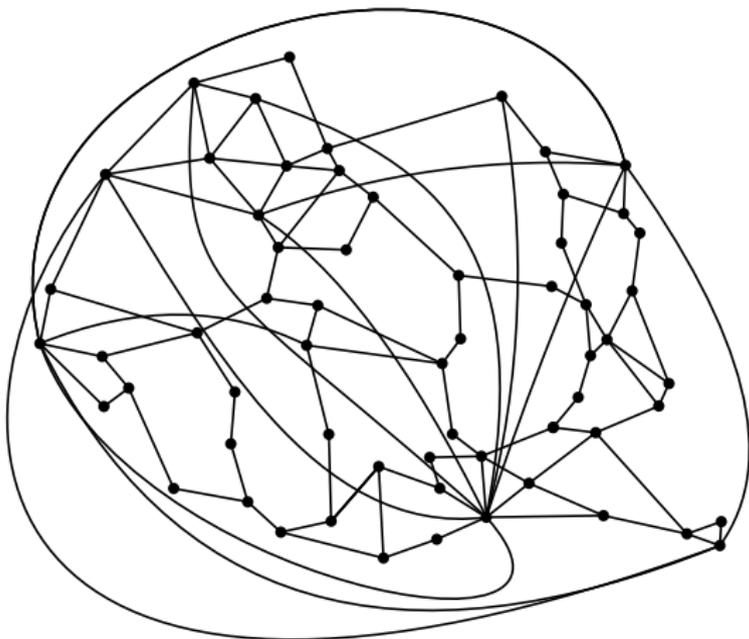
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**Plane**  $\equiv$  no crossing edges

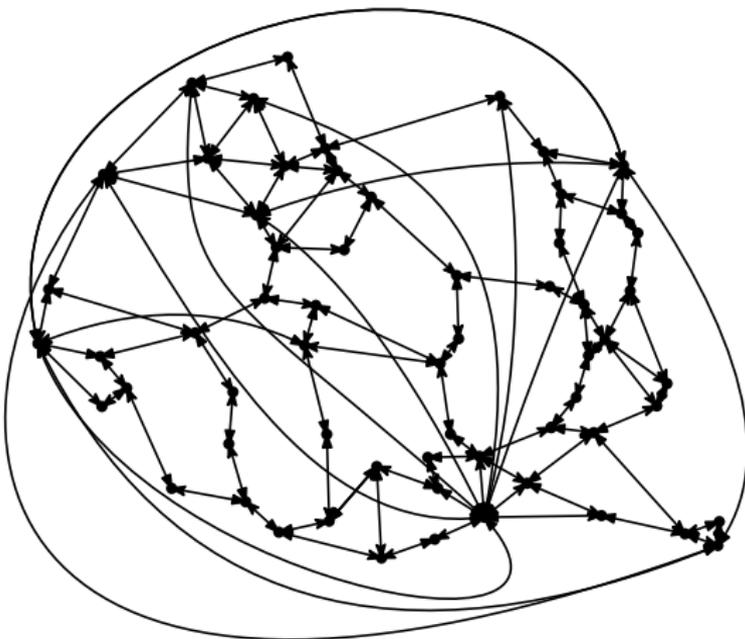
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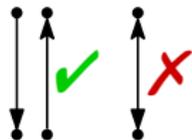
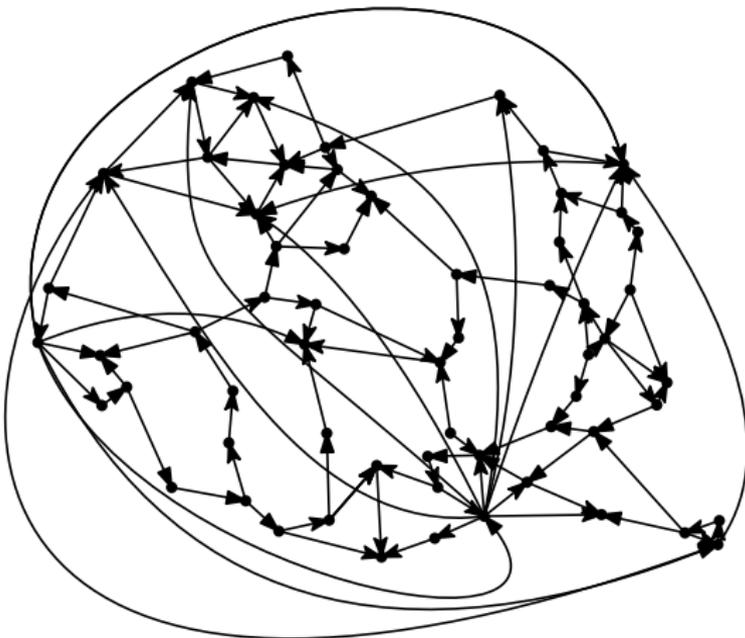
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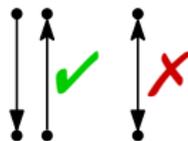
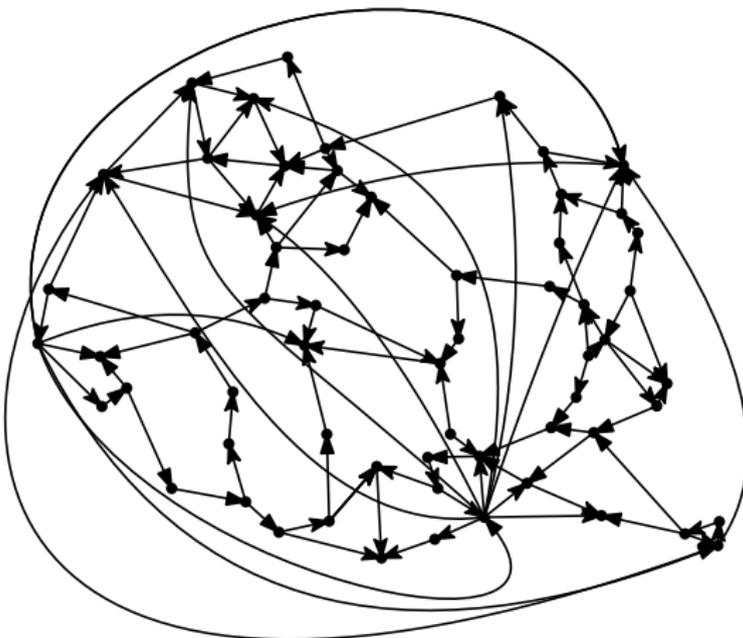
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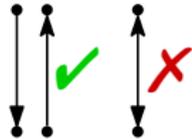
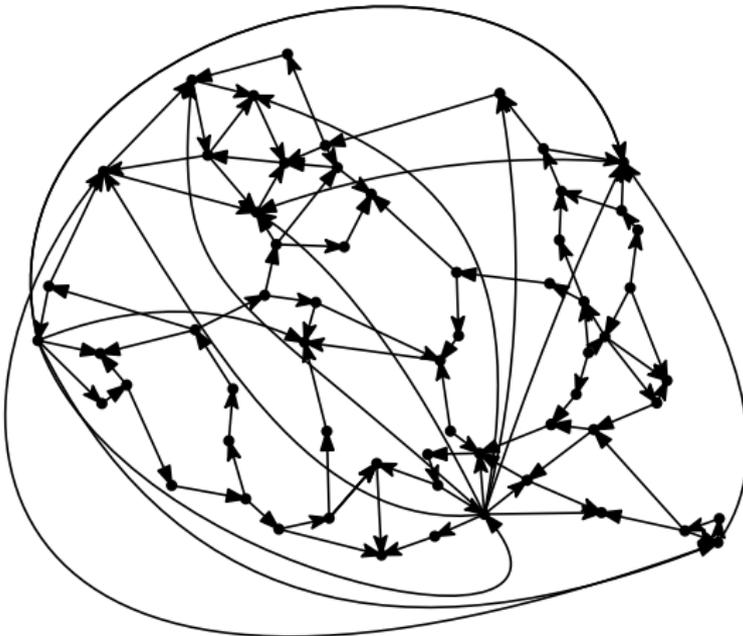
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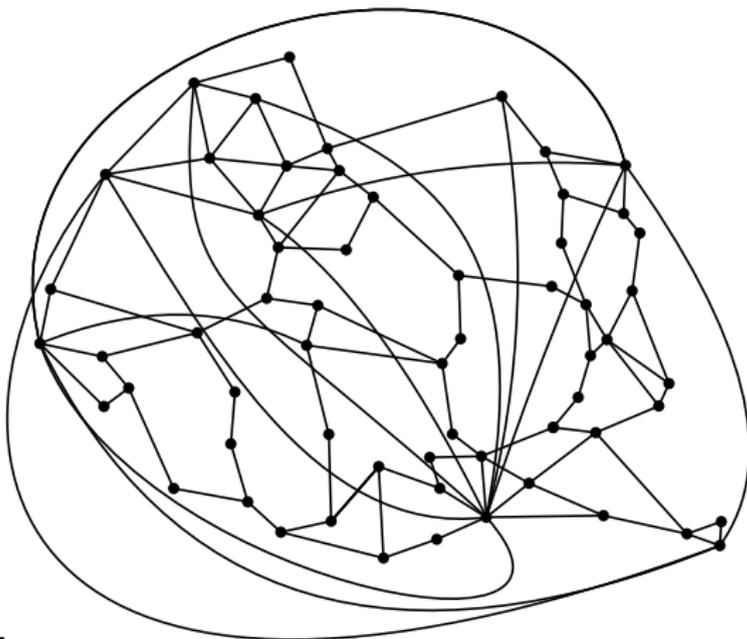
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- I Algorithms: Increase or decrease connectivity
- II Structure: Find global conditions forcing substructures

# II - Forced structures and chromatic number

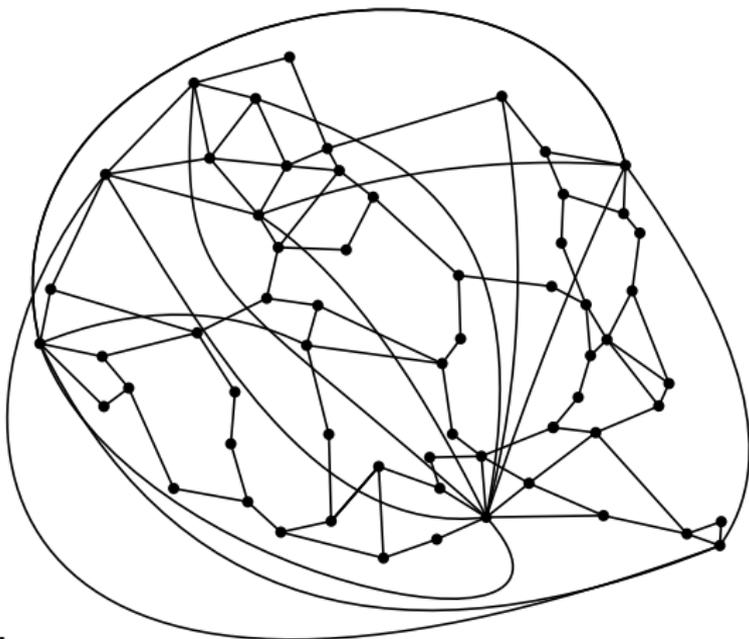
## Structural questions



### Question

Can a *global property* force *local structure*?

## II - Forced structures and chromatic number

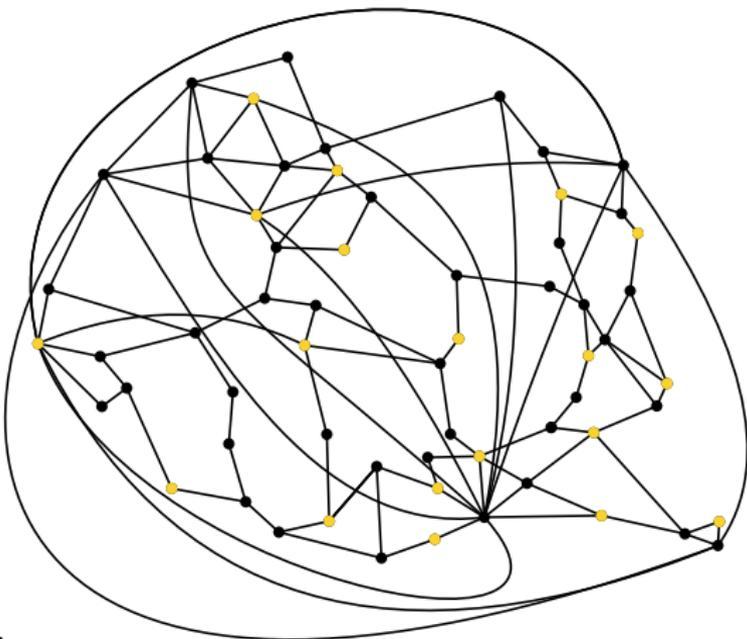


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Chromatic number  $\chi(G)$ : min. colors / no monochromatic edge.

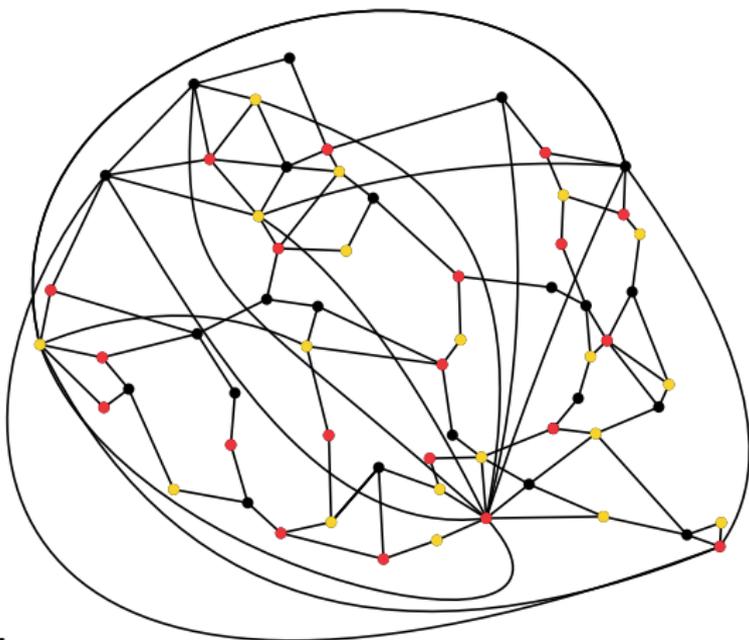


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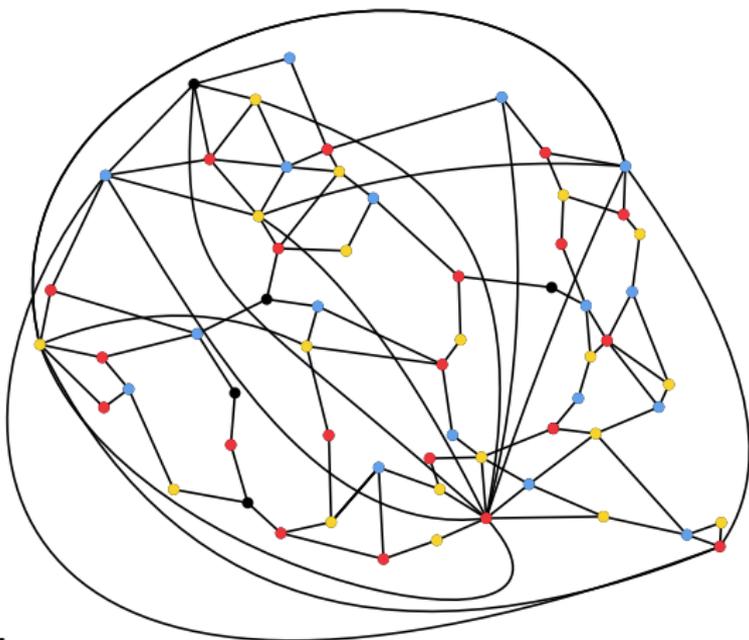


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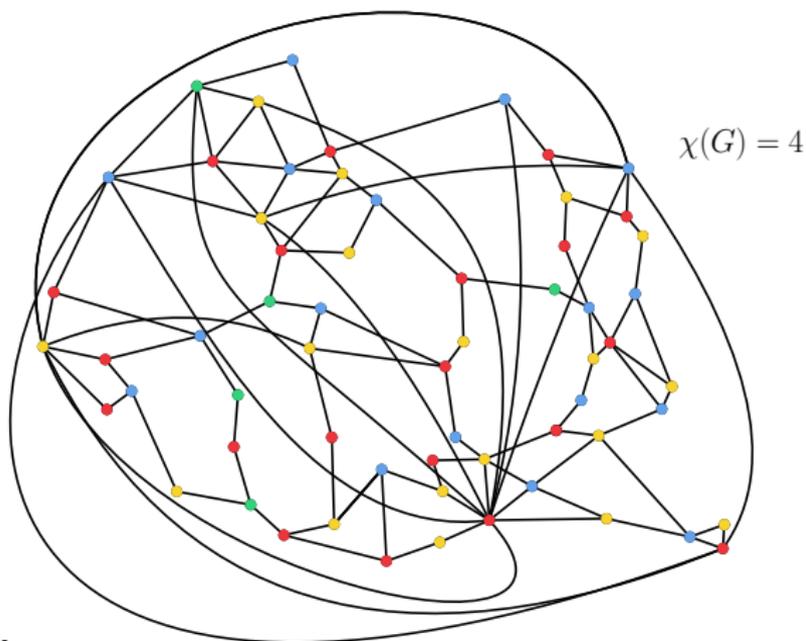


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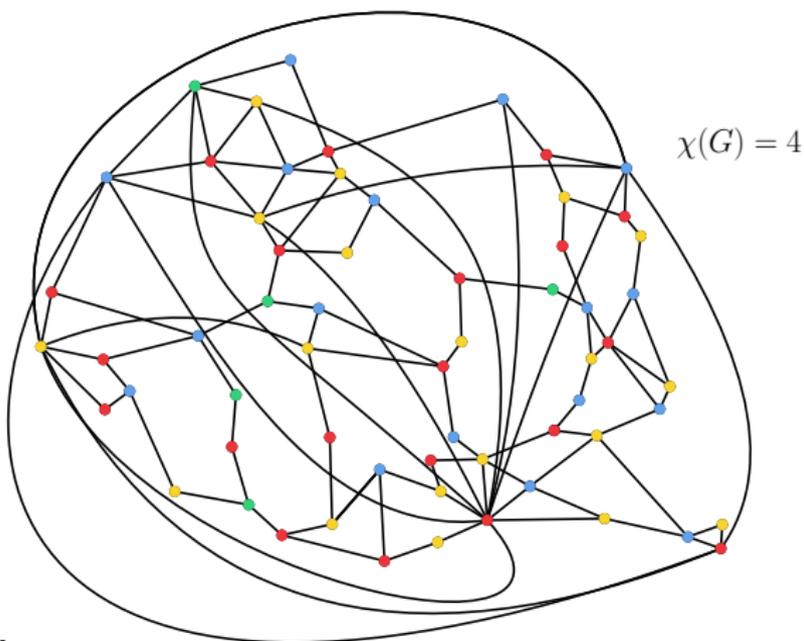


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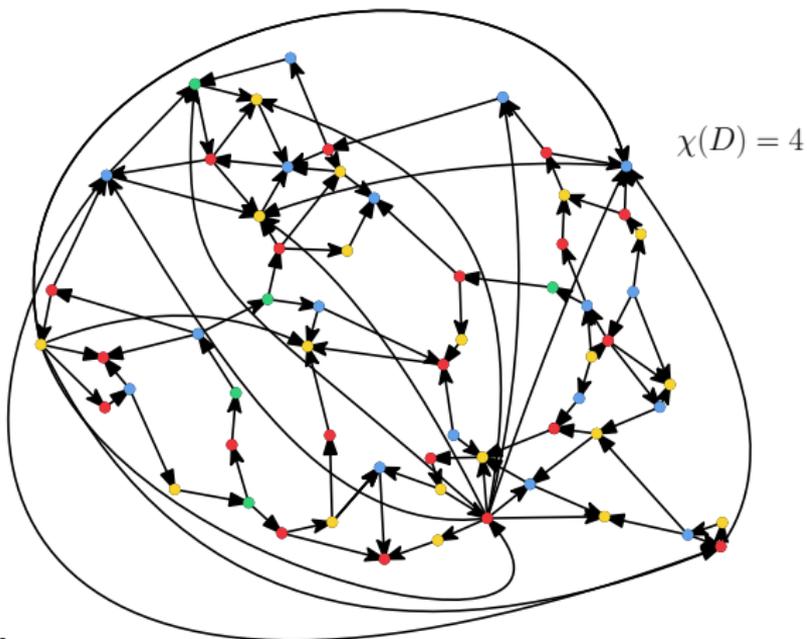


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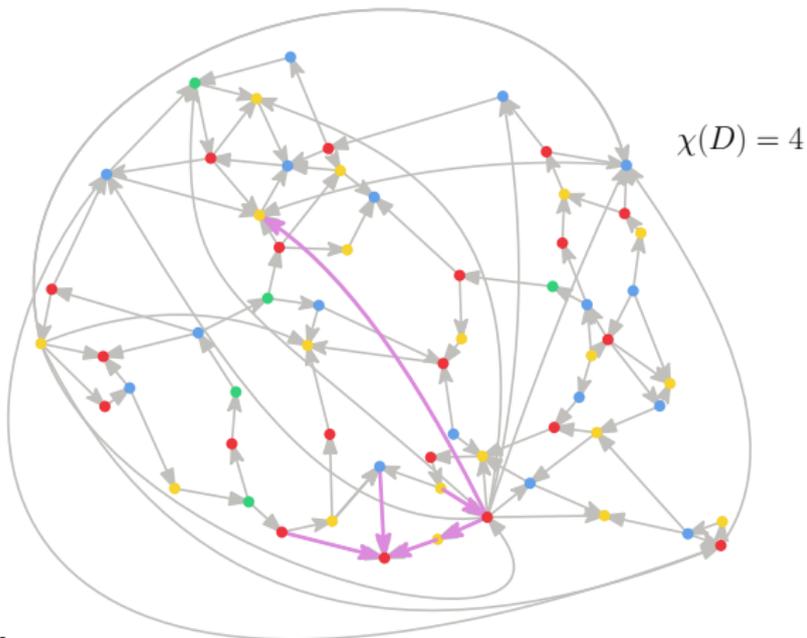


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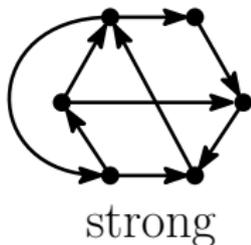
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- **$D$  strong**: all vertices have a **directed path** towards all others.

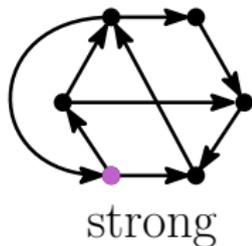
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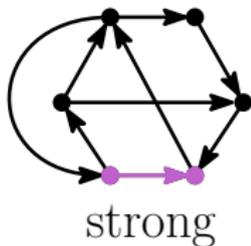
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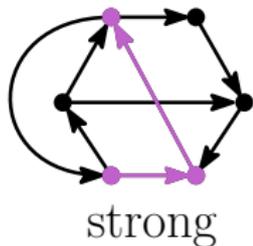
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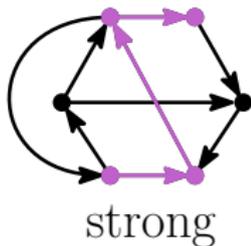
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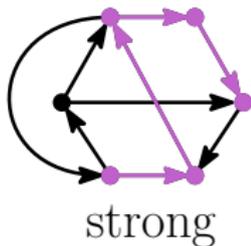
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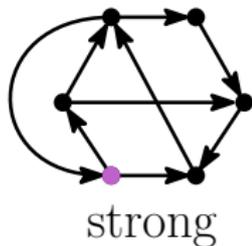
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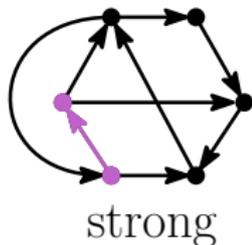
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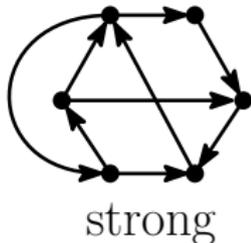
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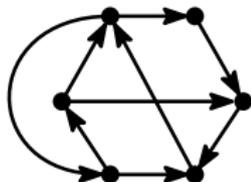
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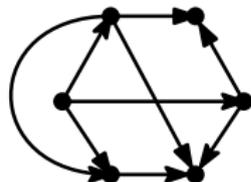


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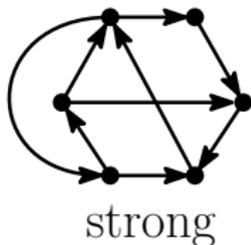
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DAG

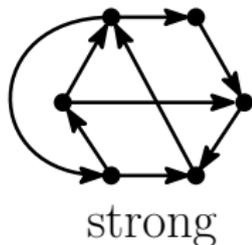
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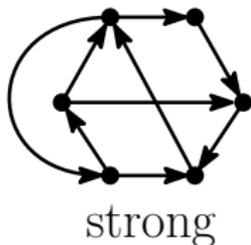


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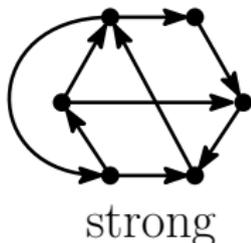
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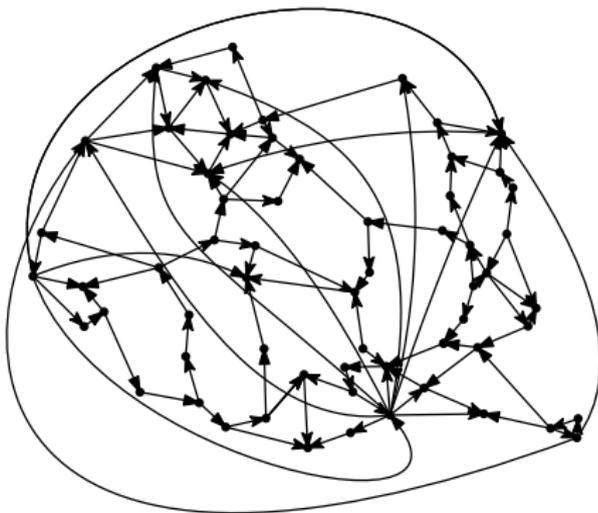
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- FPT algorithms: time  $O(f(k) \cdot \text{poly}(n)) \rightarrow$  fast for **small  $k$** .

## I - Plane strong connectivity augmentation

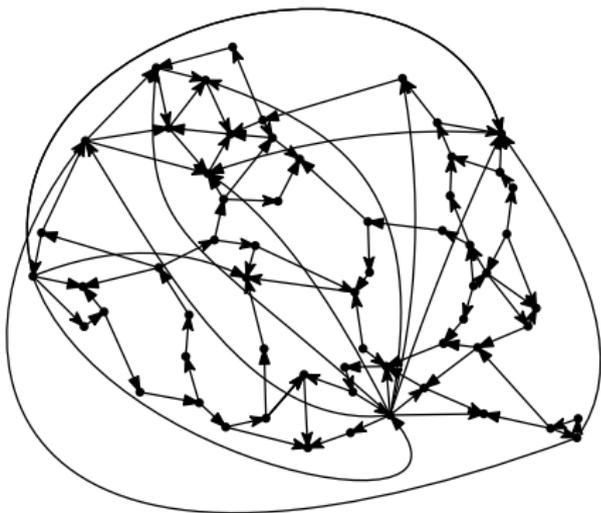
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SCA: Input:  $D$ , Output: min. new arcs  $X$  s.t.  $D+X$  strong.



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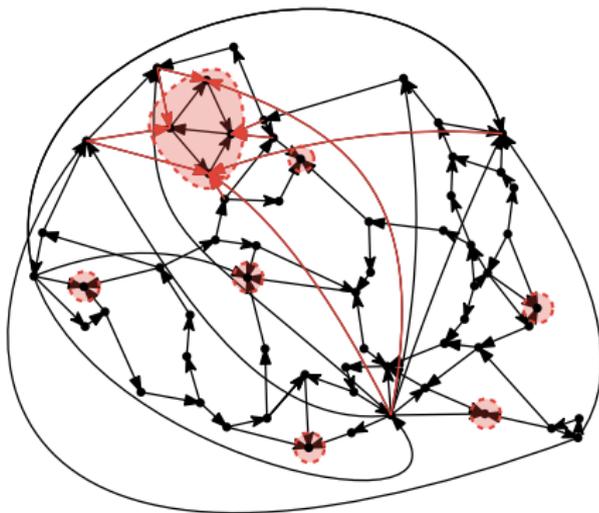
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Terminal components

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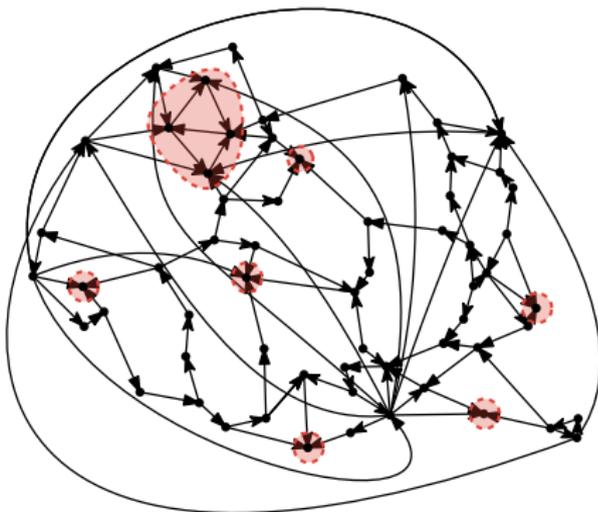
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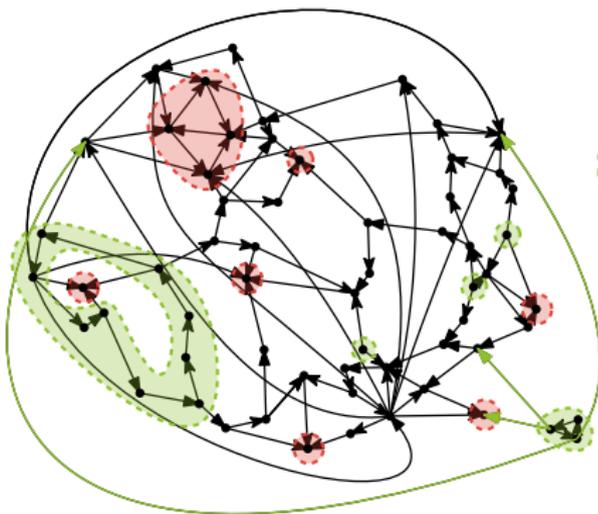
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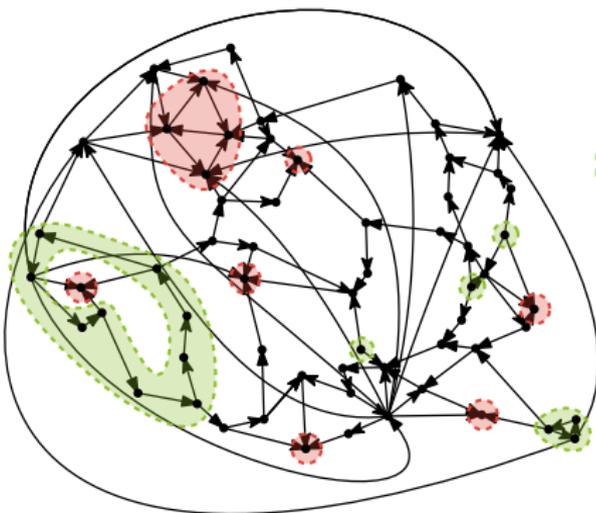
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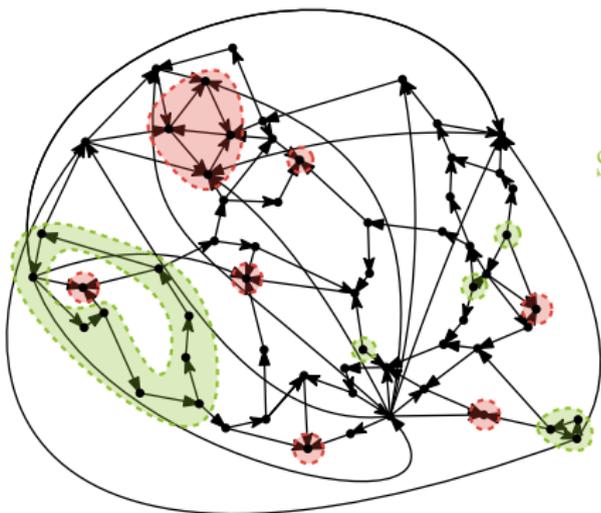
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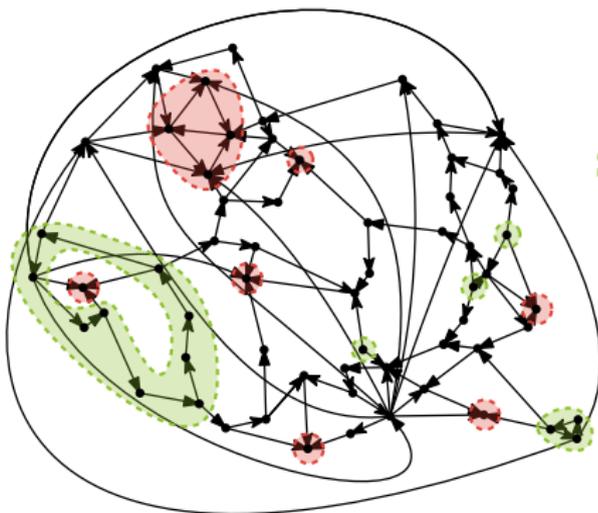
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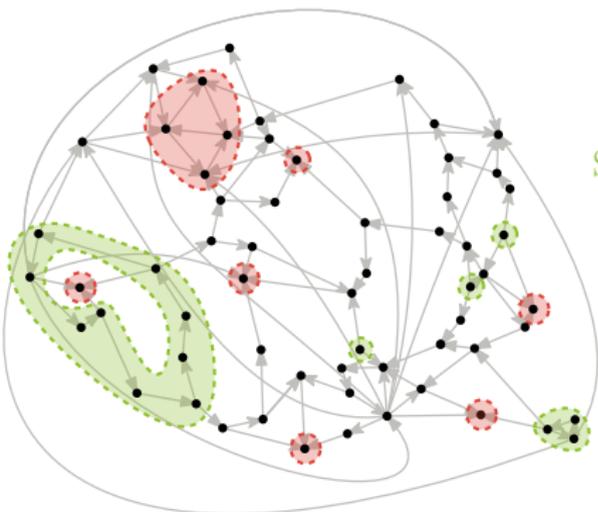
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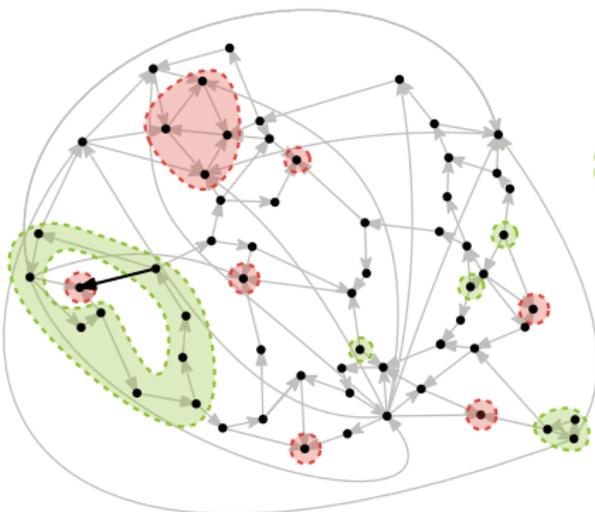
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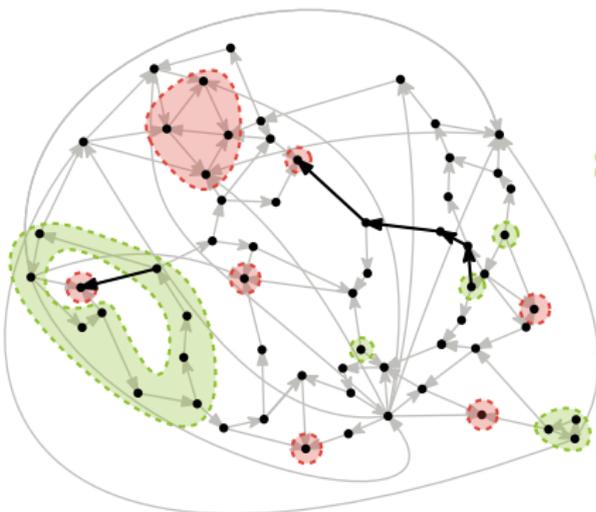
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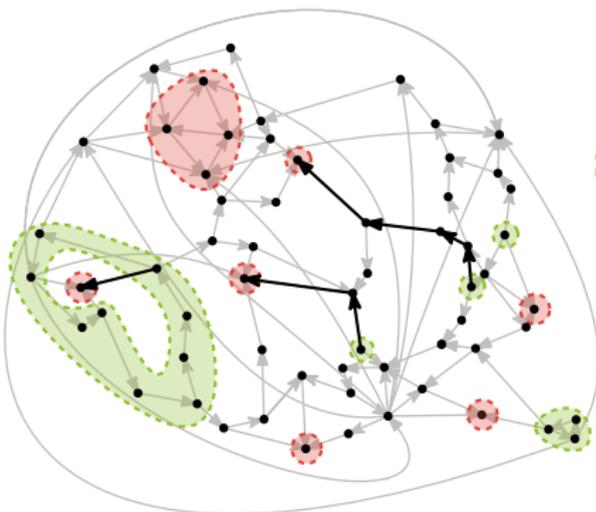
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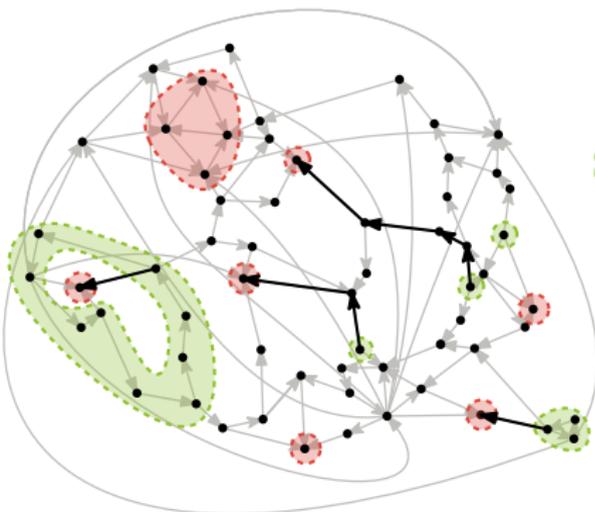
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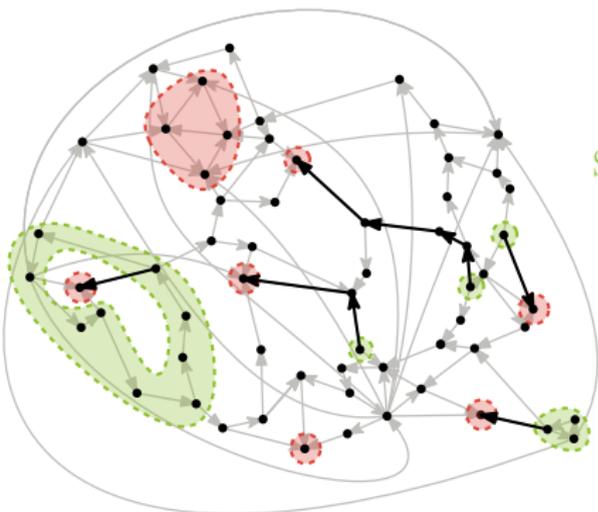
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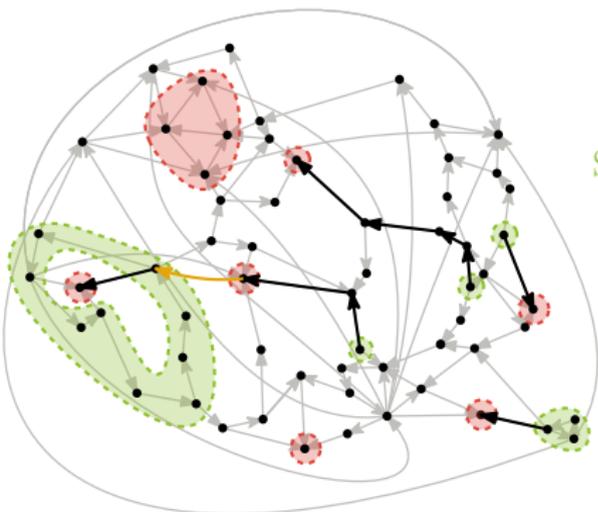
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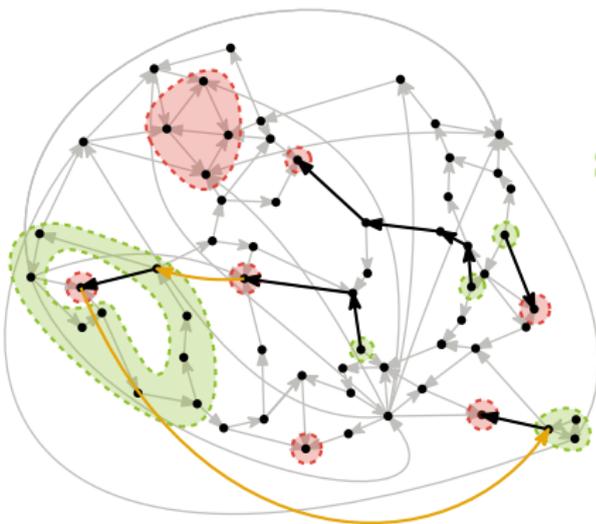
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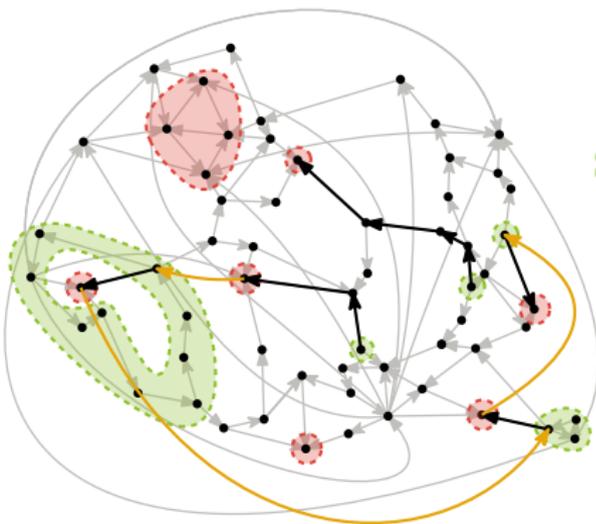
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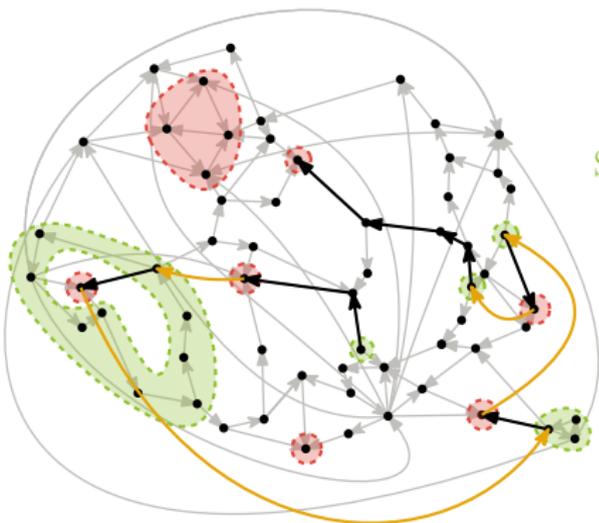
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**Terminal components**

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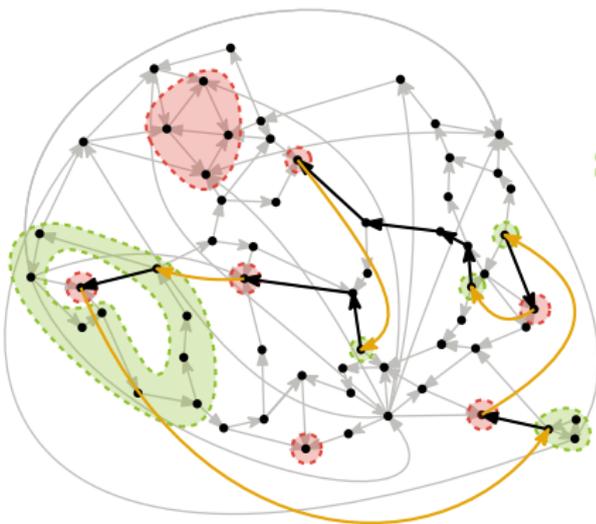
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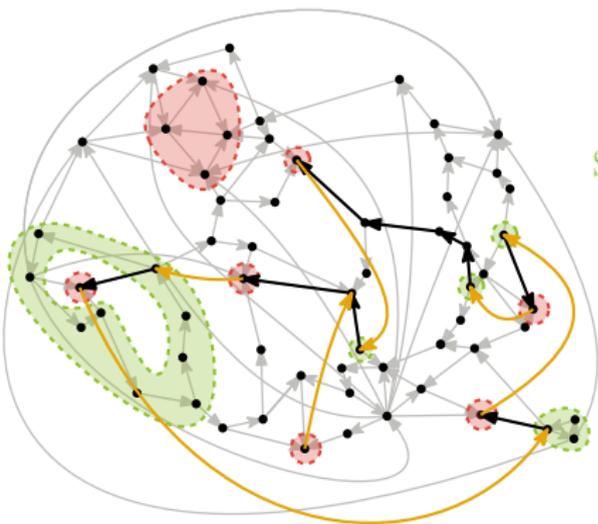
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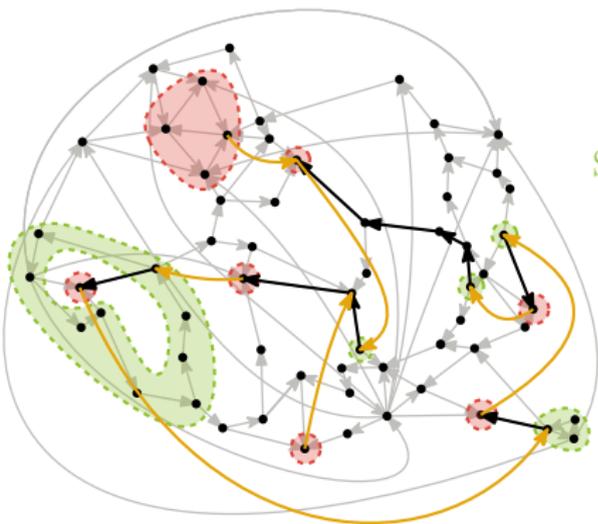
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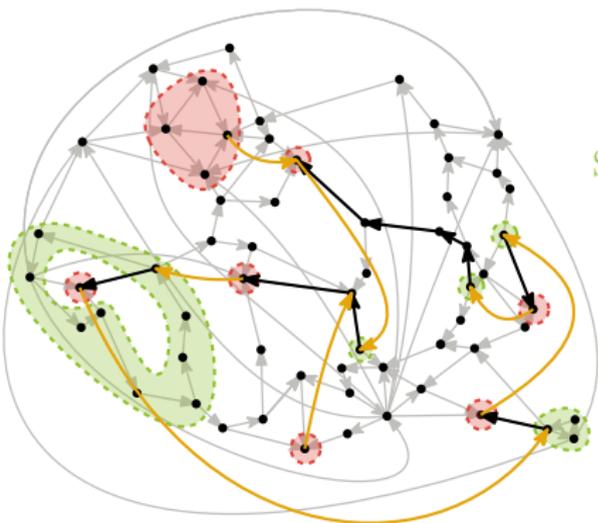
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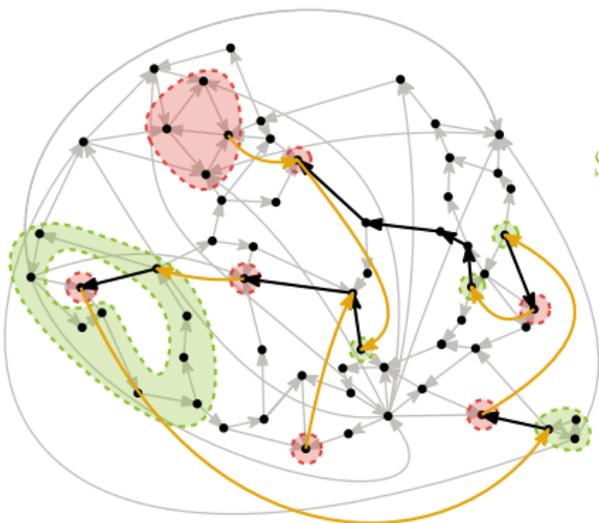
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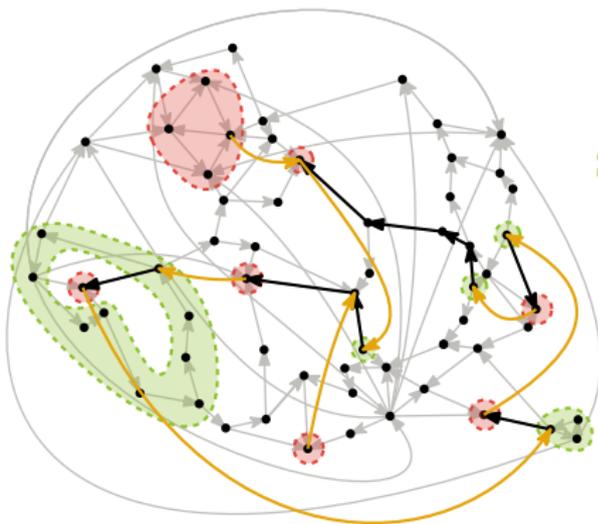
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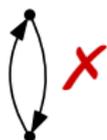
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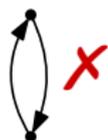
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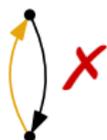
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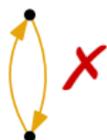
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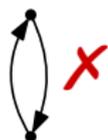
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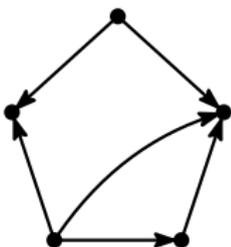
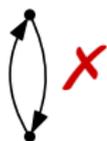
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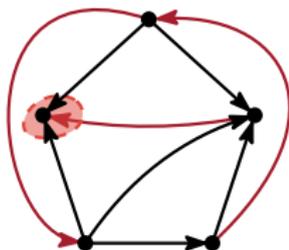
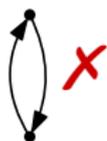
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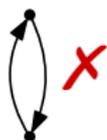
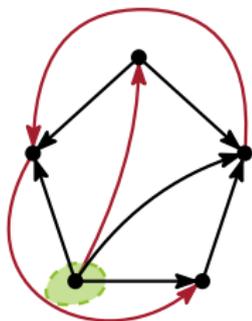
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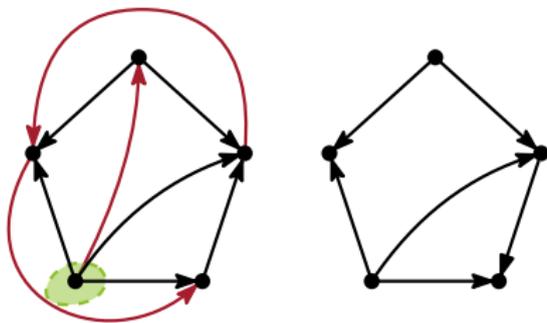
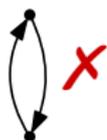
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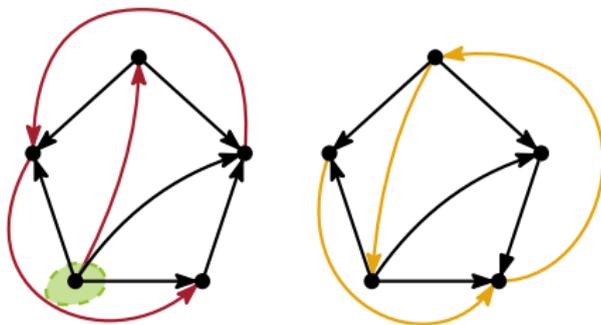
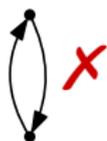
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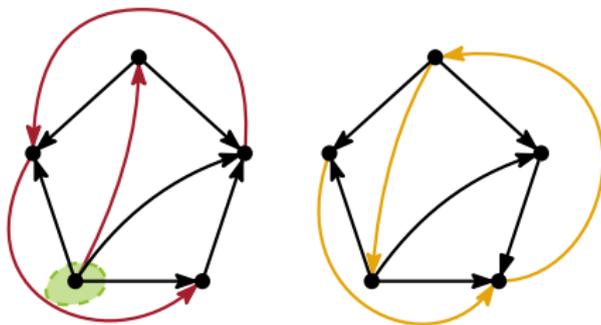
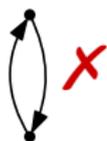
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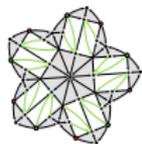
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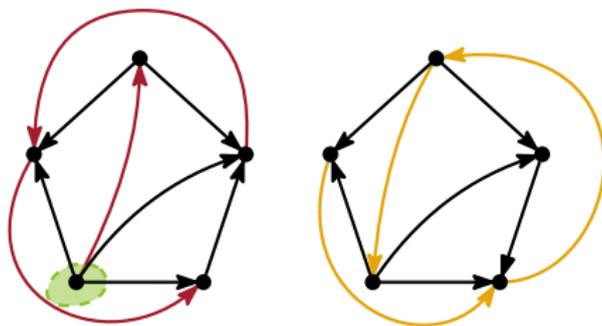
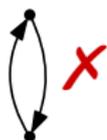
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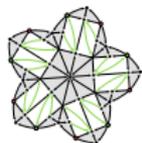
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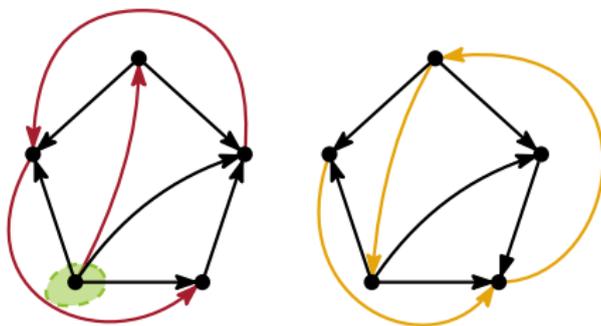
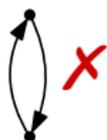


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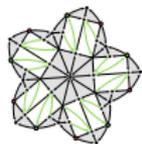
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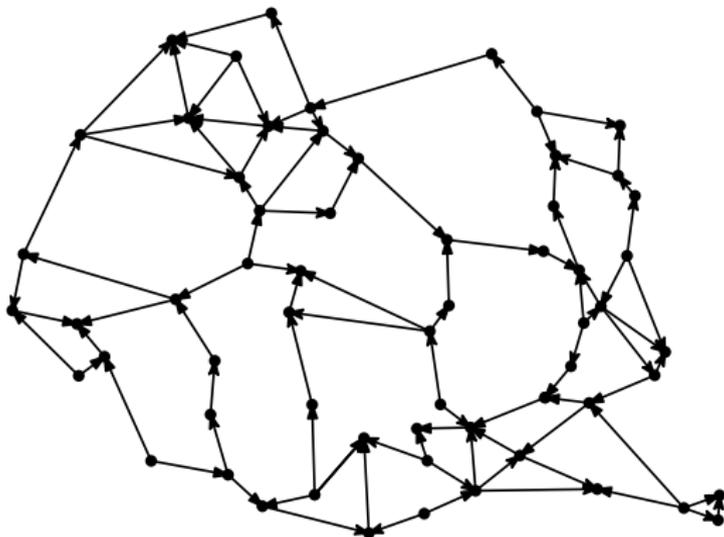
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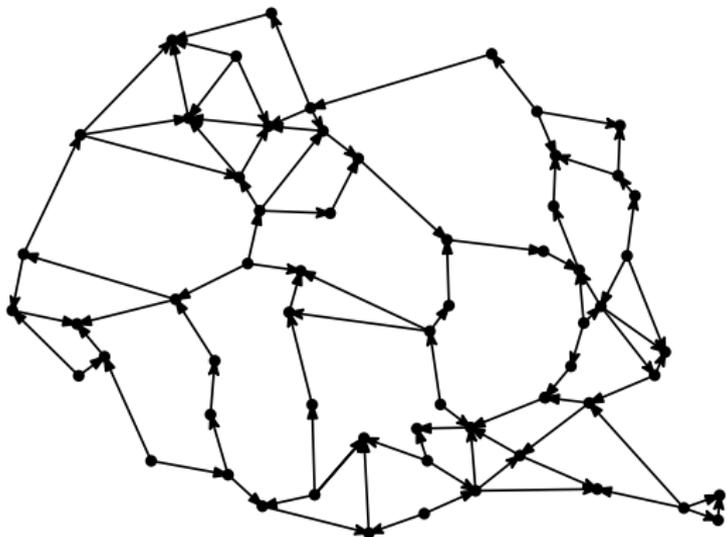
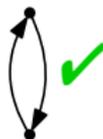
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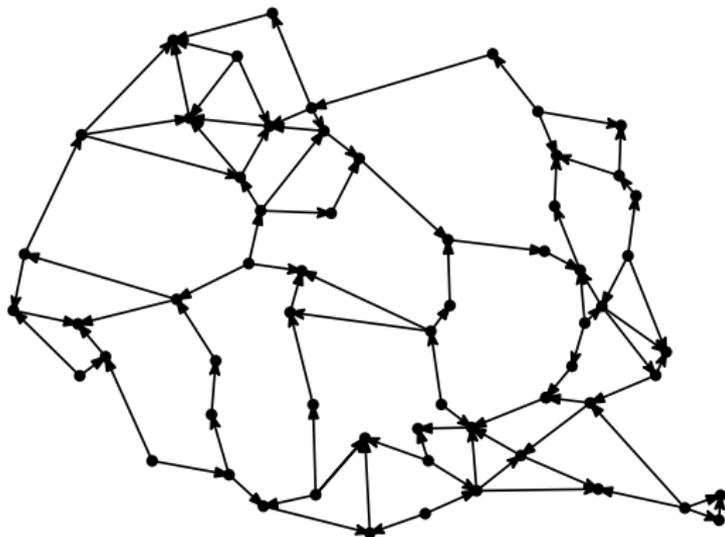
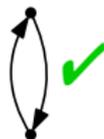
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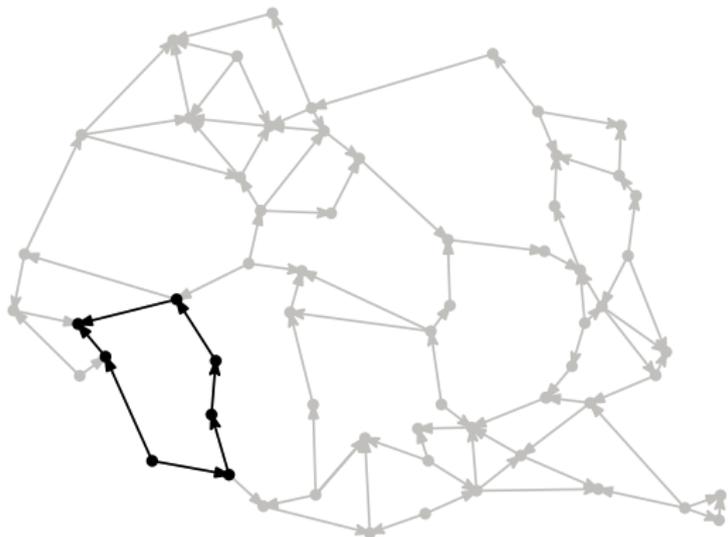
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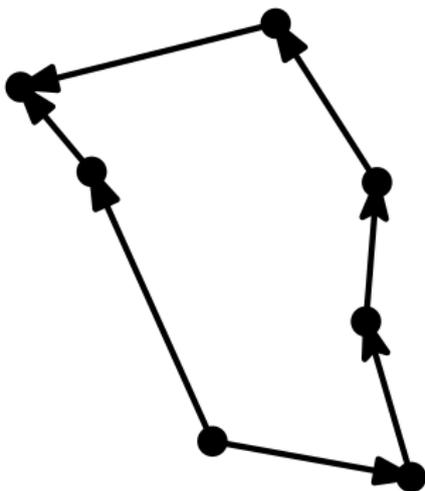
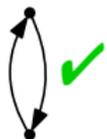
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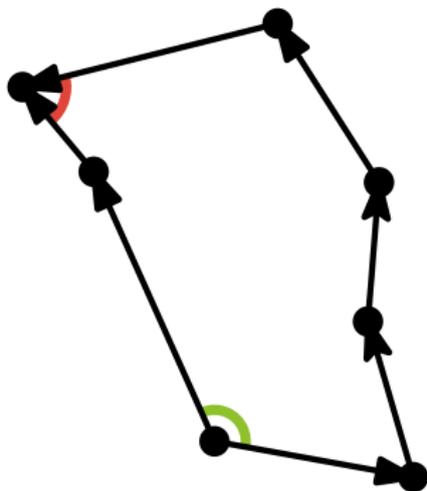
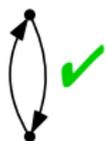
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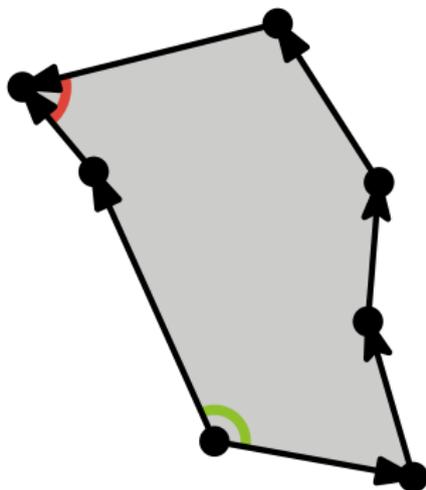
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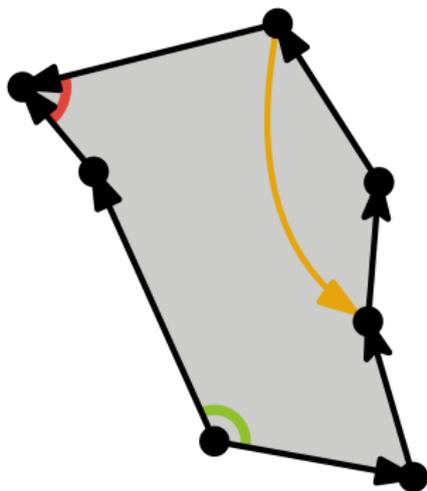
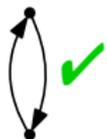
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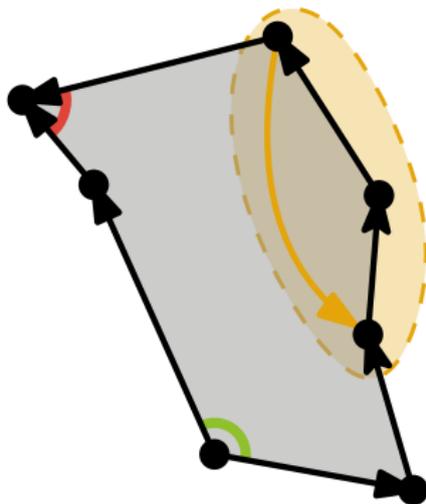
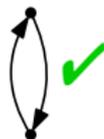
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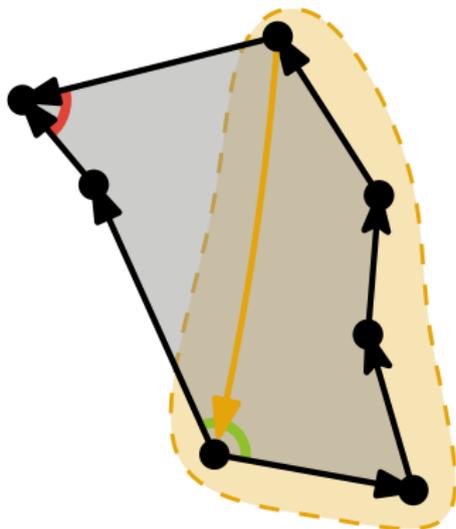
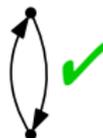
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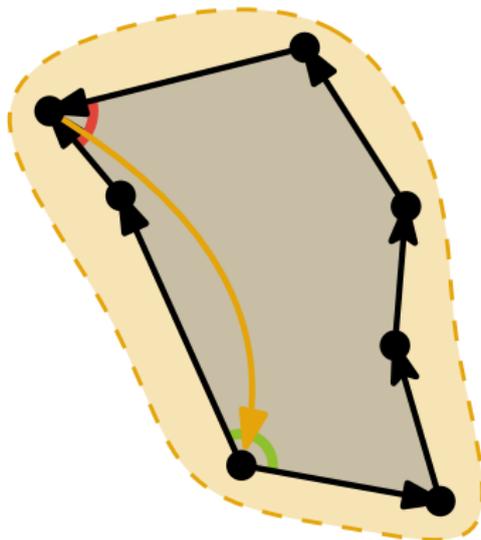
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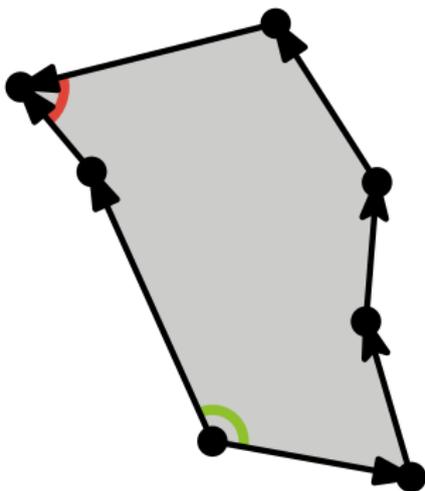
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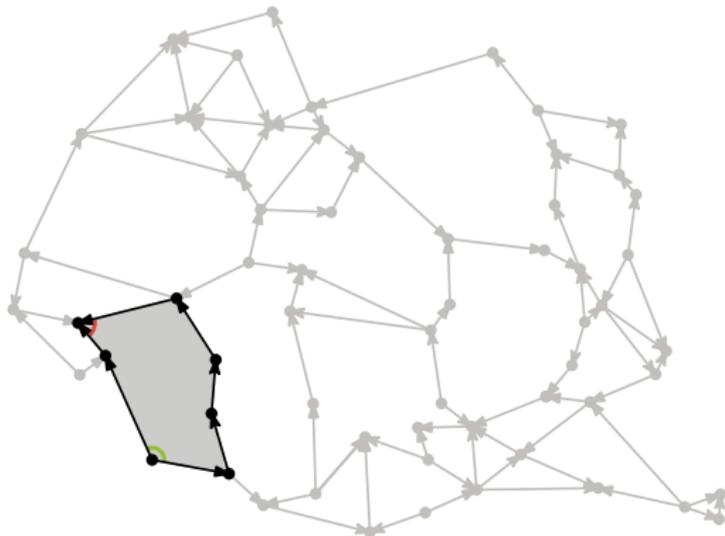
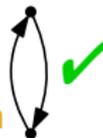
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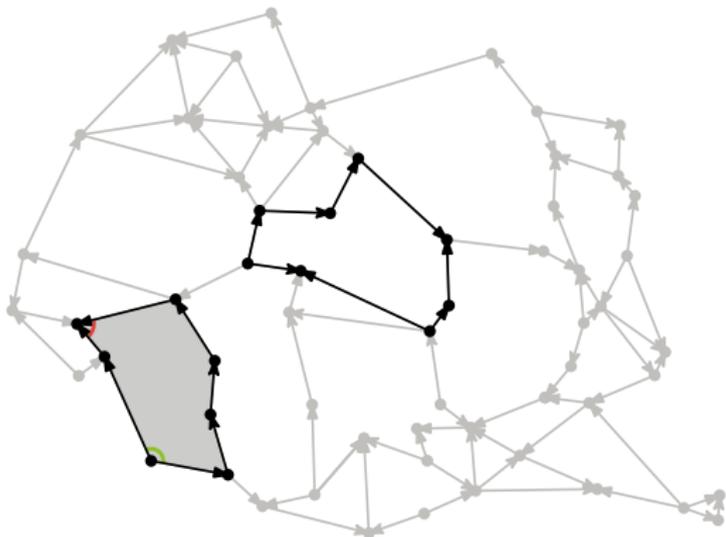
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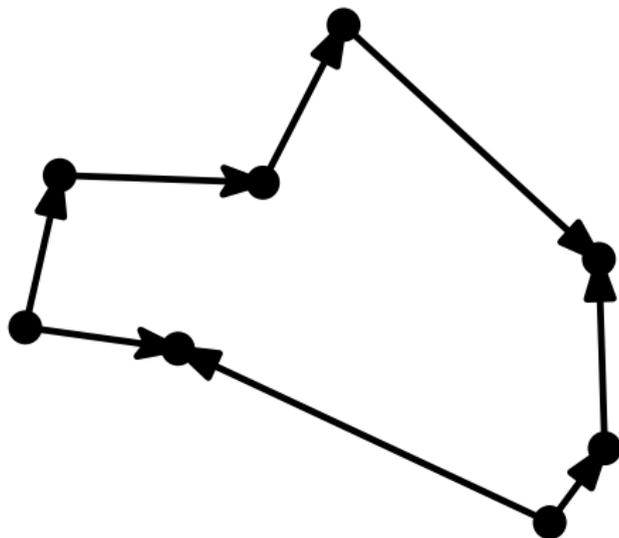
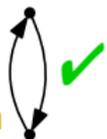
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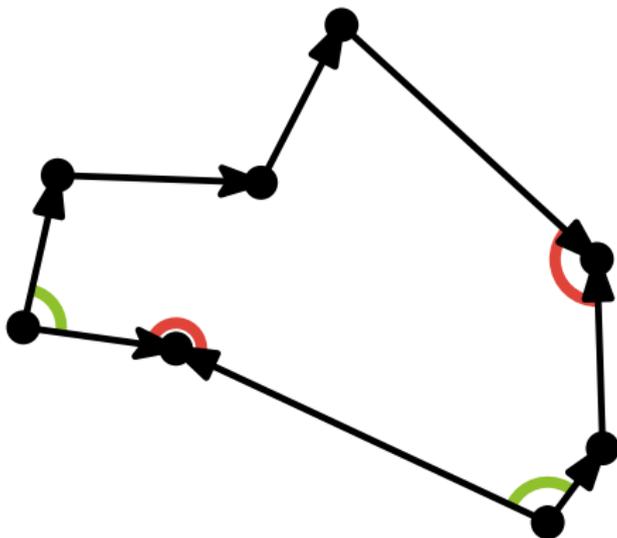
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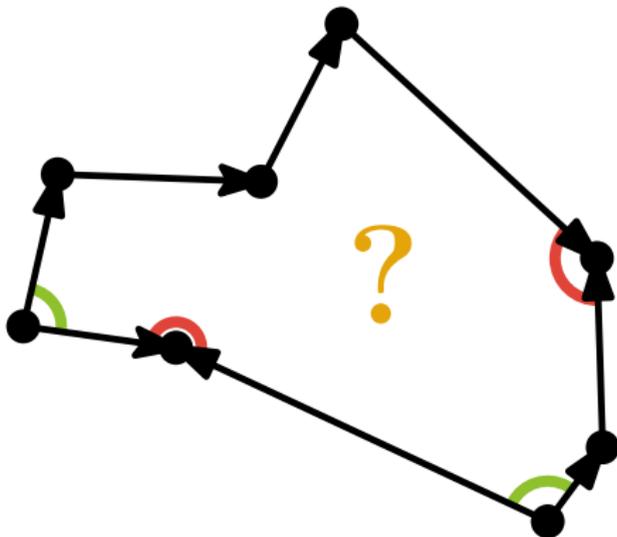
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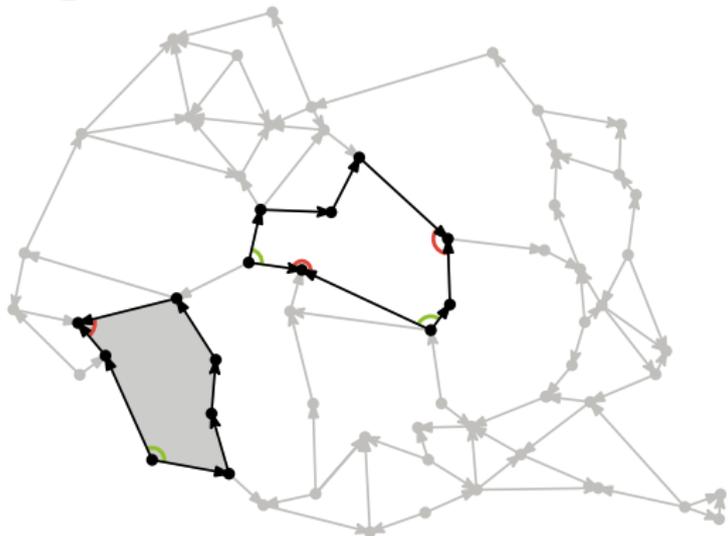
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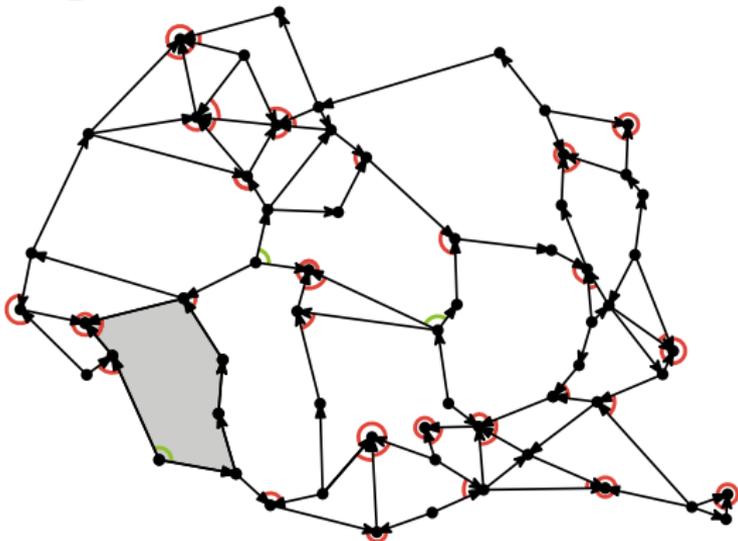
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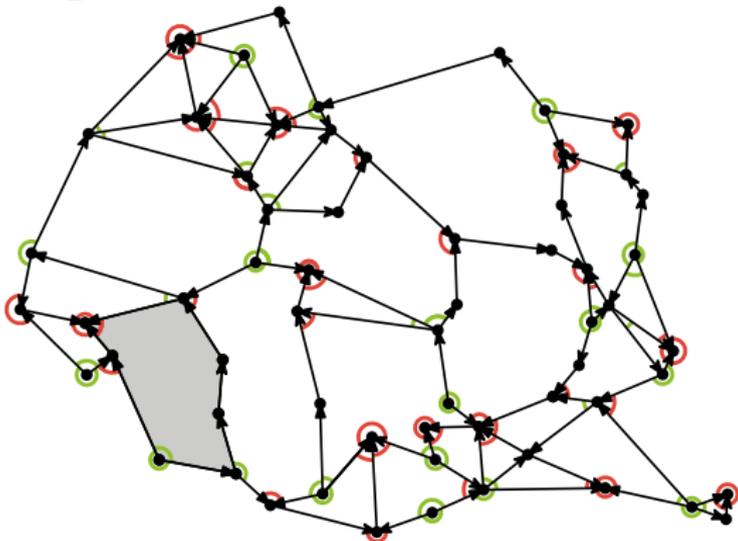
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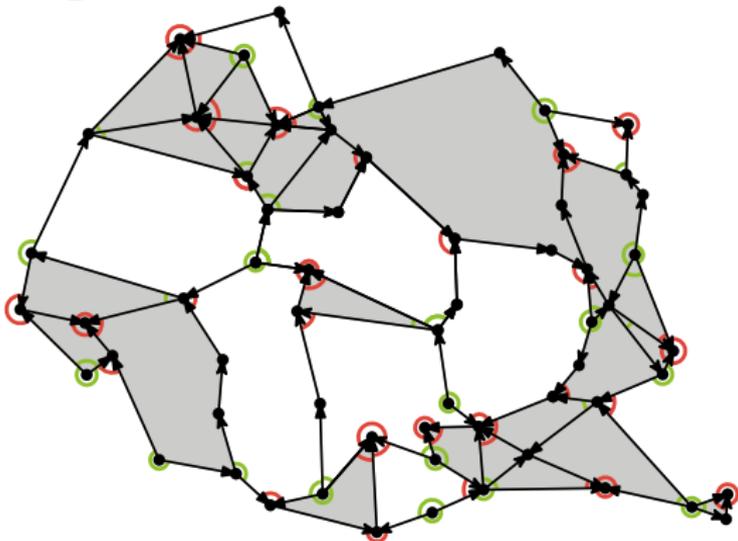
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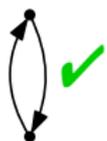
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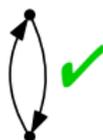
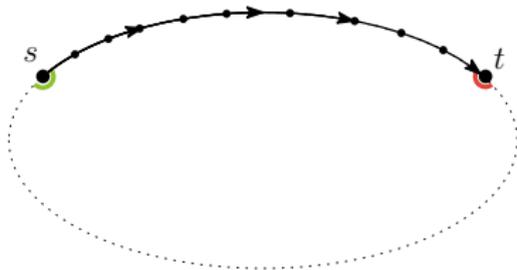
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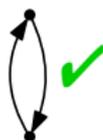
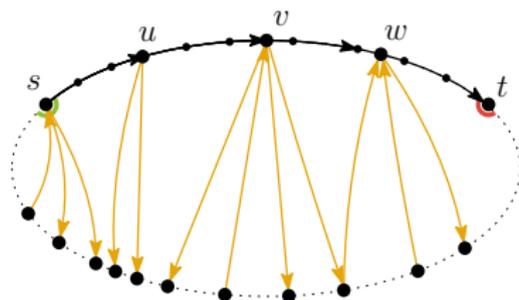
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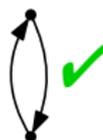
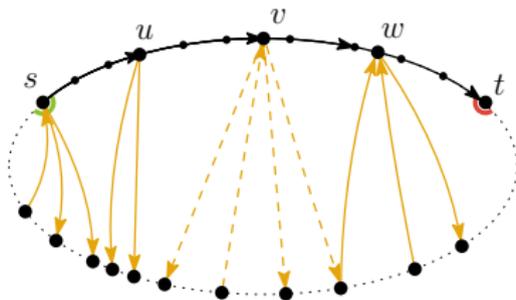
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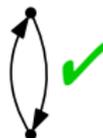
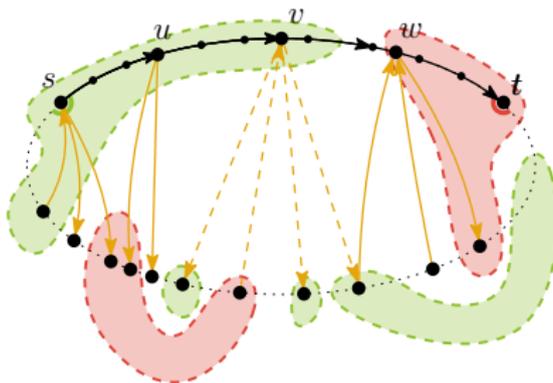
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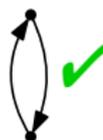
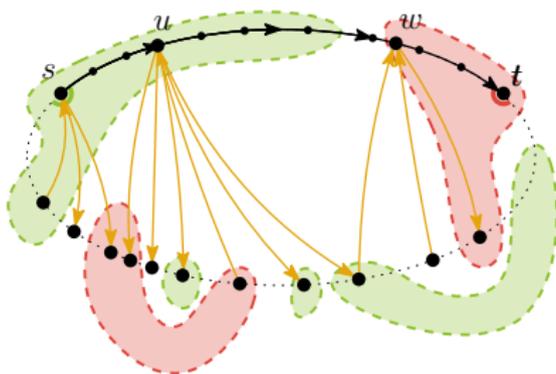
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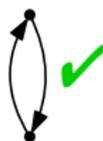
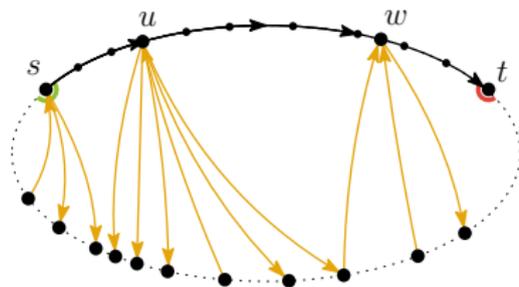
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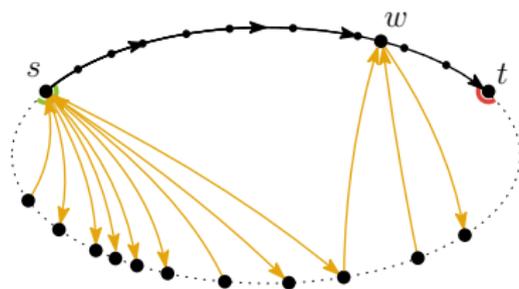
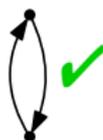
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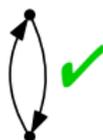
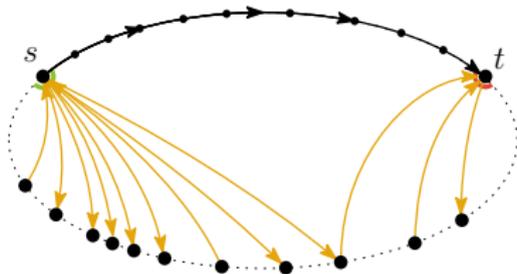
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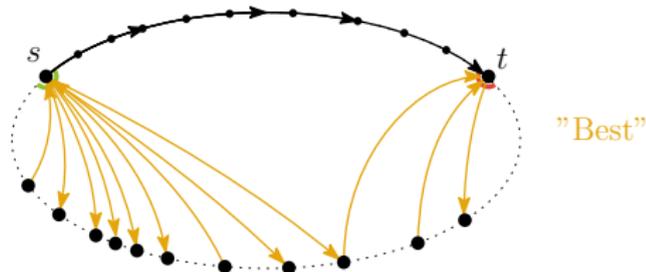
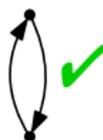
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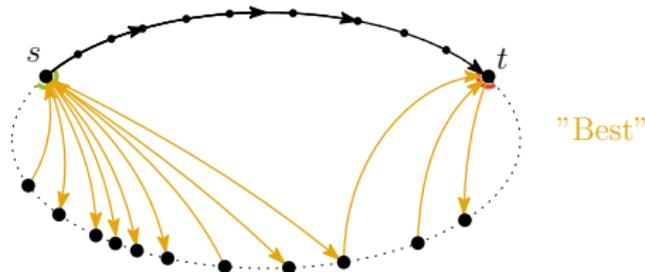
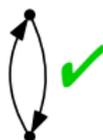
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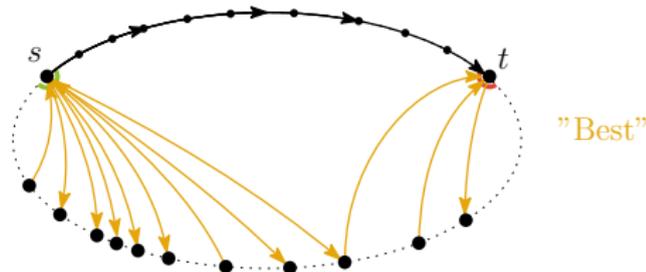
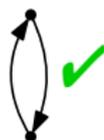


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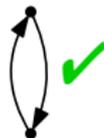
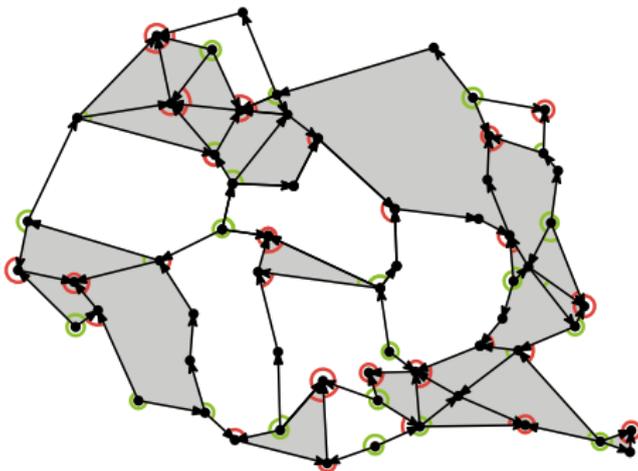
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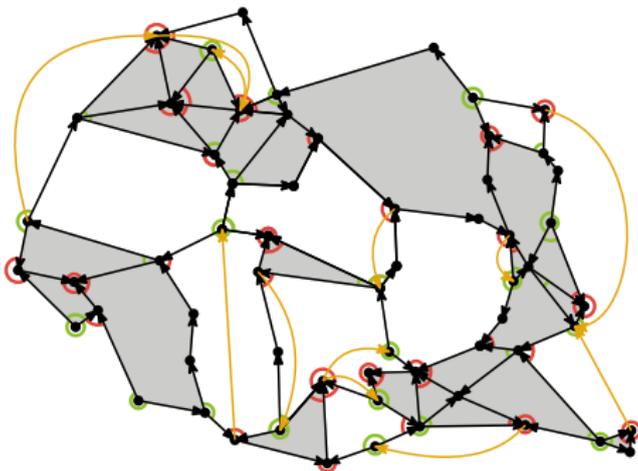
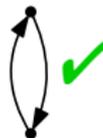
$\hookrightarrow k^{O(k)}$  different **best completions** across all alternating faces.

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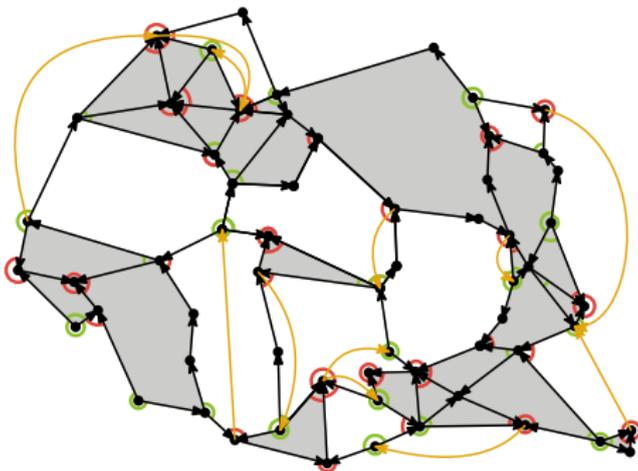
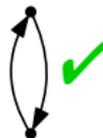
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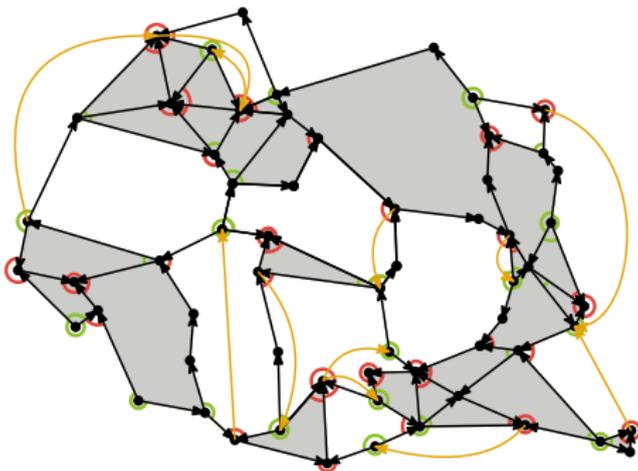
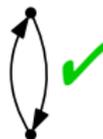
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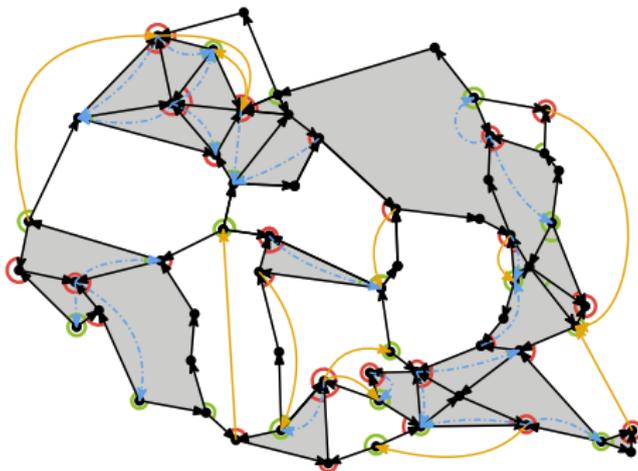
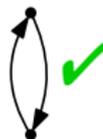
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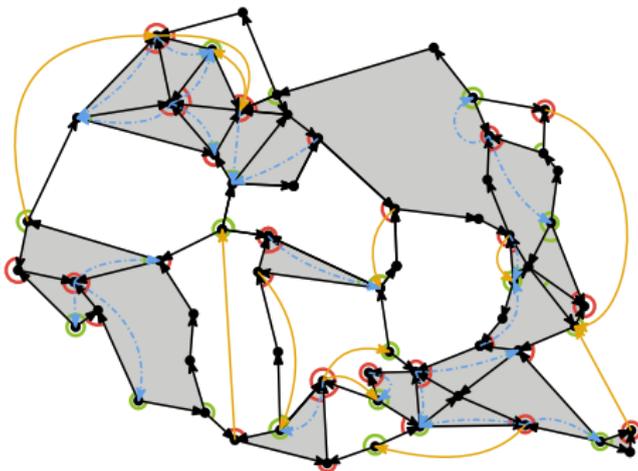
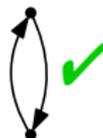
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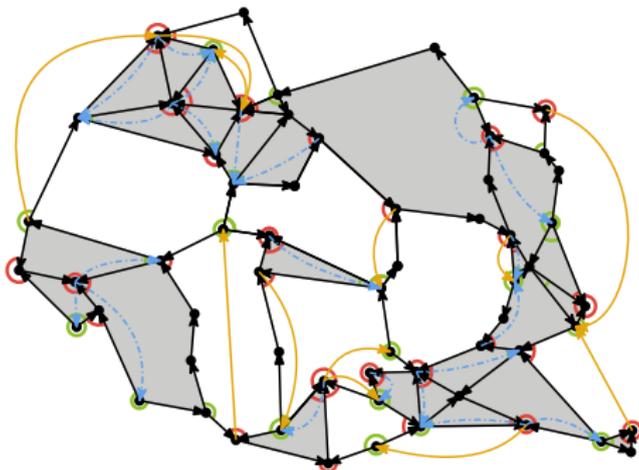
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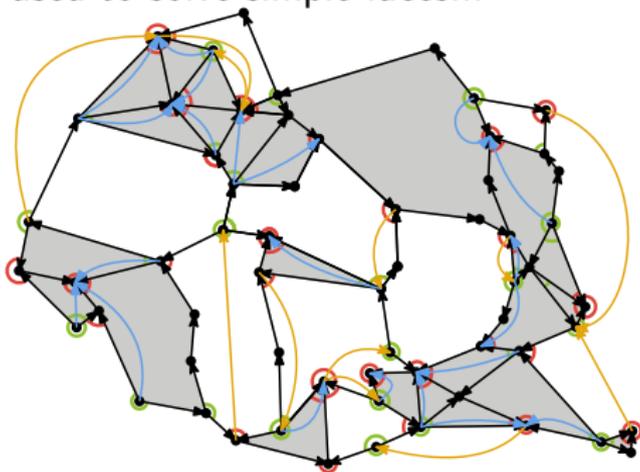
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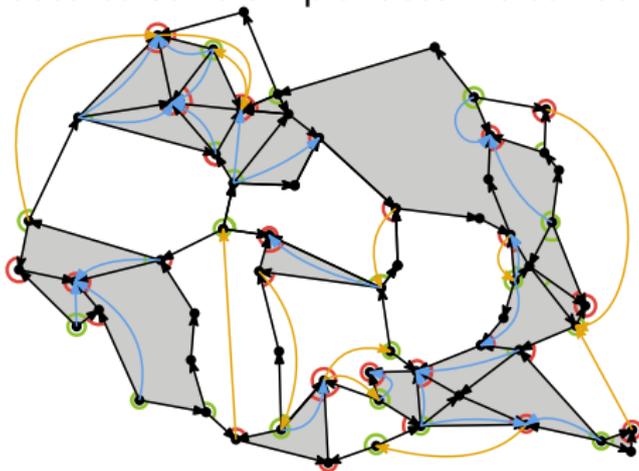


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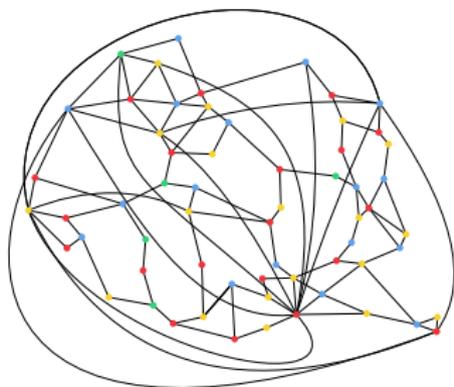
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## II - Forcing oriented subtrees with $\chi$

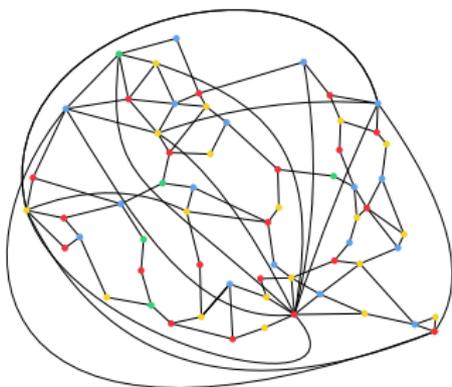
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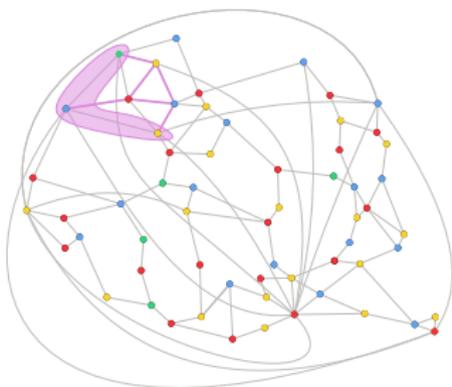
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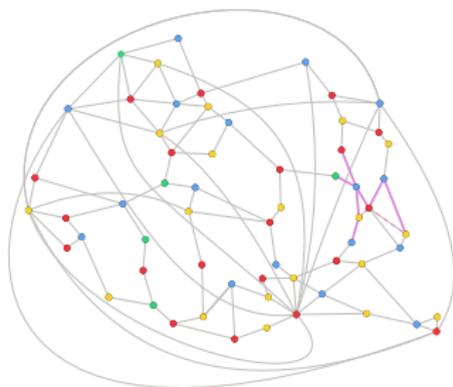


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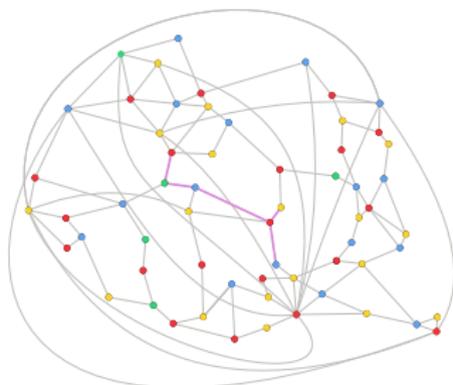


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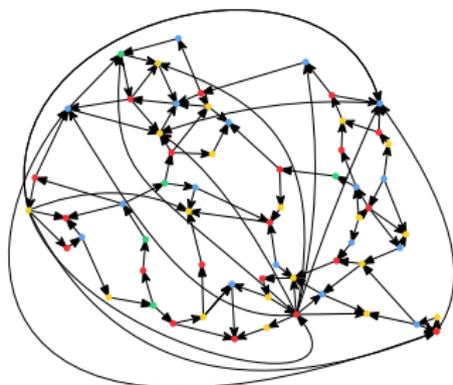


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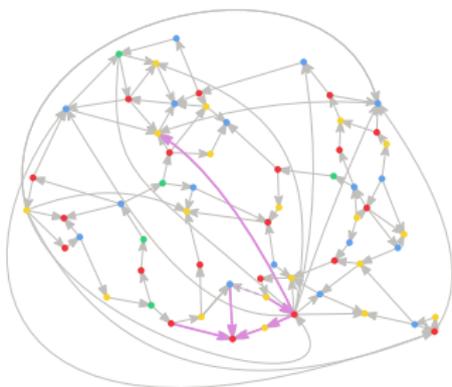


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**Theorem (Erdős, Hajnal '1966)**  
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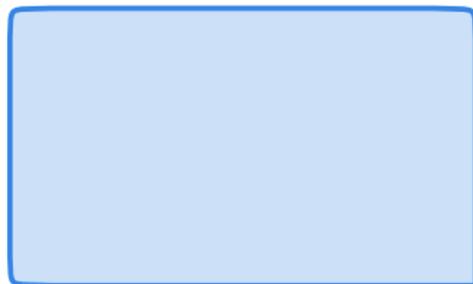
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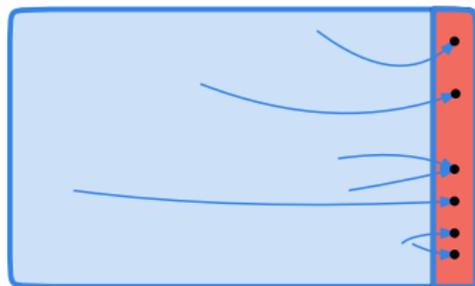
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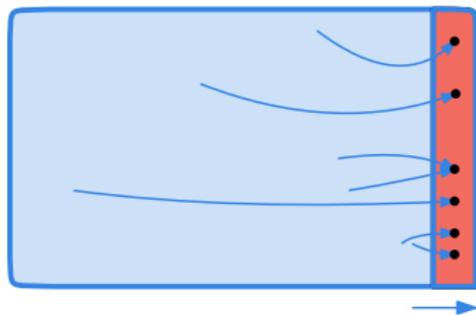
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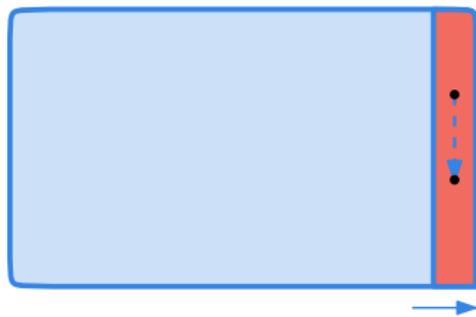
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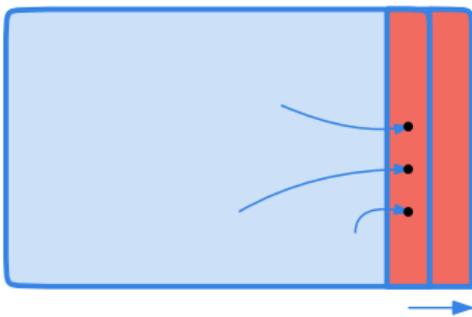
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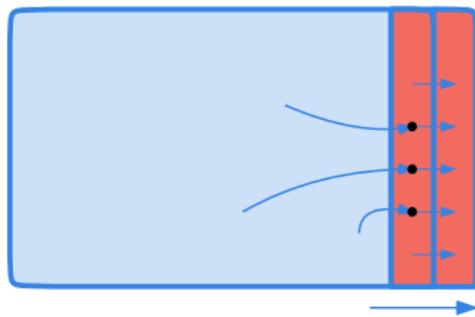
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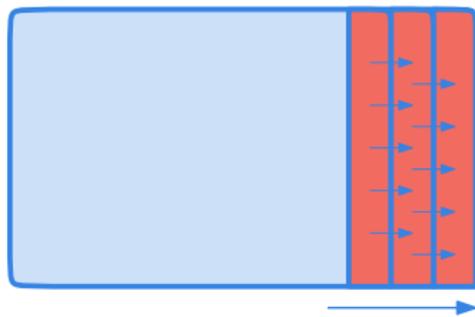
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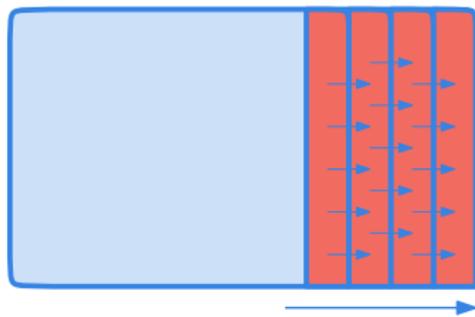
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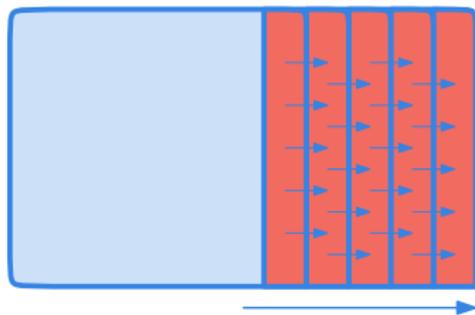
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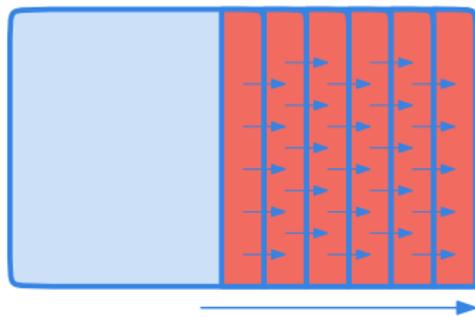
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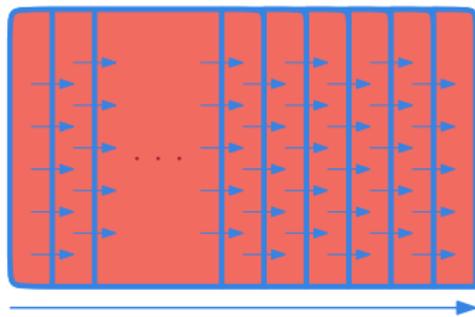
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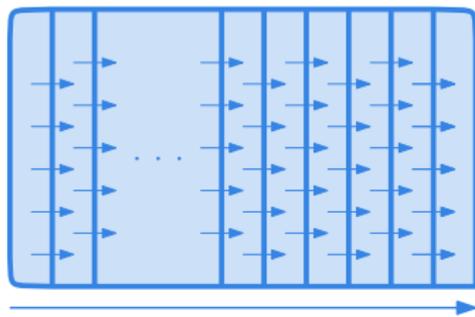
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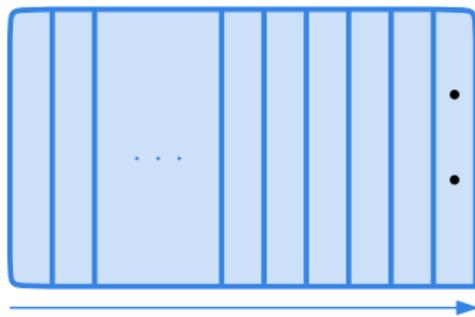
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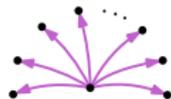
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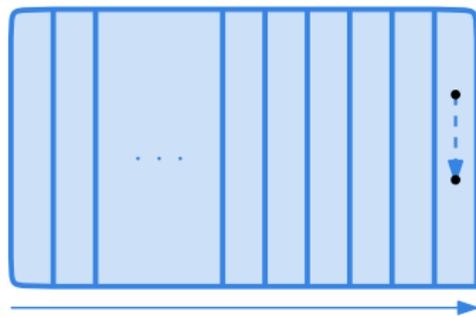
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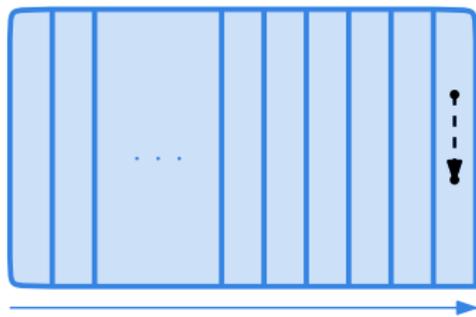
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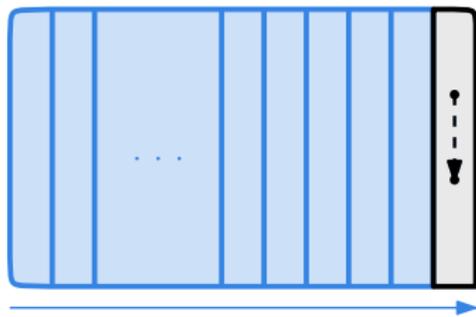
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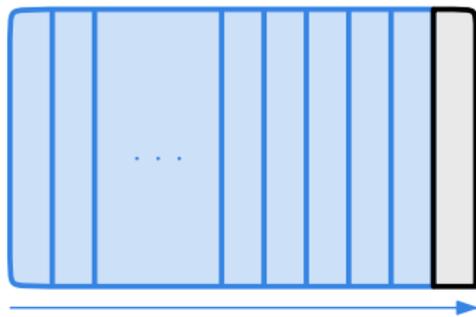
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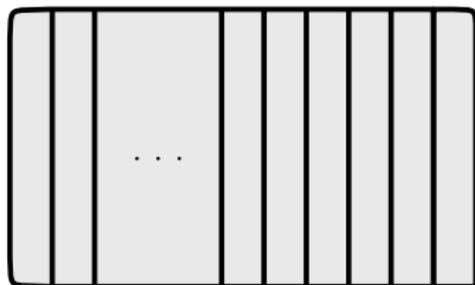
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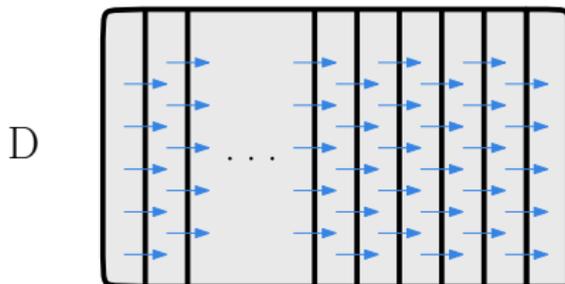
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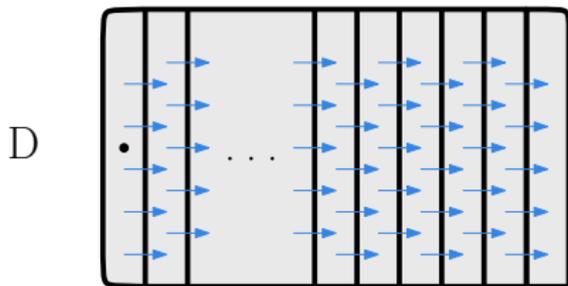
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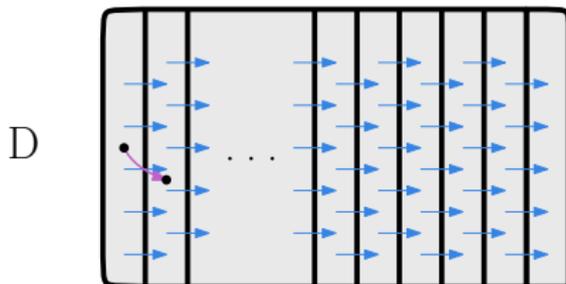
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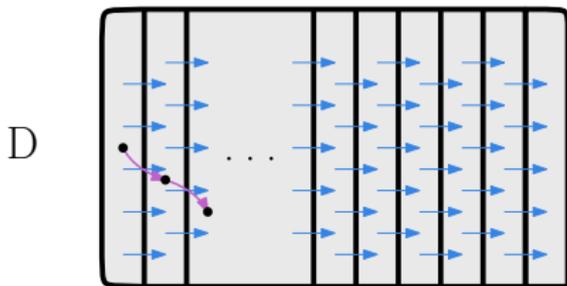
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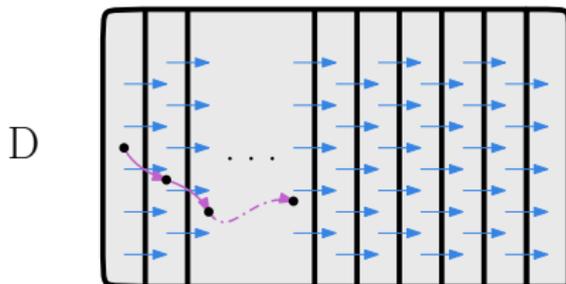
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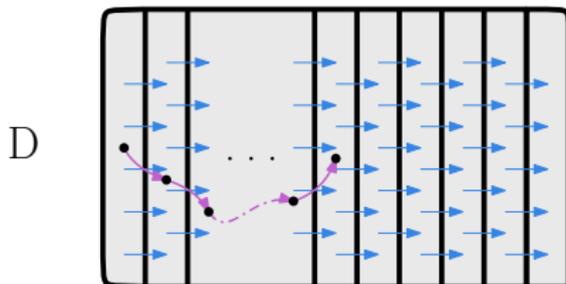
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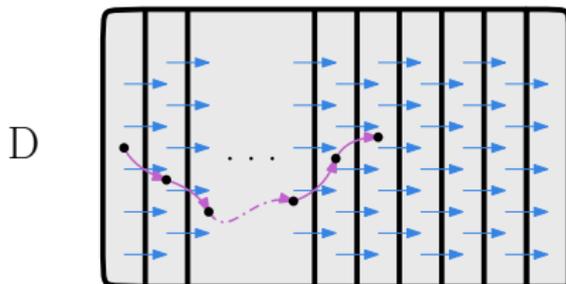
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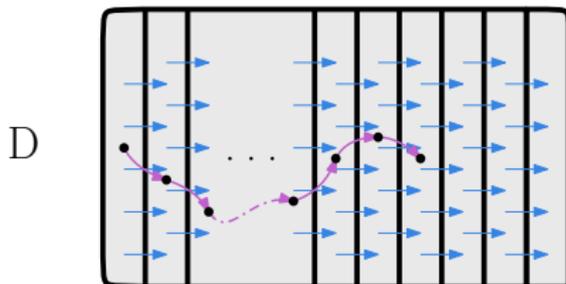
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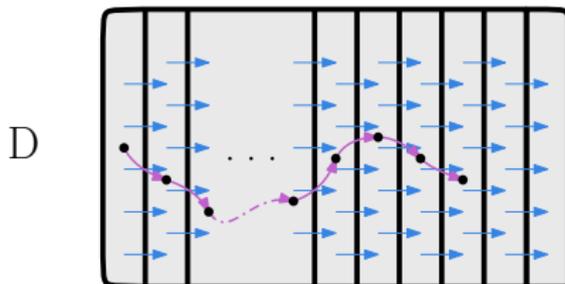
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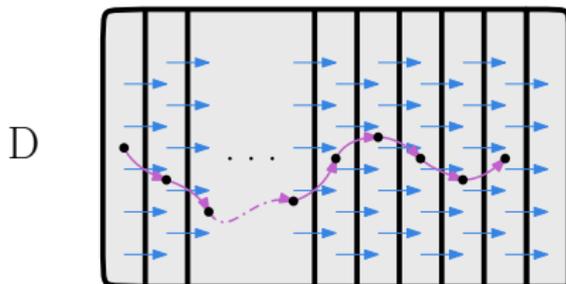
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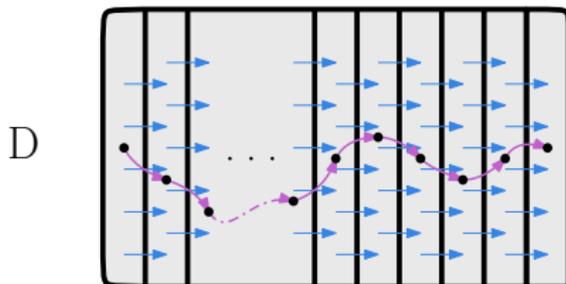
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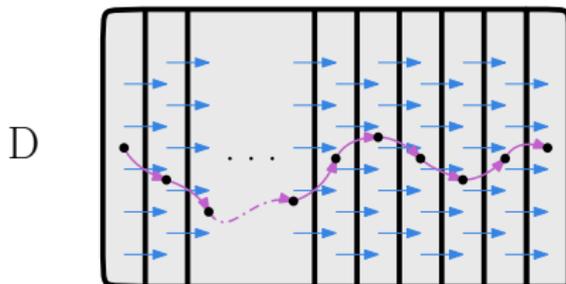
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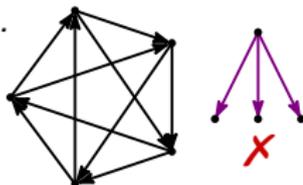
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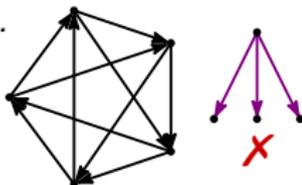


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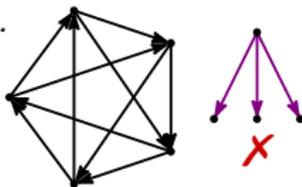
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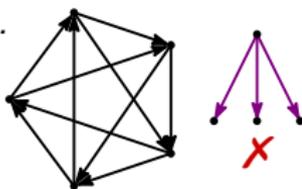
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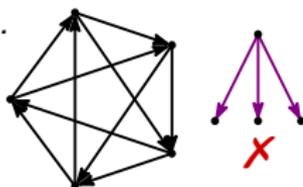
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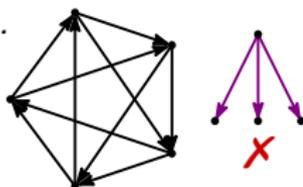
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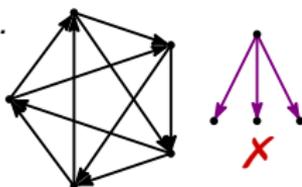
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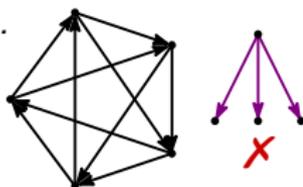
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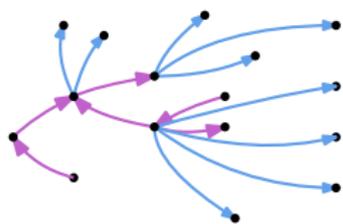
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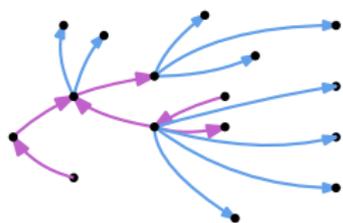
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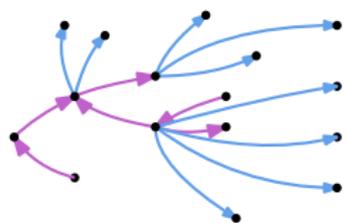
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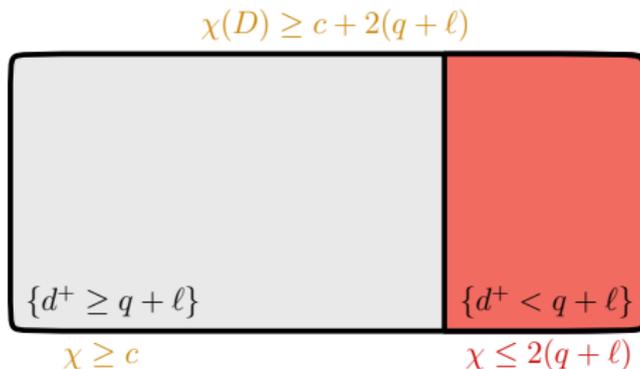
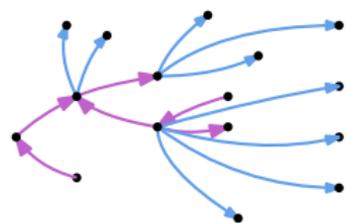
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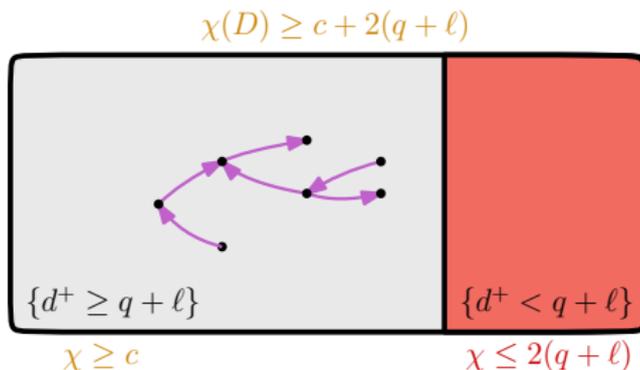
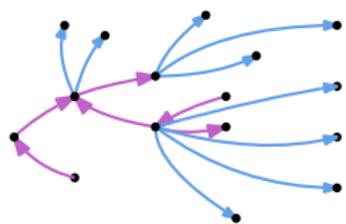
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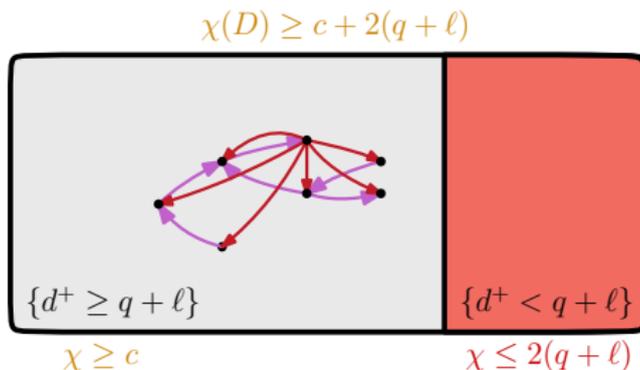
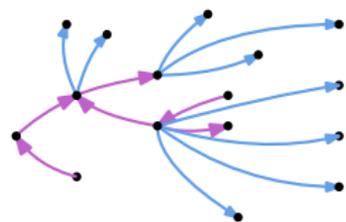
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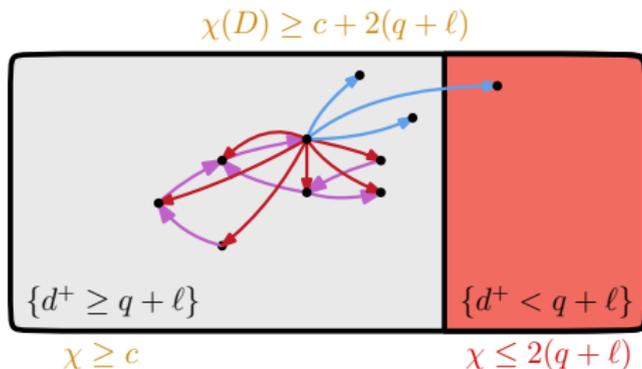
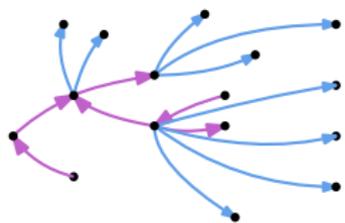
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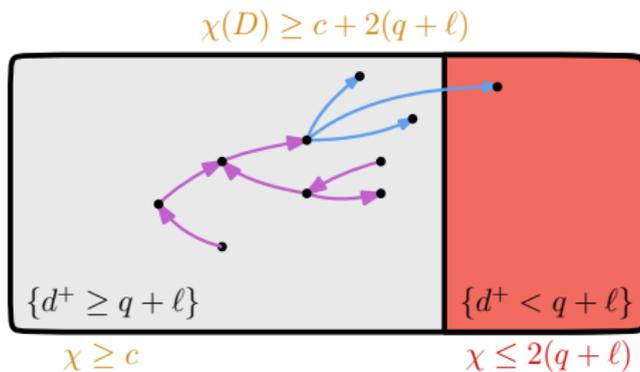
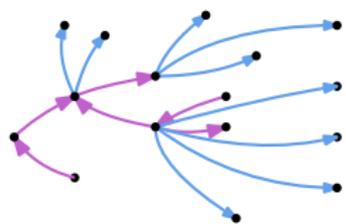
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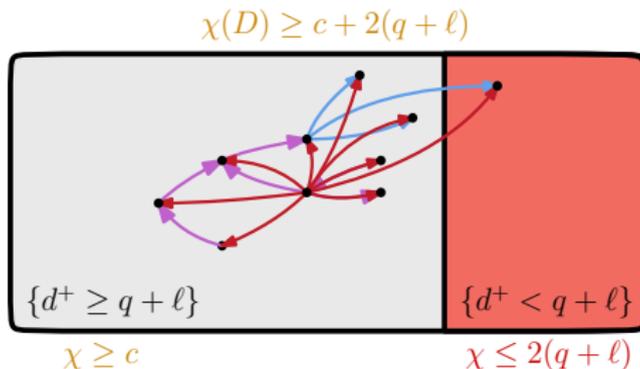
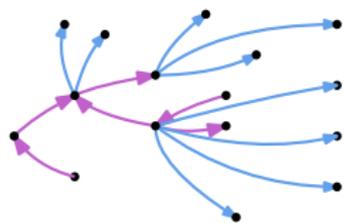
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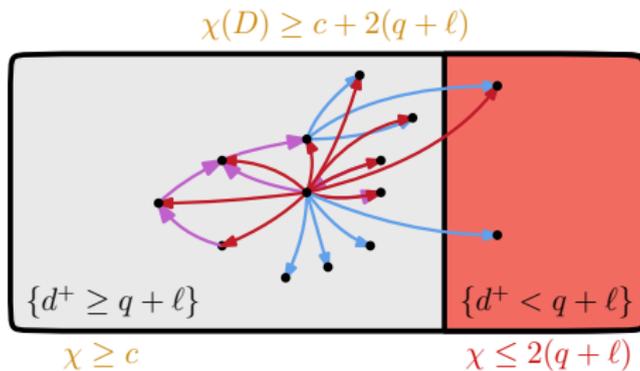
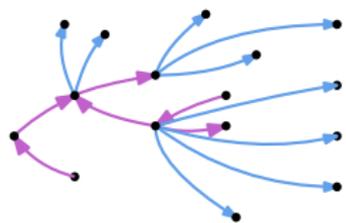
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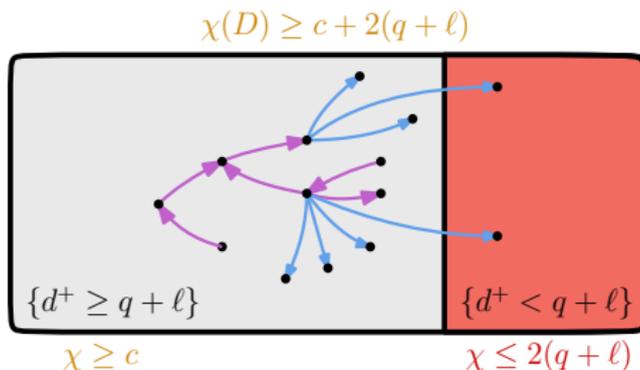
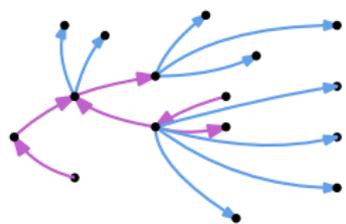
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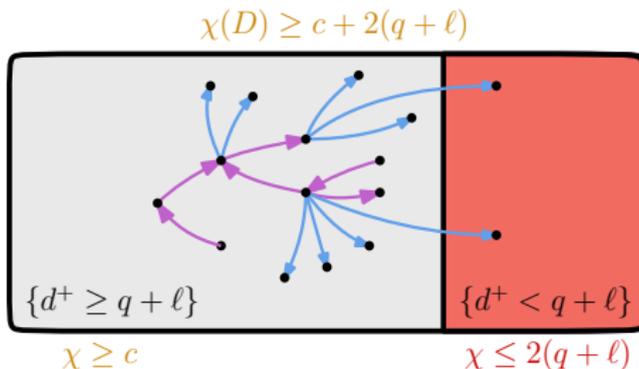
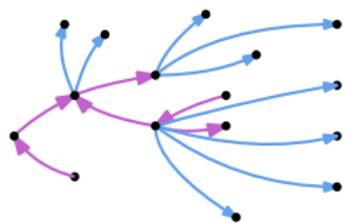
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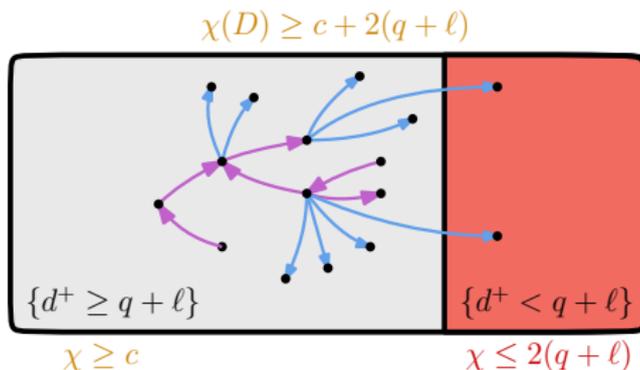
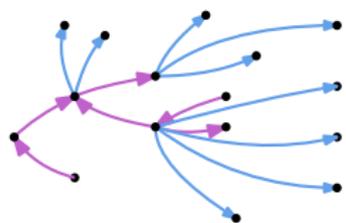
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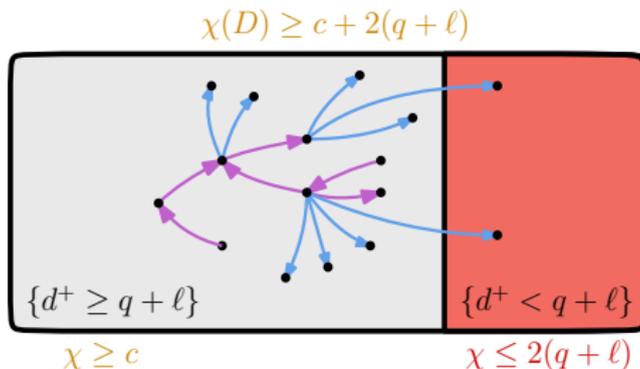
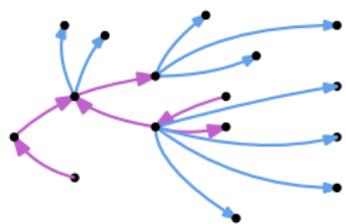
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$\chi(D) \geq c + 2(q + \ell) \implies T$  glued with  $\ell$  out-leaves

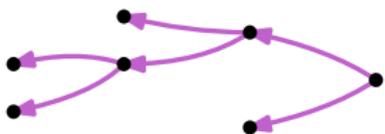


Already yields  $O(k^2)$  bound.

# Growing trees by gluing directed paths

**Lemma (Bessy, Gonçalves, R. '24)**

Assume  $\chi(D) \geq c \implies q$ -tree  $T$ ,

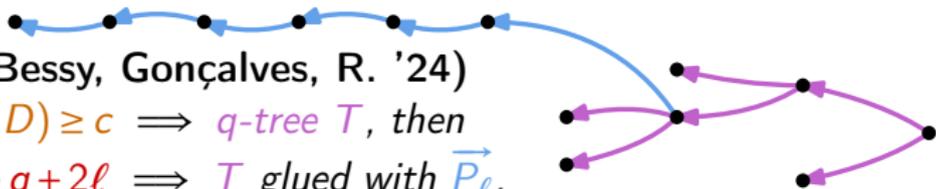


# Growing trees by gluing directed paths

**Lemma (Bessy, Gonçalves, R. '24)**

Assume  $\chi(D) \geq c \implies q$ -tree  $T$ , then

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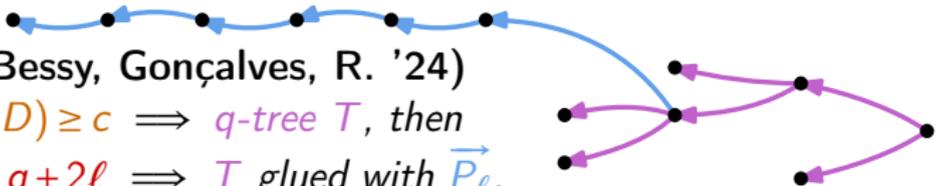


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max DAG

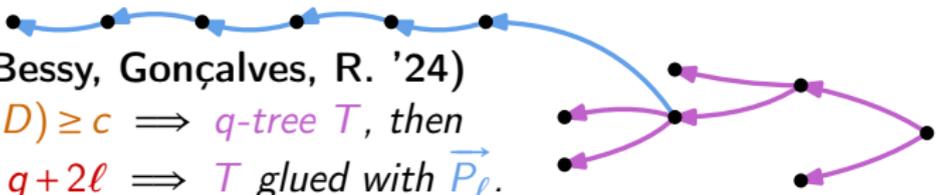


# Growing trees by gluing directed paths

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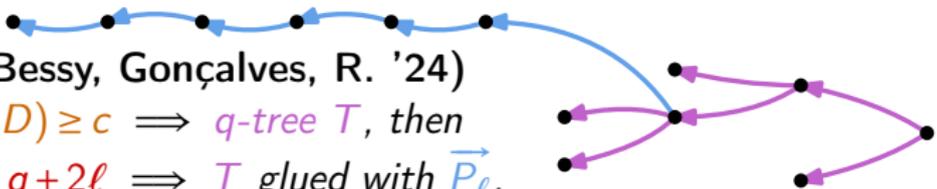
max DAG  
 $\chi \geq q + \ell \implies T + \vec{P}_\ell$   
 $\chi < q + \ell$

# Growing trees by gluing directed paths

**Lemma (Bessy, Gonçalves, R. '24)**

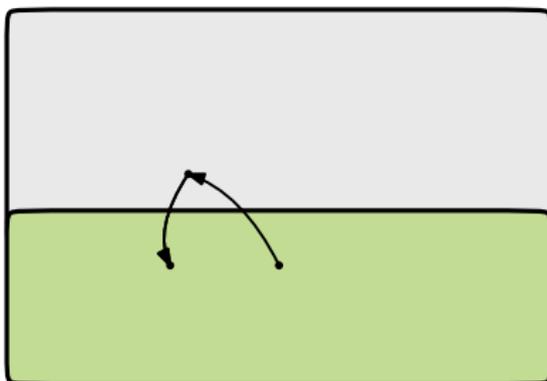
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max DAG

$\chi < q + \ell$

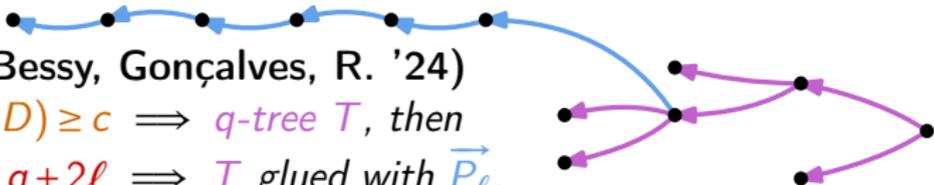


# Growing trees by gluing directed paths

**Lemma (Bessy, Gonçalves, R. '24)**

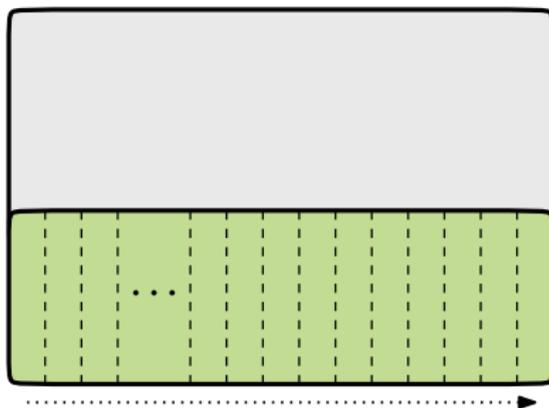
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max DAG

$\chi < q + \ell$

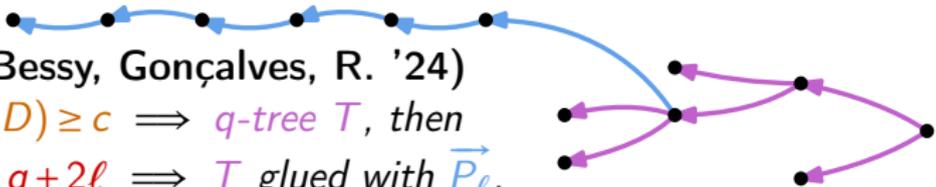


# Growing trees by gluing directed paths

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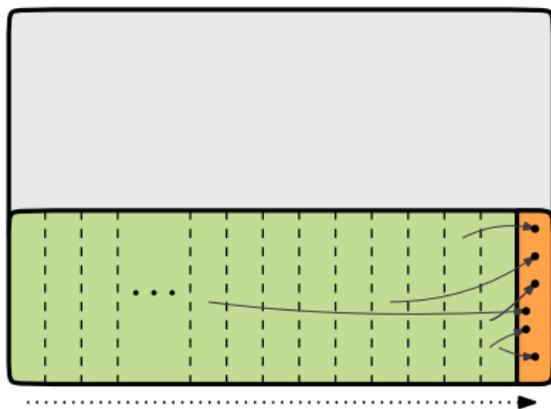
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max DAG

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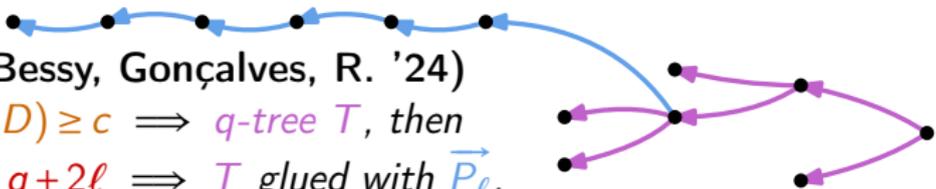


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$\chi < q + \ell$



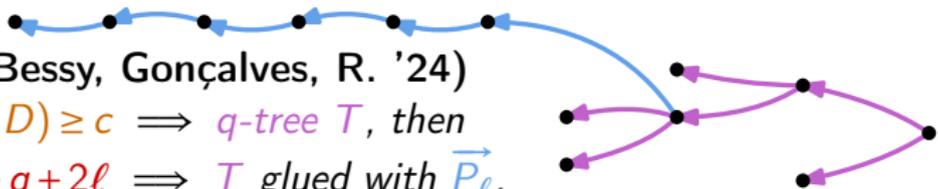
$\chi = 1$

# Growing trees by gluing directed paths

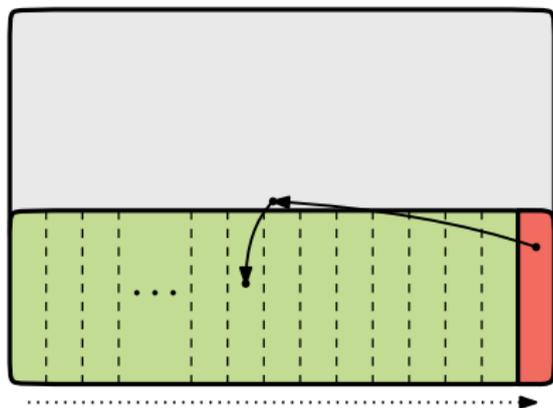
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$\chi < q + \ell$



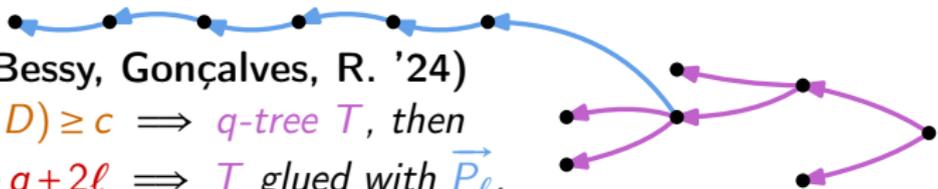
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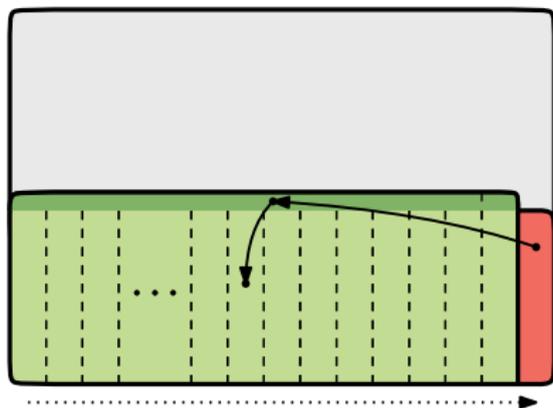
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max DAG

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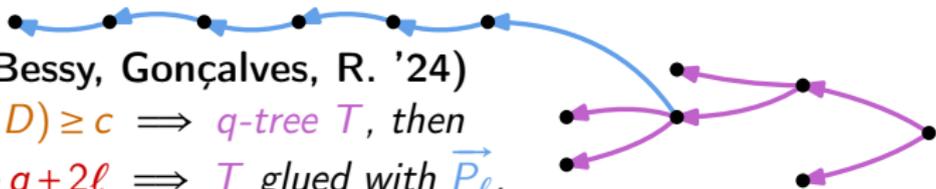
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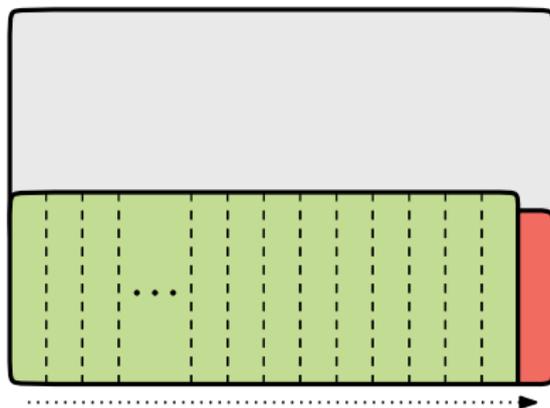
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max DAG

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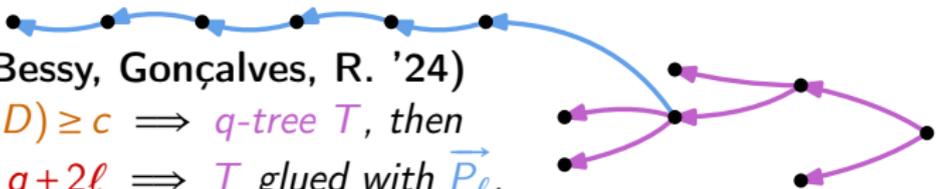
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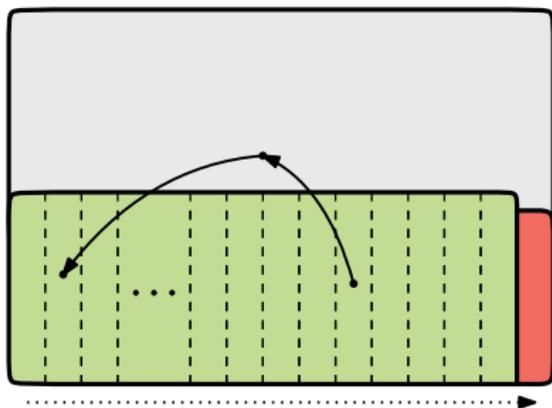
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max DAG

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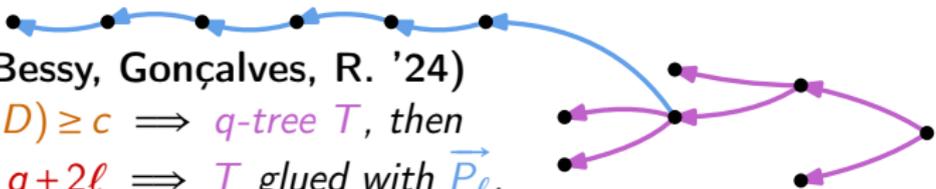
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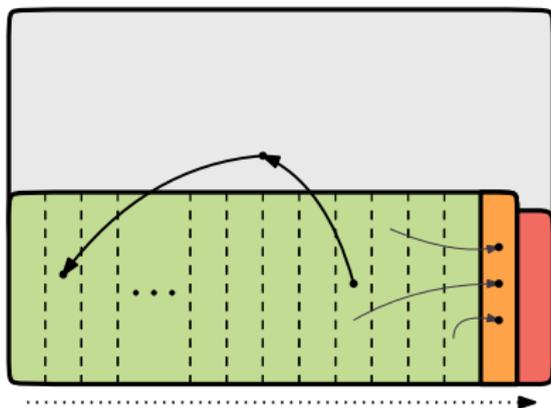
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max DAG

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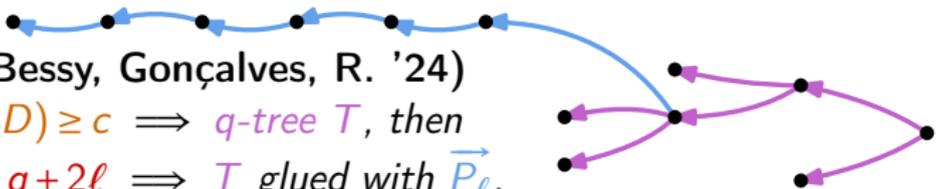
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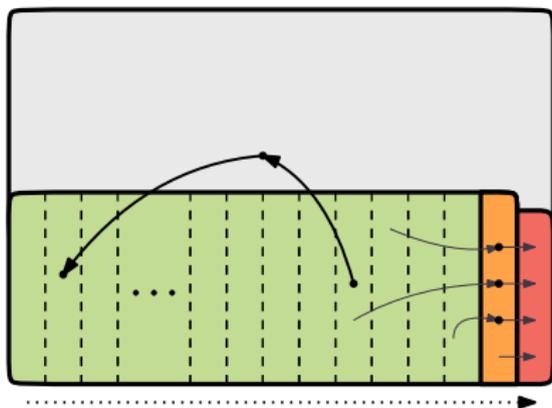
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max DAG

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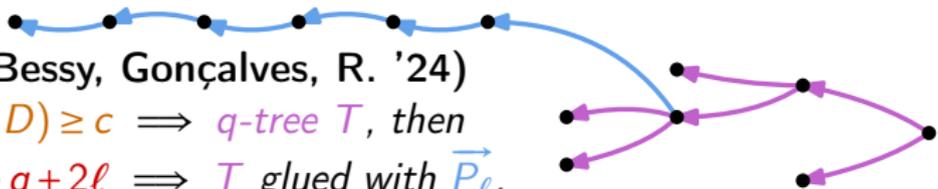
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# Growing trees by gluing directed paths

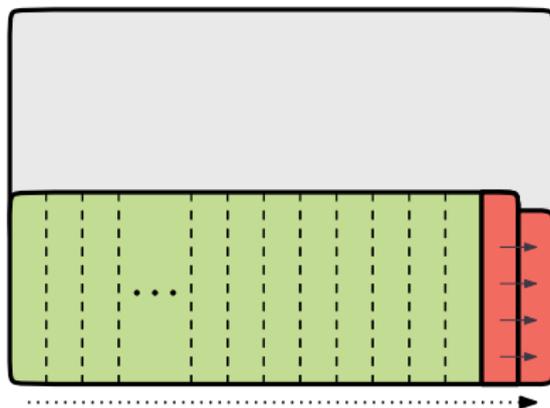
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$$\chi < q + \ell$$



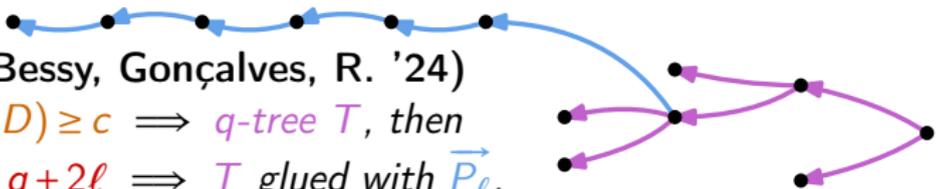
$$\chi = 2$$

# Growing trees by gluing directed paths

**Lemma (Bessy, Gonçalves, R. '24)**

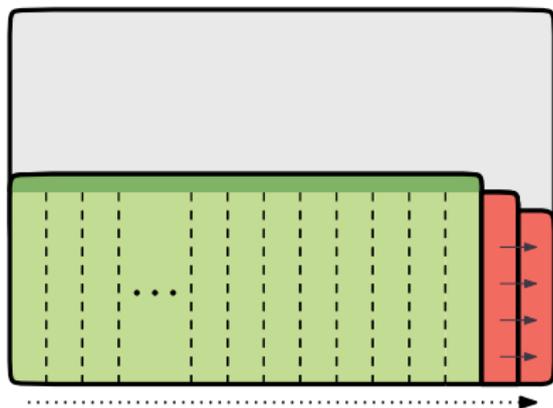
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max DAG

$\chi < q + \ell$



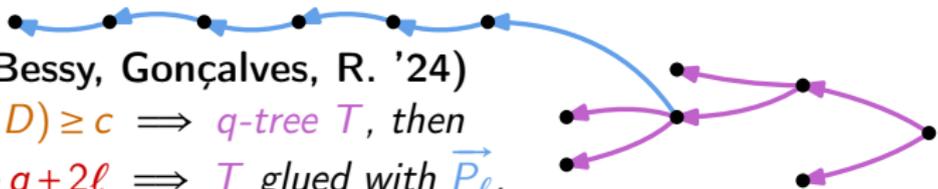
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# Growing trees by gluing directed paths

**Lemma (Bessy, Gonçalves, R. '24)**

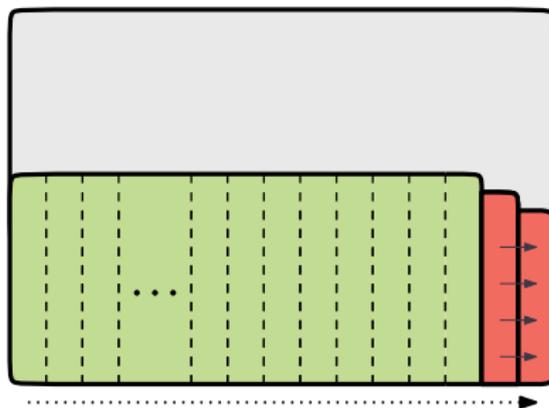
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max DAG

$\chi < q + \ell$



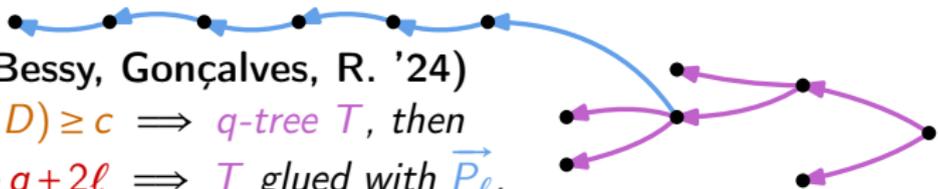
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## Growing trees by gluing directed paths

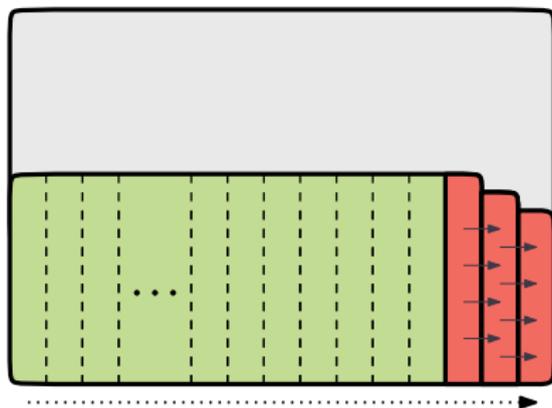
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$$\chi < q + \ell$$



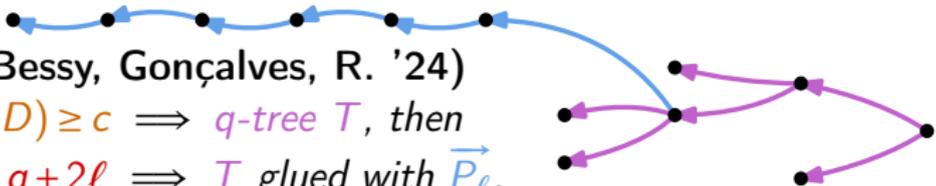
$$\chi \leq 3$$

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max DAG

$\chi < q + \ell$



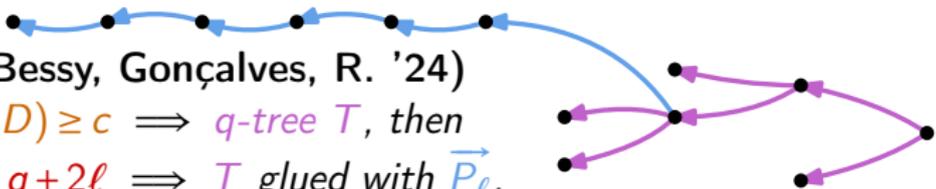
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# Growing trees by gluing directed paths

**Lemma (Bessy, Gonçalves, R. '24)**

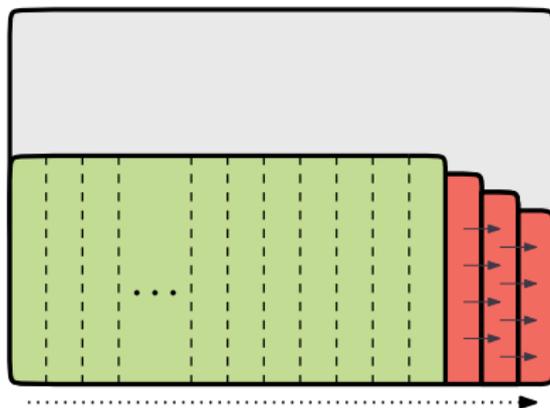
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max DAG

$\chi < q + \ell$



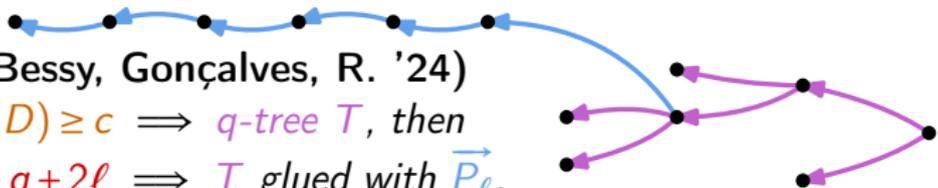
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# Growing trees by gluing directed paths

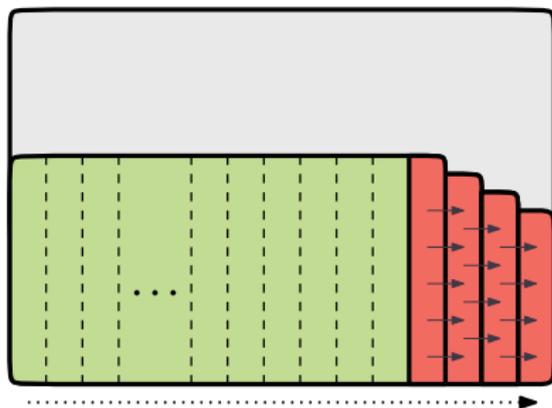
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$$\chi < q + \ell$$



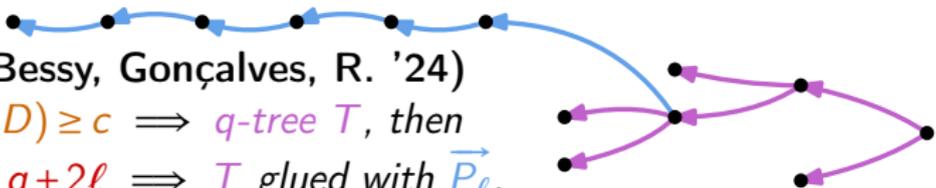
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# Growing trees by gluing directed paths

**Lemma (Bessy, Gonçalves, R. '24)**

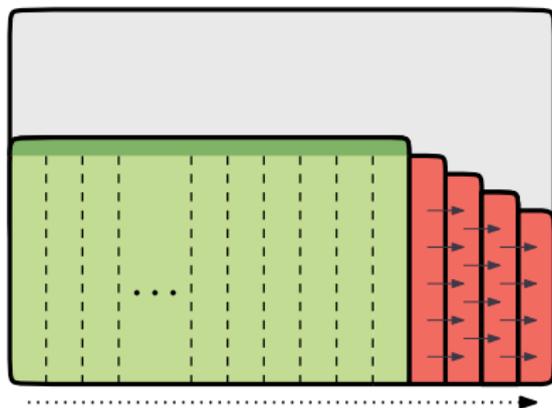
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max DAG

$\chi < q + \ell$



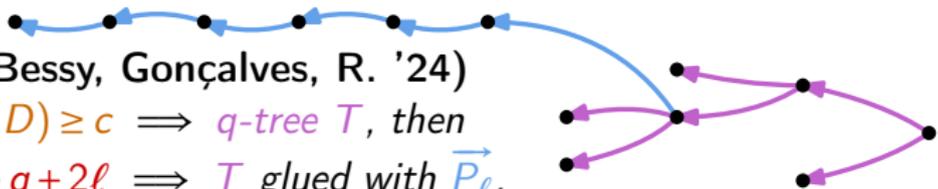
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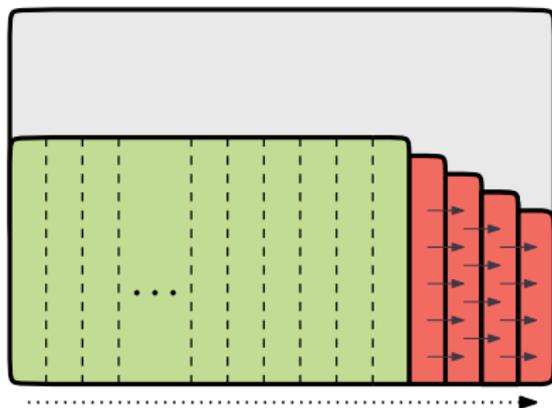
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max DAG

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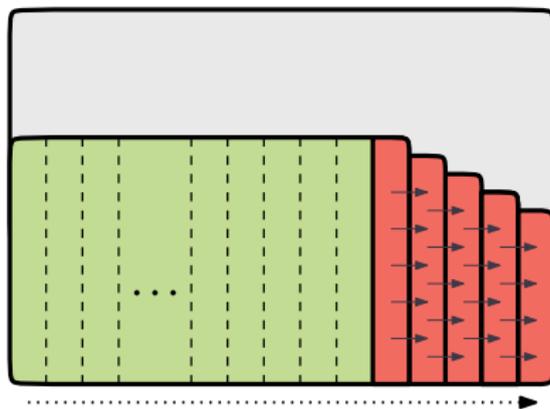
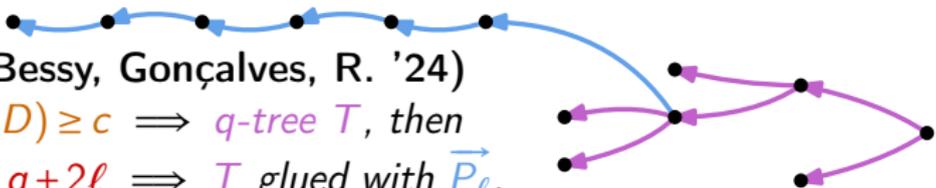
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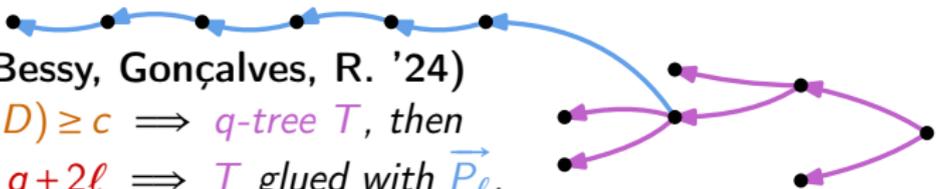
$$\chi \leq 5$$

## Growing trees by gluing directed paths

**Lemma (Bessy, Gonçalves, R. '24)**

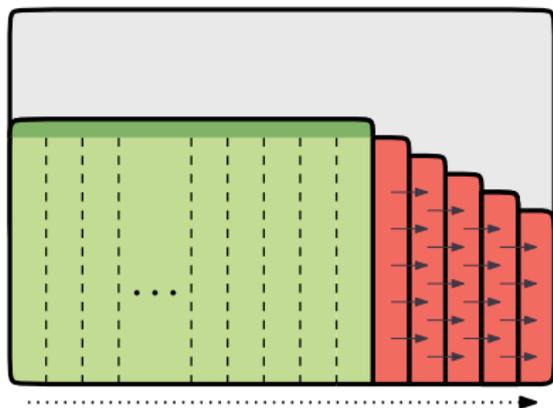
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max DAG

$\chi < q + \ell$



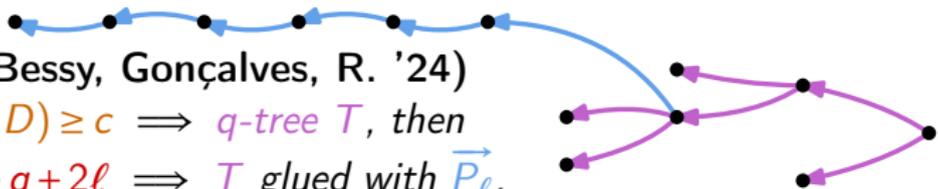
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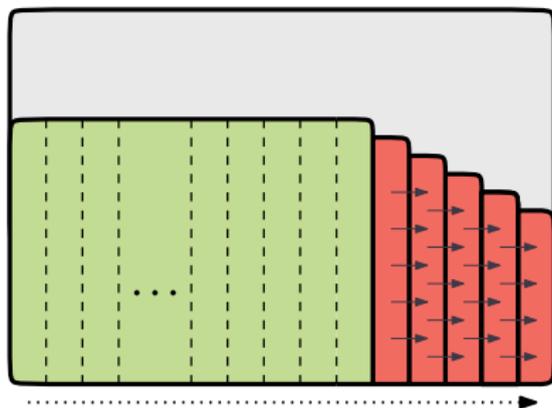
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max DAG

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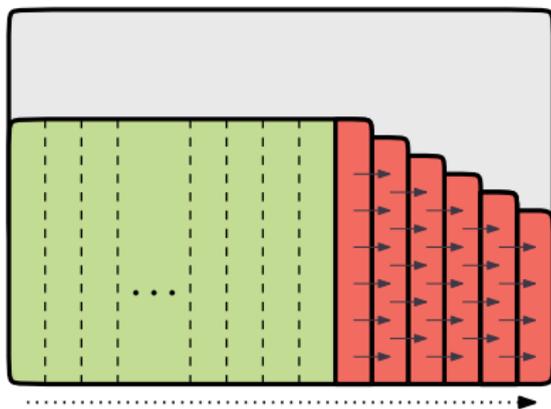
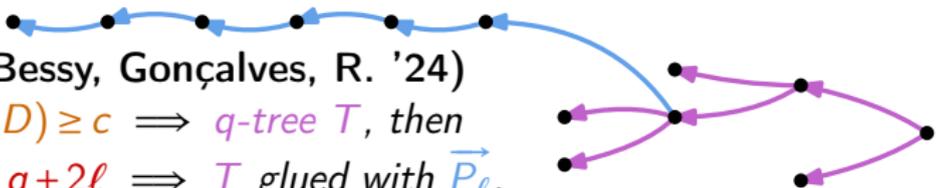
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$$\chi < q + \ell$$

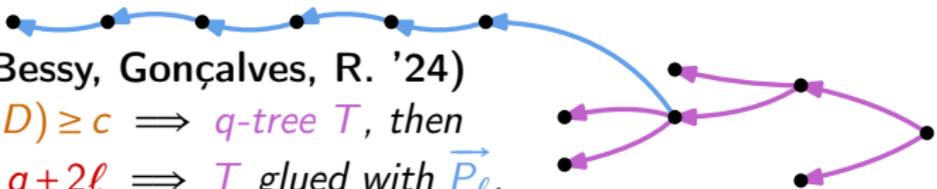
$$\chi \leq 6 = \ell$$

## Growing trees by gluing directed paths

**Lemma (Bessy, Gonçalves, R. '24)**

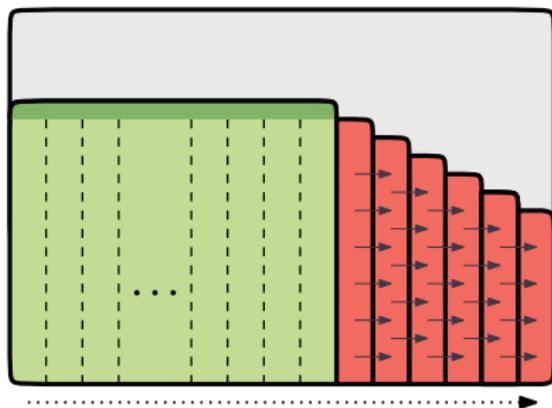
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max DAG

$\chi < q + \ell$



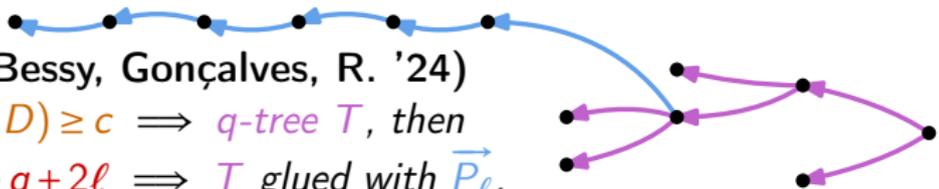
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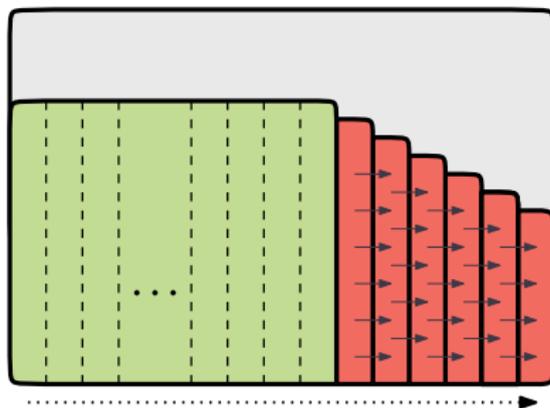
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max DAG

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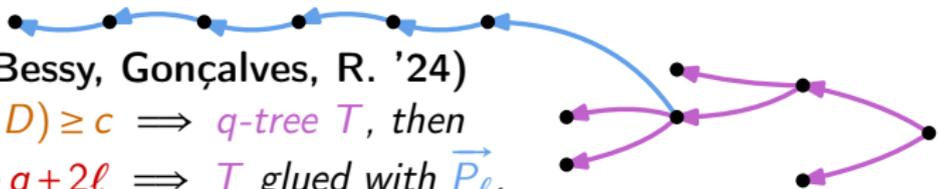
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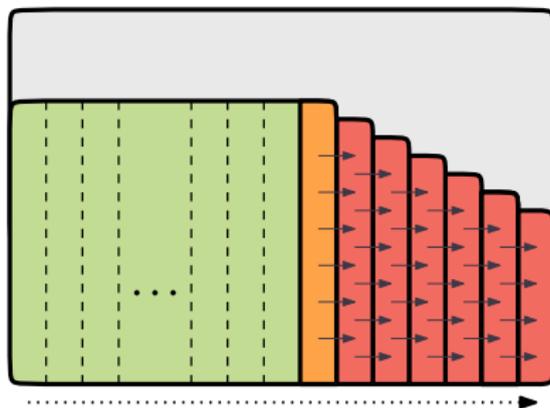
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max DAG

$\chi < q + \ell$



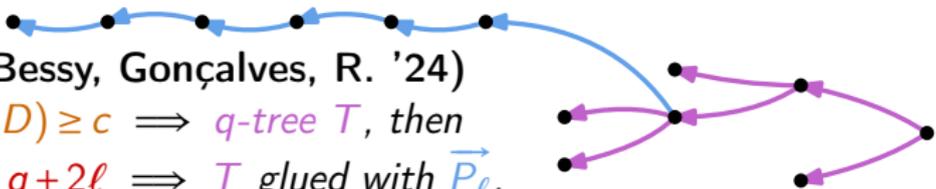
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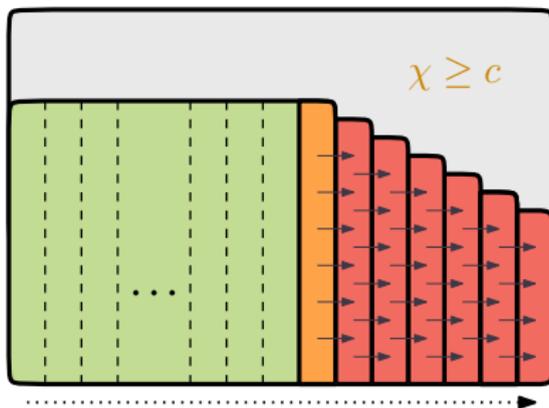
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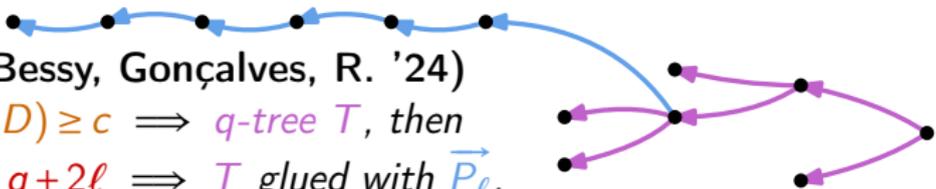
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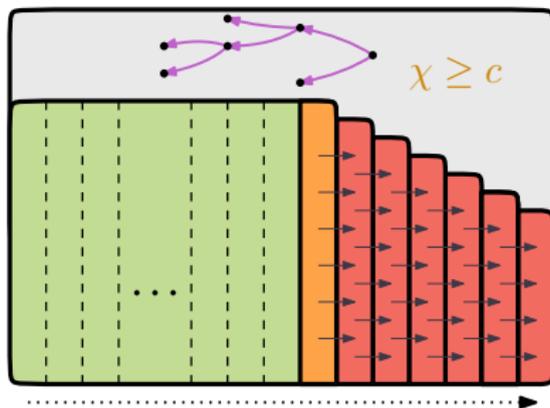
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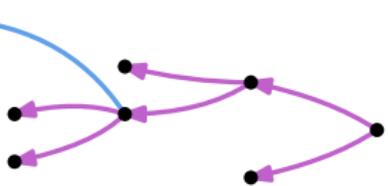
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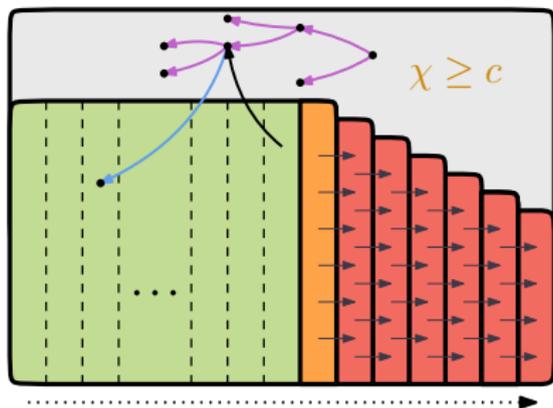
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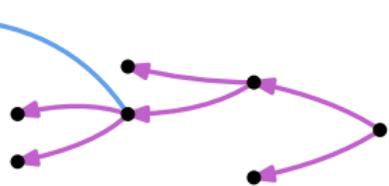
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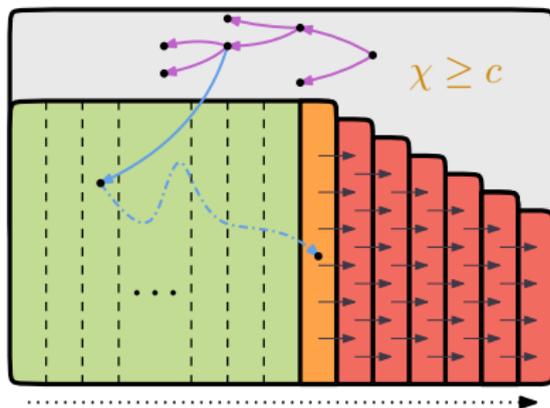
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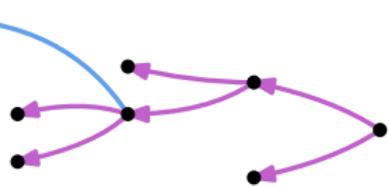
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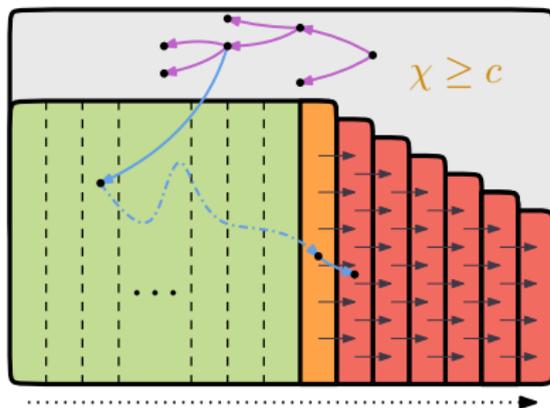
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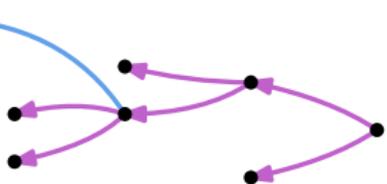
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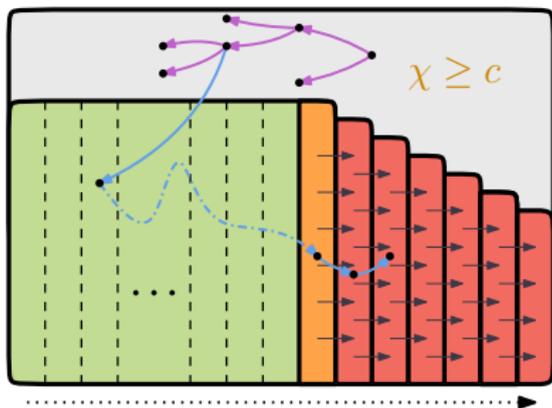
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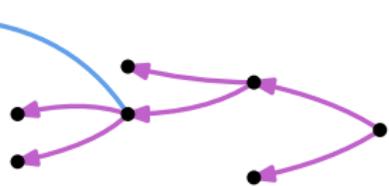
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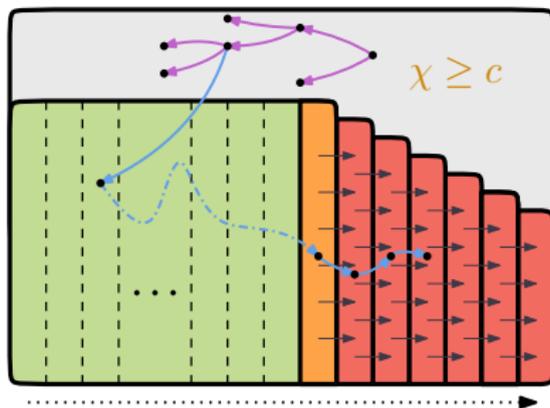
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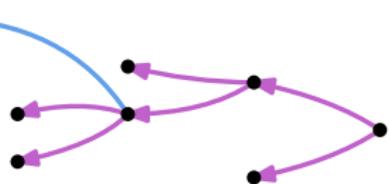
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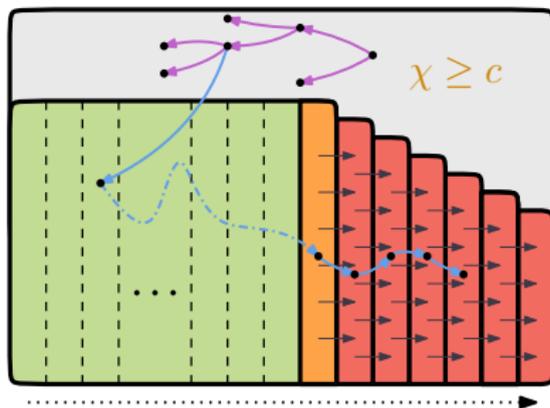
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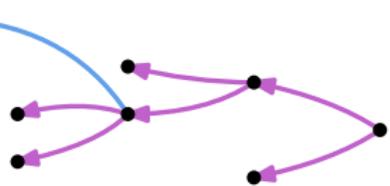
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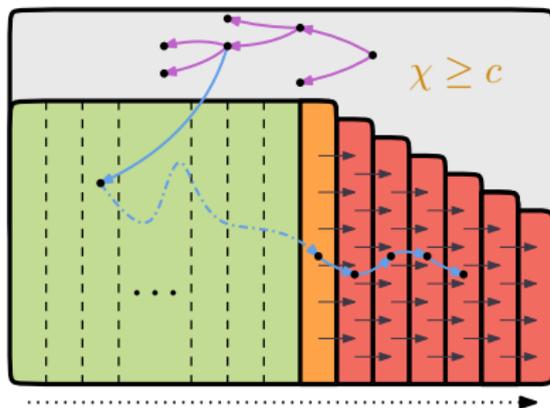
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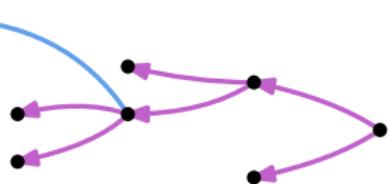
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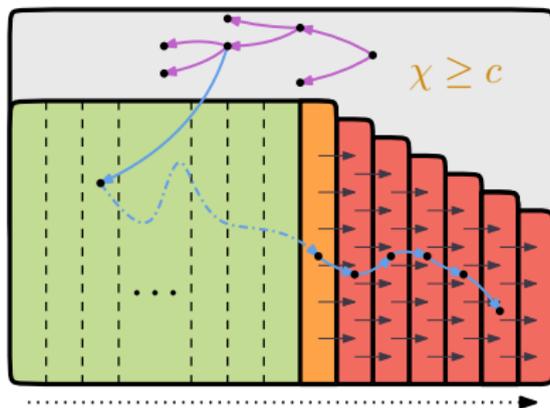
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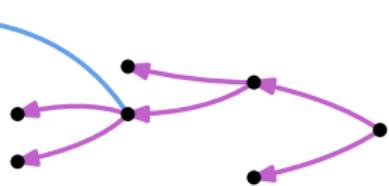
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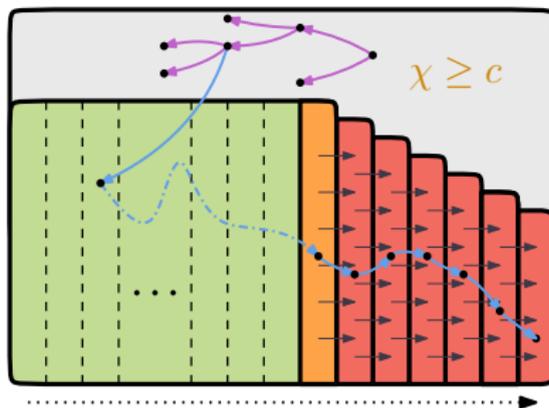
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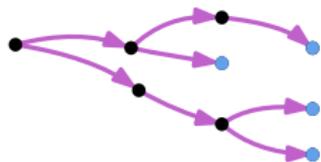
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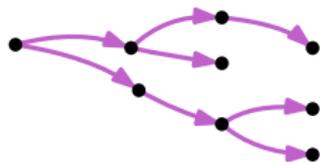
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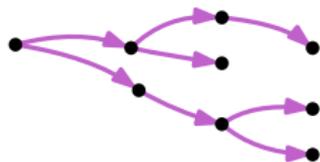


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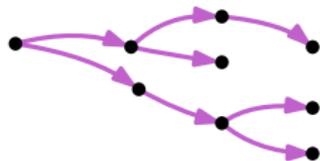


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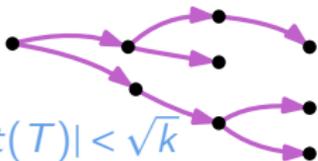
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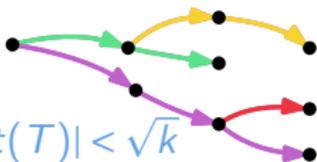
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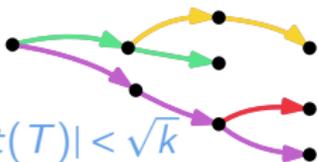
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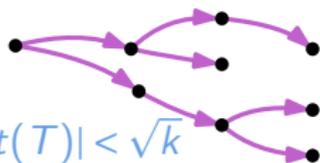
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**Theorem (Bessy, Gonçalves, R. '24)**

If  $\chi(D) \geq 4k\sqrt{k} + O(k)$ ,  $D$  contains every  $k$ -arborescence.

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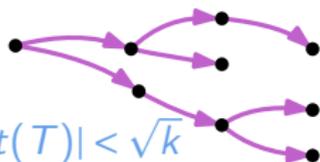
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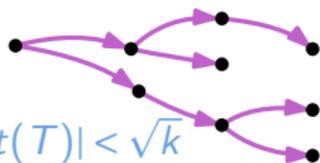
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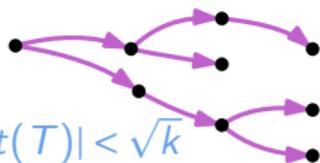
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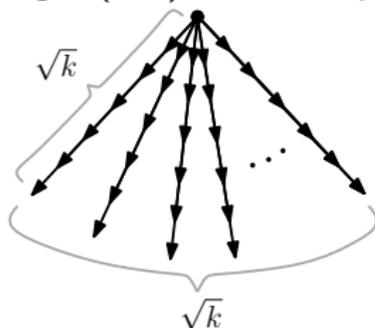
If  $\chi(D) \geq 2.93k\sqrt{k} + O(k)$ ,  $D$  contains every oriented  $k$ -tree.

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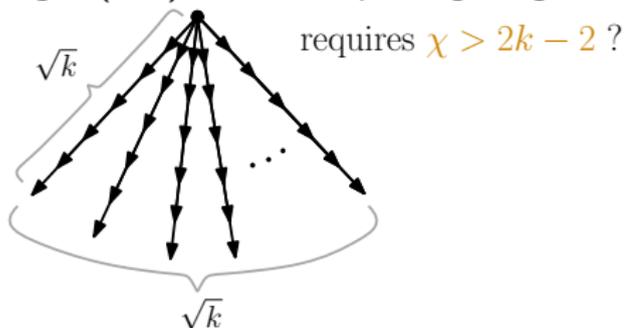
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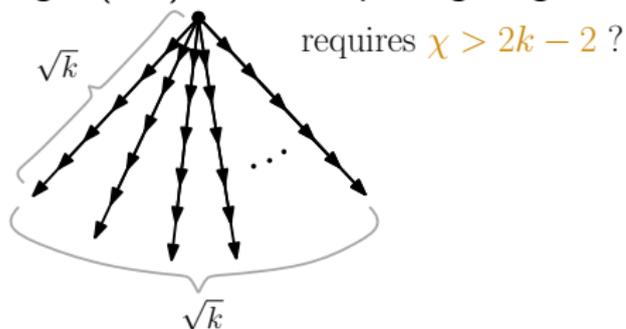
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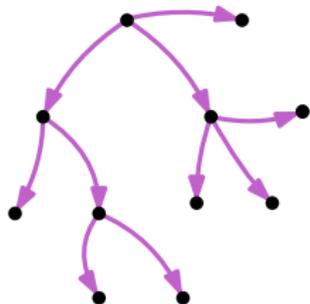


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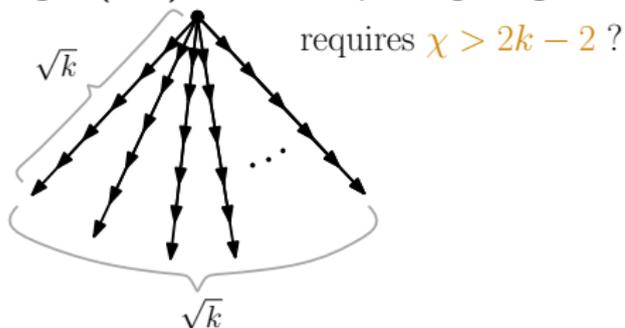


- Relaxing containment of **trees**

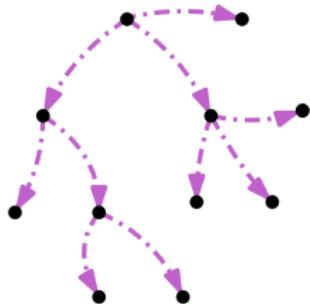


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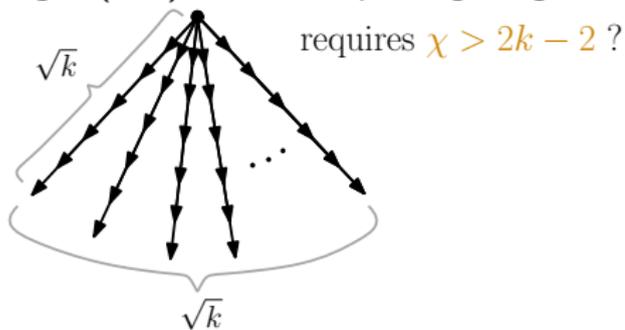


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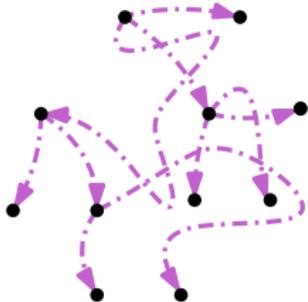


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Substructures from high  $\chi$

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## Substructures from high $\chi$

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- **induced  $\vec{P}_4/K_k$**  from **high dichromatic number** [C,M,R,P,S '22]
- Brooks-type **obstructions** from **high "generalized" dichromatic number** [G,P-A,R '24]

Thank you!

Bonus slides

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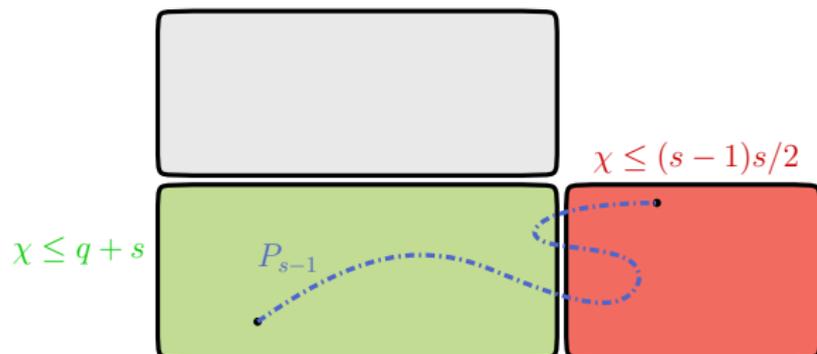
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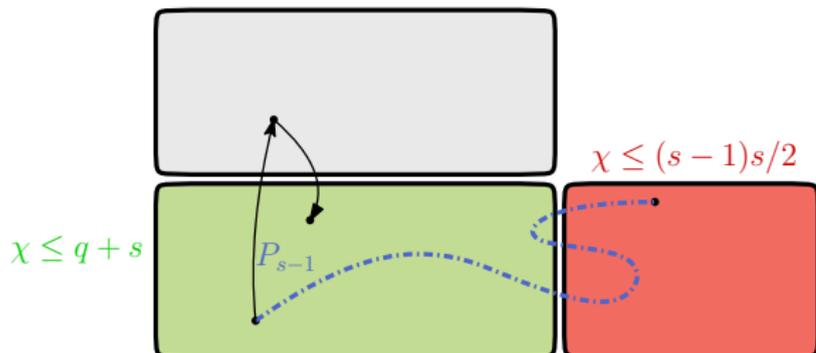
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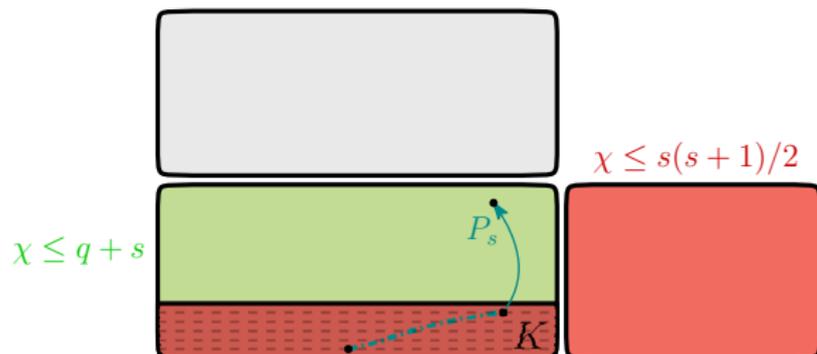
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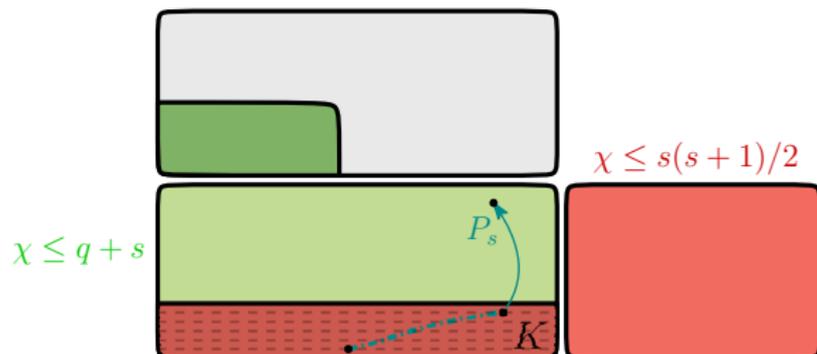
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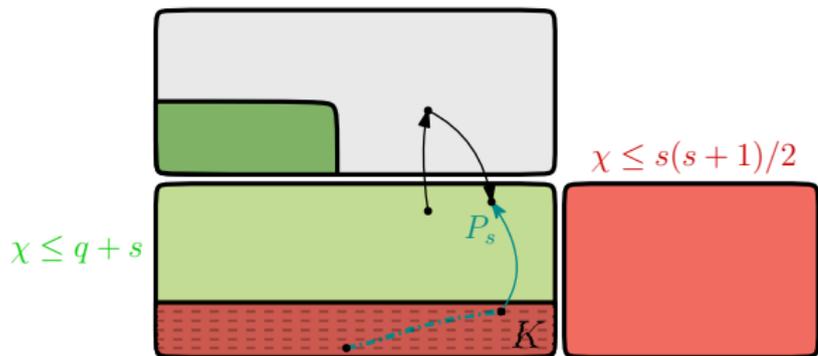
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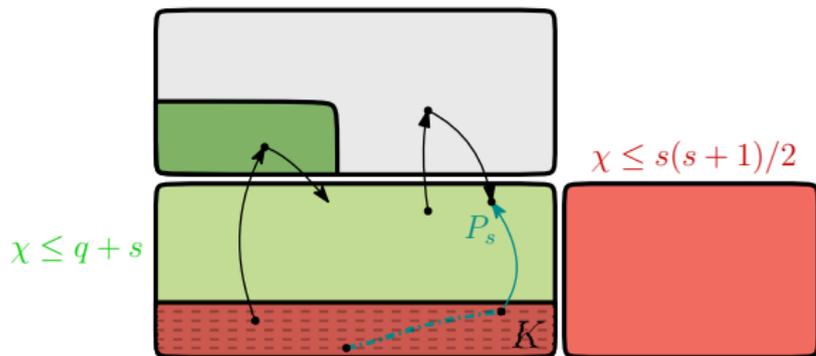
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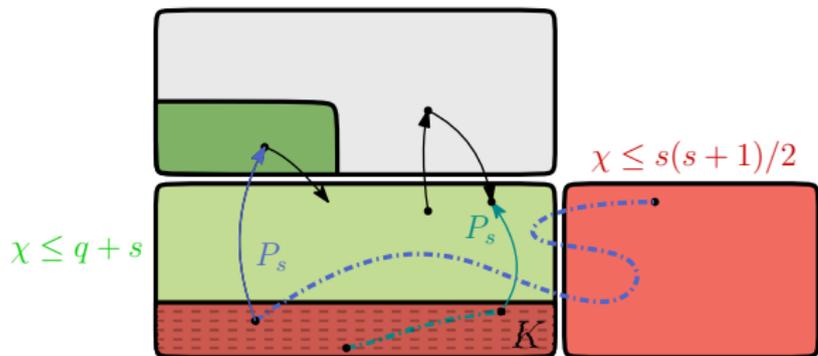
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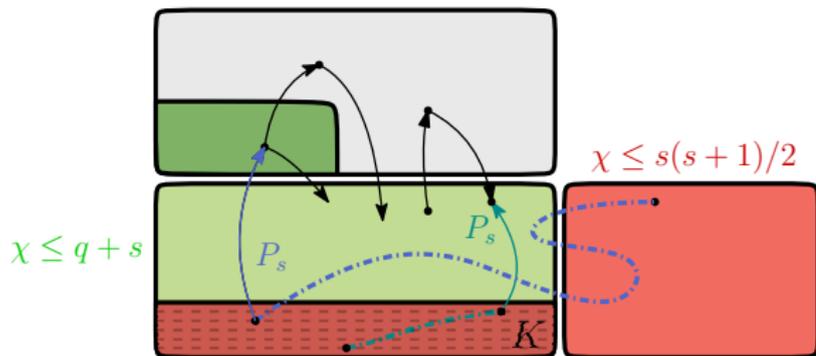
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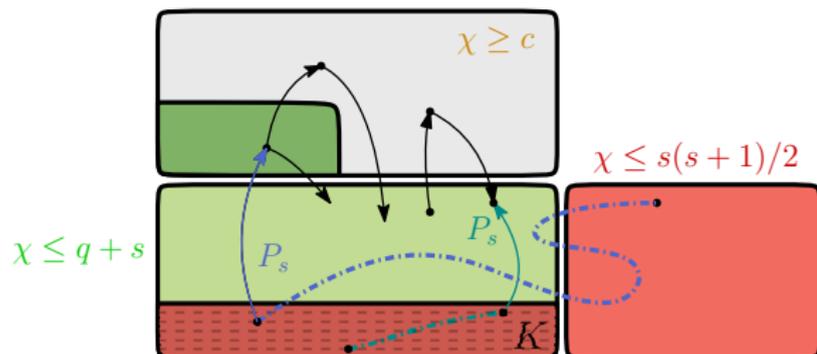
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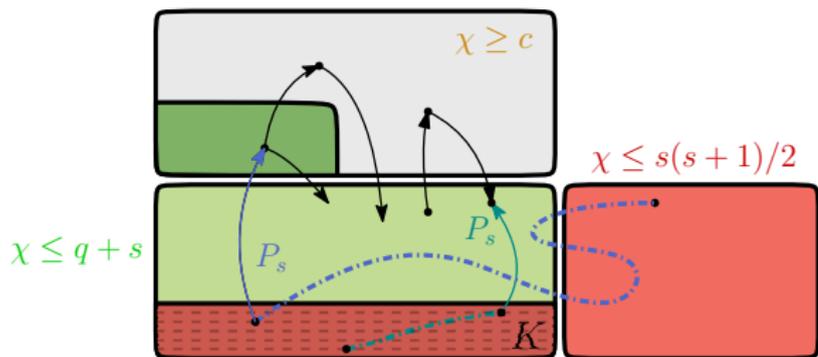
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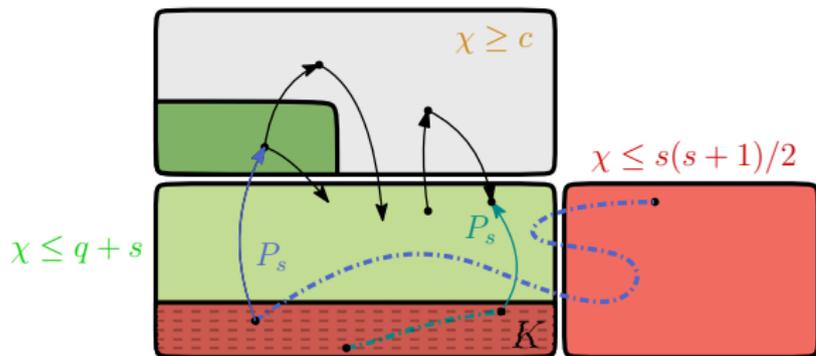
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**Theorem (Bessy, Gonçalves, R. '24)**

If  $\chi(D) \geq 2.93k\sqrt{k} + O(k)$ , then  $D$  contains every oriented  $k$ -tree.

## Articles in the thesis

- Bessy, Gonçalves, R., **Oriented trees in  $O(k\sqrt{k})$ -chromatic digraphs, a subquadratic bound for Burr's conjecture**, *WG '24 - journal submitted* (2023)
- Cook, Masarik, Pilipczuk, R., Souza, **Proving a directed analogue of the Gyárfás-Sumner conjecture for orientations of  $P_4$** , *EJC* (2022)
- Gonçalves, Picasarri-Arrieta, R., **Brooks-type colourings of digraphs in linear time**, *JGT* (2025)
- Bessy, Gonçalves, R., Thilikos, **Plane Strong Connectivity Augmentation**, *in preparation* (2025+)
- Bang-Jensen, Havet, Hörsch, Rambaud, R., Silva, **Making an oriented graph acyclic using inversions of bounded or prescribed size**, *in preparation* (2025+)