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Analysis of Voting Rules in the 1D-Euclidean Model with Strategic Candidates

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Abstract

The basic mechanism of democracy, which is nowadays one of the most common forms of government, is the election. As there are many voting rules that are used to determine the election winner, it is difficult to clearly say which one is the most suitable. To understand them better, we will perform an analysis of some of the most common voting rules, including Plurality, Borda, harmonic scoring function, Instant-runoff voting (IRV) and Condorcet methods after introducing the concept of strategic candidacy.

In this thesis, we consider the voters and the candidates as points in a one-dimensional space of Euclidean preferences. We will try to see how the distribution of candidates is changing, assuming that they have a chance to adjust their position in space in order to maximise their chances to win the election.

The results will be analysed in two aspects. First, our goal will be to see whether the simulation reaches ε-local-equilibrium, i.e. if after a certain number of election rounds all candidates reach their optimal positions and stop moving. Second, for cases that do stabilize, we will look at final distributions of candidates. It will be shown that for the vast majority of voting rules, candidates' mobility become insignificant or even reaches the value of 0, with the only exception of Plurality rule. With regard to the second aspect of our analysis, we will be able to distinguish two main behaviours—candidates tend either towards the mean values of distributions used to randomly selected the voters (harmonic scoring function, IRV) or their final distribution is shifted in the direction of the median voter (Borda, reversed harmonic scoring function, Condorcet winner).

Keywords

Computational Social Choice Theory, single-winner voting rules, strategic candidacy, Euclidean preferences, median voter

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Contents

Introduction 5

1. Preliminaries 7
1.1. Voting rules 7
1.1.1. Positional scoring rules 7
1.1.2. Runoff rules 9
1.1.3. Condorcet methods 10
1.2. The model of Euclidean preferences 11

2. Strategic positioning of candidates 13
2.1. The model 13
2.2. Experimental setup 19
2.2.1. Hardware 19
2.2.2. Default values of variables 20
2.2.3. Execution time 20
2.3. Results 22

3. Conclusions and related work 43

Introduction

Nowadays, democracy is one of the most common forms of government. The main idea behind it is to give people the power to decide about their rights and restrictions. Although originally democracy was meant to enable an active participation of all citizens with a goal of establishing the laws (which is referred to as direct or participatory democracy), it quickly became clear that with bigger nations this approach is not feasible. Thus, today we more often encounter a different form of this system, called indirect or representative democracy. With indirect democracy, citizens choose a group of representatives who speak for the whole nation. To implement this form of government into life, a need of formalizing ground rules that enable choosing people who are a good representation of citizens' needs and beliefs occurred. Hence the public elections.

The election is the basic mechanism of democracy and commonly occurs at the state and national level, as well as in smaller communities such as schools or clubs. During election, community members vote for their most favoured candidate or create a ranking of preferences based on their subjective beliefs. Usually, when deciding who to support, voters take into account how closely the nominee's views reflect their own opinions, as their goal is to select their representatives.

As simple this concept may seem, conducting elections in a fair way is not that straightforward. Voting rules need to be exactly specified in an electoral system to avoid any falsifications and ensure fairness. A function that maps voters' preference rankings to election results is called a decision rule. As there are many variants of such functions, it is hard to decide which one is the best fit for given situation. Although announcing those who got the highest amount of votes as election winners is the most intuitive solution, it turns out that there are many more options. Their application area depends not only on election type but is also country specific. It follows from the fact, that it is still unclear which voting rules are the most equitable and able to make as many citizens as possible feel represented.

In this thesis, we will perform an analysis of some of the most common voting rules, including Plurality, Borda, harmonic scoring function, Instant-runoff voting (IRV) and Condorcet methods. What is more, a concept of strategic candidates that can adjust their beliefs to maximise their chances to win will be introduced. By simulating a sequence of multiple election rounds in a one dimensional space of Euclidean preferences, we will try to determine which position is the most beneficial for the candidate to take in order to get the largest number of votes. Proposed model with all the simulations is open-source and available in github repository joannapaszkowska/StrategicCandidates.

Results will be examined in two aspects. First, our goal is to see whether the simulation reaches equilibrium, i.e. if after a certain number of election rounds all candidates reach their optimal positions and stop moving. Second, for cases that do stabilize, we will consider and compare final distributions of candidates. As it will be shown, results are also not unified in this aspect, as for some voting rules candidates tend to move toward the position of median voter, whereas in other cases they oscillate around the mean location of voters' distribution.

Results and conclusions of conducted experiments can facilitate the understanding of decision rules, as knowing how the behaviour of strategic candidates differ depending on a mapping function may reveal differences in the mechanisms behind the most common decision systems.

Chapter 1

Preliminaries

The key concept in the process of choosing a representative is an election—a procedure in which voters collectively select a winner from the proposed candidates.

Let V be a set of voters and C—a set of candidates. We define an election as a pair (C, R), where $R = \bigcup_{v \in V} \{r_v\}$ consists of voters' preference rankings. For each voter $v \in V$, r_v is a linear order of candidates, which determines voter's preferences. The highest ranked candidate in r_v is v's most preferred candidate.

To convert linear orders of candidates into election results, it requires a function (an algorithm) referred to as a voting or decision rule. With the input of a set of candidates and voters' preference rankings, the goal of those algorithms is to return an election winner. There are many well-known single-winner voting rules, some of which will be analysed in this thesis.

In practical cases, a tie-braking mechanism is needed to select one clear winner, as choosing more than one candidate as a winner is usually not a possible option. For convenience, implementations of all discussed in this thesis voting rules always announces lexicographically first candidate as a winner in case of a tie. Consequently, we can define a single-winner voting rule as a function that takes an election (C, R) as an argument and returns one winning candidate $w \in C$.

1.1. Voting rules

This section describes three most commonly studied classes of election rules.

1.1.1. Positional scoring rules

Positional scoring rules are defined by scoring functions, which based on the position in a voter's preference ranking, grant candidates with a certain number of points.

Let m be a size of the set of candidates C, $[m] = \{1, 2, ..., m\}$. Scoring function $\gamma\colon [m] \to \mathbb{R}$ defines the corresponding positional voting rule. The value of $\gamma(i)$ determines how many points gets a candidate ranked in i-th position in some preference order.

Thus, candidate's $c \in C$ total score is a sum of $\gamma(pos_v(c))$ for every $v \in V$, where $pos_v(c)$ is candidate's c position in voter's v ranking. The following condition must hold: if $i < j$, then $\gamma(i) \geq \gamma(j)$, so the higher the candidate's position, the more points they get. The winner is a candidate with the highest score.

Examples of common positional voting rules are [Brandt et al. (2016)]:

Plurality Plurality is one of the most popular voting rules. It is commonly used among smaller, less formal communities such as clubs (e.g. when selecting a chairman) or

voter	preference order
v_1:	$a \succ b \succ c \succ d \succ e$
v_2:	$a \succ b \succ c \succ d \succ e$
v_3:	$b \succ c \succ d \succ e \succ a$
v_4:	$c \succ b \succ d \succ e \succ a$
v_5:	$d \succ b \succ e \succ c \succ a$
v_6:	$e \succ b \succ d \succ c \succ a$

Table 1.1: A sample election with the candidate set $C = \{a, b, c, d, e\}$ and 6 voters $\{v_1, v_2, v_3, v_4, v_5, v_6\}$.

schools (e.g. for choosing a class or a student council president), but also in politics, for instance during national elections in countries such as the United Kingdom, Canada, India or the United States.

Given a set C of candidates and voters' preference rankings R, we define a plurality scoring function for $i \in [m]$ as follows:

$$\alpha(i) = \begin{cases} 1 & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Plurality voting takes into consideration only first position in a raking and ignores the further part of the order. Consequently, in practice, voters usually instead of determining their whole preference rankings, indicate just one person with their highest level of support. However, to unify the notation of representations of decision rules, we will consider plurality as a ranking-based system.

The winner of an election in plurality voting is a candidate ranked first by the largest number of community members.

Borda Another well known example of positional scoring rules is Borda count. Borda scoring function is more diversified, compared to the plurality one. With m being the number of candidates, let the scoring function β be defined as such: $\beta(i) = m - i$ for $i \in [m]$.

Borda system grants points every candidate except the least preferred one. Hence, it may result in selecting a winner accepted by a wider range of community members.

Example 1.1.1 *Consider the election from Table 1.1. Let* $plurality(c)$ *be a plurality score of candidate* c *and* $borda(c)$*—their Borda score.*

Using scoring functions defined above, we have $plurality(a) = 2$ *and* $plurality(b) = plurality(c) = plurality(d) = plurality(e) = 1$. *Consequently, the plurality voting rule would return candidate* a *as this election winner. However, although* a *is ranked first in two of the preference rankings, it is least preferred by the other 4 voters, which plurality ignores.*

On the other hand, after calculating Borda scores using $\beta(i) = 5 - i$ *scoring function, we get* $borda(a) = \beta(1) + \beta(1) + \beta(5) + \beta(5) + \beta(5) + \beta(5) = 8$, $borda(b) = 19$, $borda(c) = 13$, $borda(d) = 12$ *and* $borda(e) = 9$, *which actually leaves candidate* a *with the smallest number of points and announces* b *as a winner.*

Besides those best-known and commonly used examples of positional voting rules, a scoring function γ can be defined in many different ways, as long as it complies with a condition that for $i < j$, $\gamma(i) \geq \gamma(j)$. One of the scoring functions that shows good properties in case of social

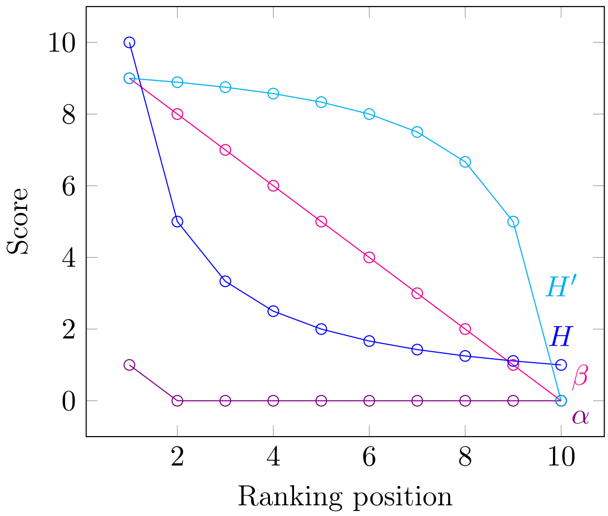


Figure 1.1: Function graphs of scoring functions: α—plurality scoring function, β—Borda count, H—harmonic function (scaled by 10) and H'—reversed harmonic function (scaled by 10).

welfare, which was discussed by Boutilier et al. (2015), is the harmonic scoring function. This function is also known as the Dowdall method and is sometimes used during real-world election as an alternative to the Borda count (Fraenkel et al. (2014)). Consequently, in forthcoming experiments we will also consider voting rules with scoring functions defined using harmonic weights.

Harmonic function Let us define a function that outputs a candidate with the highest sum of points received from all voters. With m being the size of C, $[i] = \{1, ..., i\}$, we define a scoring function used by the voting rule in question using a vector of harmonic weights:

$$H(i) = \frac{1}{i}$$

Harmonic function reversed Under this voting rule, we define a scoring function H':

$$H'(i) = 1 - H(m - i + 1) = 1 - \frac{1}{m - i + 1}$$

and select a candidate with the highest score calculated using H' as a winner.

That way, set of positional voting scoring functions used to conduct experiments consists of different kinds of functions, including linear, concave and convex ones as shown in Figure 1.1.

1.1.2. Runoff rules

In preferential systems, voters also rank all the candidates from the most to the least preferred one and a positional voting rule is used to determine the winner. However, it is not always true, that the winner shall be announced after a single election round. In order for a candidate to defeat their opponents, they need to get a specific number of votes. Otherwise, the election is taken to the next round.

count	preference order
6	$a \succ b \succ c$
5	$b \succ a \succ c$
3	$c \succ b \succ a$
1	$c \succ a \succ b$

Table 1.2: A sample election with candidate set $C = \{a, b, c\}$ and following ballot counts.

IRV A common single-winner voting rule is Instant-runoff voting (IRV). In order to become a winner under this voting rule, a candidates needs to get the majority of votes, i.e. $\lfloor n/2 \rfloor + 1$, where n is the number of voters. Votes are allocated as follows: first, every voter gives one point to their most favourite candidate just like in case of Plurality. If the majority of votes is received by some candidate, the winner is announced. Otherwise, a candidate with the lowest Plurality score is eliminated. If the score of the eliminated one is grater than 0, that is the eliminated candidate was the most preferred amongst all opponents for a nonempty set of voters, the votes are transferred to the next candidates in preference orders of such voters. If the quota of $\lfloor n/2 \rfloor + 1$ is reached after transferring the votes, the winner is chosen. In other case, we continue by eliminating the next least preferred candidate and transferring the votes exactly like in the previous case. The process is repeated until we get a candidate with the majority of votes.

Example 1.1.2 *Let us consider an election presented in Table 1.2. As total number of votes equals to 15, quota that needs to be reached by a candidate in order to become a winner is* $\lfloor 15/2 \rfloor + 1 = 8$.

1. *First, we calculate Plurality scores based on input ballots. Thus, candidate a gets 6 points, candidate b—5 points and candidates c—4 points. After checking that the highest ranked candidate does not reach the quota, the IRV rule needs to be taken to the second round.*
2. *Next, a candidate with the lowest score, i.e. candidate c, needs to be eliminated. Votes that were cast for c are transferred to subsequent candidates in respective ballots. That way, candidate b will be granted with 3 more votes and a–an extra one.*
3. *The scoring after eliminating c looks as follows: candidate a gets 7 points (1 of which comes from becoming first in a preference ranking after eliminating c) and candidate b—8 points (3 of which are a result of being second best after c).*
4. *Candidate b with the score of 8 reaches the quota and is announced the winner of the election.*

1.1.3. Condorcet methods

Another class of voting rules are Condorcet methods—the functions that select a winning candidate referred to as a Condorcet winner. A Condorcet winner is a candidate who would win under plurality system in a two-candidate election with any other candidate as an opponent. Intuitively, it means, that $w \in C$ is a Condorcet winner if for every $c \in C \setminus \{w\}$, the majority of voters would choose w over c.

It is a fact, that Condorcet winner does not always exist. Thus, there are many voting rules that differ in their behavior when such a winner is not present. The most popular voting rules which are Condorcet include Copeland, Minimax or Schulze methods. Copeland score for candidate $c \in C$ is the number of candidates that are less preferred that c by the majority of voters. Formally, results under Copeland method may be computed using an $m \times m$ matrix M, where m is the number of candidates. M_{ij} is equal to 1 if candidate i is higher ranked than candidate j by the majority of voters, 1/2 if the votes are split evenly and 0 otherwise. We assume that M_{ii} is also 0. The Copeland score for candidate i is equal to $\Sigma_{j \in C} M_{ij}$. The candidate with the highest Copeland score is announced as the Copeland winner. Moreover, if such candidate receives a score of $n - 1$, where n is the number of voters, the winner is Condorcet.

The Minimax score, on the other hand, takes into consideration a minimal value from the numbers of voters that prefer candidate c over another candidate. Let us consider a function $score$. For every two candidates $i, j \in C$, $score(i, j)$ is defined as the number of voters that ranked candidate i higher that candidate j in their preference rankings. The Minimax winner is the candidate that received the score of $min_{i \in C}(max_{j \in C} score(j, i))$.

In this thesis, we will further examine the latter of mentioned Condorcet rules, that is the Schulze method.

Schulze For $a, b \in C$, let $P(a, b)$ be the number of voters that ranked candidate a higher than b in their preference ranking. For example, using the rank ballots from Table 1.1, we can calculate that $P(a, b) = 2$ and $P(b, a) = 4$.

A path from candidate a to b is a sequence of candidates such that every subsequent candidate is less preferred than a previous one. Formally, path $p = c_1, ..., c_n$, where $c_1 = a$, $c_n = b$ and for every $1 < i < n - 1$, $P(c_i, c_{i+1}) > P(c_{i+1}, c_i)$.

Strength of path p is the minimal value of $P(c_i, c_{i+1})$, $1 \leq i < n$ and strength of a pair of candidates $a, b \in C$ is a strength of the strongest path between a and b. Let us denote it by $S(a, b)$. If a path between two candidate does not exist, $S(a, b) = 0$.

We say that candidate a gets higher score than b if and only if $S(a, b) > S(b, a)$. An election winner under Schulze method is a candidate a that fulfills $\forall_{c \in C \setminus \{a\}} S(a, c) \geq S(c, a)$ and it is proven that there is always such a candidate. Consequently, Schulze method returns a winner for every election.

Schulze method has many beneficial properties that make this voting rule widely used. It has found its applications in cities such as Turin, but also is used by many organizations, also computer science related ones.

1.2. The model of Euclidean preferences

A basic mathematical model of elections was presented by Davis et al. (1965). It defines a rule behind creating voters' preference rankings as follows. Having an election (C, R), a preference profile $r \in R$ is n-Euclidean if we can assign to voters and candidates positions in n-dimensional space, so that the following formula is satisfied:

$$pos_v(c_1) < pos_v(c_2) \implies ||x(v) - x(c_1)||_n < ||x(v) - x(c_2)||_n,$$

where $c_1, c_2 \in C$, $v \in V$ and $x(a) \in \mathbb{R}^n$ is a vector of coordinates assigned to a voter or candidate a.

In other words, in n-dimensional Euclidean preference model, the closer the candidate is to the voter in the n-dimensional space, the higher they are ranked in voter's preference profile, which can represent political preferences quite realistically. Euclidean domain was applied to

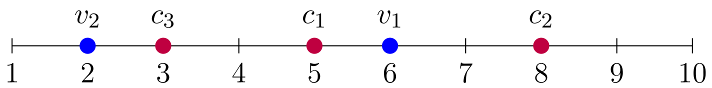


Figure 1.2: Example representation of voter's v_1 preference ranking $c_1 \succ c_2 \succ c_3$ and voter's v_2 preference ranking $c_3 \succ c_1 \succ c_2$ in a one dimensional Euclidean preferences domain.

model elections e.g. in the works of Elkind, Faliszewski, et al. (2017) or Godziszewski et al. (2021) to illustrate the behaviour of multi-winner voting rules and compare the distribution of winners under those rules in a 2D space.

Deciding if a given preference profile is n-Euclidean is one of the problems that has been studied several times. It has been proven that for one-dimensional space, there exist polynomial time algorithms which allow to resolve this issue (Chen et al. (2017), Elkind and Faliszewski (2014)). However, for any other fixed $n > 1$, this problem is NP-hard, as Peters (2017) has shown that deciding if preference profile is n-Euclidean preferences model is equivalent to the ETR (Existential theory of the reals) problem.

The simplicity of this model has made it commonly used e.g. in economics or marketing. In order to visualise behaviour of chosen voting rules, in this thesis we will use a one dimensional model of Euclidean preferences. Having an election (C, R), we can consider sets of candidates and voters as sets of coordinates in a 1D space. For each pair of points x_1 and x_2, the distances between them $d(x_1, x_2)$ is denoted as $|x_1 - x_2|$. The closer to zero $d(x_1, x_2)$ is, the more similar preferences of voters' at those positions are.

Each ranking $r_v \in R$ is created based on the distances between voter v and the candidates, which is illustrated by the example in Figure 1.2.

Chapter 2

Strategic positioning of candidates

In this chapter, we will define the one-dimensional Euclidean preference model that will be used to simulate elections with strategic candidates. We will also discuss all the parameters, algorithm and metrics that will be then essential for conducting an analysis of chosen voting rules.

2.1. The model

Strategic behaviours is a known concept in computational social choice. Mechanism which can be used to manipulate election results exist for practically every rule, however some of them may be more resistant than the others. Among other things, this issue in particular was a subject of many works, which were focusing both on strategic voters (e.g. Feddersen et al. (1990) and Myatt (2007) discuss strategic voting under plurality rule, where voters do not have to vote for the closest candidate, but their goal is to maximise some utility function), as well as candidates, which will be mainly considered in this thesis.

The concept of strategic candidates may refer to various aspects. In this thesis, we will focus on a variant which enables the candidates to change their position in the space of preferences. Nevertheless, there exists different variations of strategic candidacy, for instance, assuming that candidates have their own preferences over their opponents, they can manipulate their votes or even the fact of their participation in an election in order to change its results. This issue was raised by Dutta et al. (2001), who discussed how candidates entering and quitting the election regrading their own preference rankings over other candidates can affect the results. Their work showed that for no non-dictatorial and unanimous voting rule, all candidates entering the election is a Nash equilibrium (i.e. no such voting rule is resistant to strategic candidacy). What is more, it also showed candidacy games with no strategy equilibrium at all.

The above results were further extended to other voting rules by Lang et al. (2013). It has been shown, that a pure strategy equilibrium always exists for Copeland method and for all Condorcet methods with 4 candidates. Nevertheless, it was also proven that for candidacy games with more than 4 candidates, such equilibrium may not occur under Plurality or Minimax methods.

Regarding the concept of local equilibrium under voting rules, we can also look at a model introduced by Plott (1967). In this article, the authors consider a committee and a set of variables, whose values can be altered. Each of the decision makers has a defined set of utility functions regarding each variable. Their goal is to maximise total utility. To do so, changes to the values of variables are proposed and majority voting rule is used to determine if those changes are accepted. The equilibrium is reached if no other change would be accepted by the majority—i.e. the existing state is optimal. It is shown, that under presented model, the equilibrium is not reached—even after indefinite number of iterations there are still some changes that can be made to the values of variables. Nevertheless, the results are not as negative as it may appear, because the range of those alterations seem to visibly flatten and become rather insignificant as time passes. Thus, the equilibrium can be estimated and with that, the candidates may start to strategically alter their utility functions in order to lead to their initially most favoured state.

The Hotelling-Downs model

Mentioned strategies and not the only ones that may come effective in terms of manipulating election results. When analysing single winner voting rules in Euclidean preferences domain and observing tendencies in winner's positions on a line a question arises: what if the candidates could predict which position to take to maximise their chances to win? Is it possible (based on some available data) to adjust one's views and election postulates in a way that will guarantee victory? The model that can allow to find answers to those questions has been known for several decades and was first introduced by Hotelling (1929) and later on by Downs (1957).

Hotelling, who was an economist, noticed the following phenomenon: politicians tend to choose their place on a political stage so that they are in a position where the median voter is. This position depends on choice of electoral system. The example presented by the economist that illustrates this tendency is locating ice-cream stands by two vendors at the sea site. Based on this example, we can observe that vendors, instead of moving further away from each other in order to avoid competition, gravitate towards the same location. This model was further extended by Downs, who described similar tendencies on a political stage. Its extension to the Euclidean preferences domain with majority voting was presented by Enelow et al. (1984). In this thesis, we will also discuss a wider spectrum of voting rules.

The behaviour described by Hotelling and Downs is common in real life and can be observed in many situations. Even big corporation e.g. fast food ones tend to follow this tendency. It is common to open new locals in close proximity to other restaurants of the same type. However, at the same time, it may seem that locating a business in area with no competition would maximise owner's earnings. A framework based on the Hotelling-Downs model was introduced by Waldfogel (2008), who discusses the concept of median consumer and its influence on local private goods.

Modifications of the Hotelling-Downs model

The Hotelling-Downs model has been widely studied since its introduction. The model has been also considered with various modifications. For example, Sengupta et al. (2008) takes into consideration the fact that a candidate may quit the election.

The other one which was recently proposed (Feldman et al. (2016)) tries to reflect the real life behaviour of human voters more accurately than the original model. It takes into consideration the fact that voters may decide not to vote when none of candidates is good enough, i.e. none of the candidates is positioned close enough to the voter in the space of preferences. What is more, it touches on one more key observation—with a few candidates having similar positions, the choice of the most preferred one amongst them may be rather random, so the model allows dividing points equally between them in such cases.

Model definition

For our thesis, we will also consider a modification of the Hotelling-Downs model. Hence:

1. Let us consider one set of candidates and p sets of voters represented by their position on the number line. After every iteration of the simulation we independently conduct p elections (each one with the same set of candidates and one of the p sets of voters). Multiple sets of voters are needed in order to calculate the density of winners and estimate the probability of winning for candidates at each coordinate. What is important, sets of voters remain immutable, whereas the one set of candidates can be altered after each iteration (which is the clue of strategic candidacy).

2. Candidates, in contrast to the original model, are unaware of the exact positions of voters. In order to decide where to locate in the space of preferences, they will take into consideration election polls.

   Strategic voting models with incomplete information has been previously a subject of research, as in real-world situations election manipulators do not have full knowledge on voters' preferences. Endriss et al. (2016) and Tyszler et al. (2011) discuss how limiting information affects strategic voting. During our analysis, we will transfer this limitation to strategic candidates, who will have access only to the results of a sample of elections conducted amongst given society (i.e. with voters from specified distribution).

   After each iteration, i.e. after conducting p independent elections (with p different sets of voters and the same set of candidates), only one randomly chosen candidate has a chance to change their position based on the number of winners at each location from p previously conducted elections.

3. What is more, candidates' range of motion is limited. We consider a set ϵ, that consists of allowed offsets—values by which chosen candidate can move left or right on the number line to maximise their chances to win the election.

After each iteration with p elections, a randomly chosen candidate can either decide to move or to stay in their current position. In order to make this decision, we define a utility function U. This function, based on results of past elections (or the most recent poll), evaluates how good given position in a 1D space is in terms of maximising candidate's chances to become a winner. In our model, to calculate function U, we take into consideration distribution of winners from the set of elections held in previous iteration. As for each of the p elections in every iteration we consider the same set of candidates, chosen candidate's position shift will be also applied for every of the p elections in the next iteration.

Let us define function U. Let n be the size of set of elections held in every iteration and $\mathbb{D}$ a subset of $\mathbb{R}$ that consists of numbers with at most one decimal. Formally, $\mathbb{D} = z/10|z \in \mathbb{Z}$. Function $winners : \mathbb{D} \to \mathbb{N}$ is defined for every coordinate $d \in \mathbb{D}$ and returns number of candidates that won an election while their position in space was located within range $[d - 0.05, d + 0.05)$. Following from that, function $U : \mathbb{R} \to \mathbb{R}$ looks as follows:

$$U(x) = \frac{winners(round_1(x))}{n},$$

where function $round_1 : \mathbb{R} \to \mathbb{R}$ rounds number to one decimal place.

In other words, we can say that function U describes density of winners in a close approximation to given coordinate.

Algorithm

Each experiment is conducted for a unique pair of selected voting rule and distribution of election participants. Let p be the number of independent elections held in each of the tests and m be the size of sets of voters and candidates.

At the beginning of each experiment we randomly choose points in a one-dimensional space using chosen distribution. They will represent positions of m candidates and p sets of m voters.

The main program loop looks as follows:

Algorithm 1 Election with strategic candidates

Ensure: p—number of elections in each iteration, m—number of candidates and voters participating in election, k—number of iterations per experiment, $\varepsilon = \{0, 0.1, 0.2, 0.3, 0.4, 0.5\}$
Require: $i = 0$
while $i < k$ **do**
  Compute winners for each of p sets of voters (according to chosen voting rule).
  Randomly choose one candidate c
  for all $\varepsilon \leftarrow \epsilon$ **do**
    Compute values of $U(c + \epsilon)$ and $U(c - \epsilon)$
  end for
  Change the position of U to the one with the highest utility U amongst computed ones.
  $i = i + 1$
end while

Model adjustments

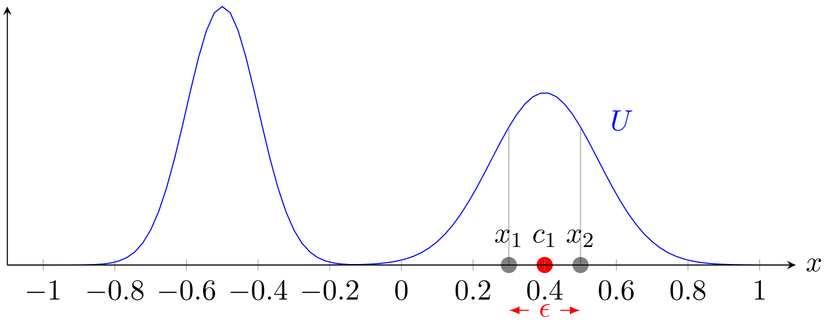
We modified the Hotelling-Downs model as described so that it can reflect single winner election according to the following interpretation. During an election, it is not true that the candidates know exact positions of the voters. They may only observe some tendencies or predictions, for example based on the past elections. Analyzing historic data may be a hint for candidates and can help to determine what kind of preferences prevail in the society.

It is also not true, that candidates in real life can adjust their position in the space of preferences arbitrarily. What it means, is that strategic candidates should not be able to move anywhere, hence the ϵ set of allowed offsets.

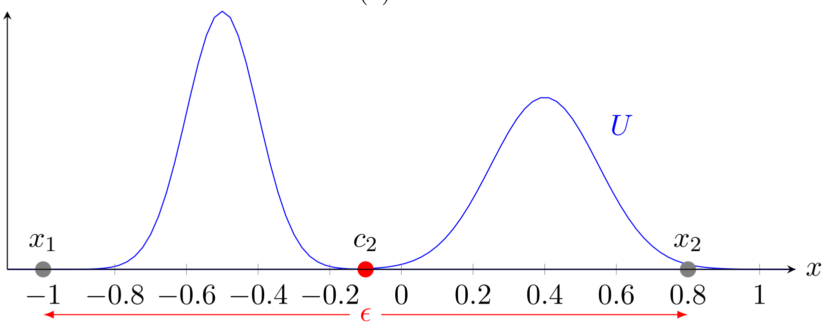
The first design of our model assumed that ϵ is only a single allowed value of offset, instead of the set of values. Initial experiments tried using fixed ϵ with different values per single experiment as said offset—when deciding if a candidate at position x wants to move, values of utility function U at positions x, $x + \epsilon$ and $x - \epsilon$ were considered. Candidates final position was the one with the highest utility.

However, after analyzing results of those experiments, it turned out that it is not the best strategy. Such a narrow view on the neighbourhood resulted in very slow changes in candidates' distribution—the chance that utility at x is already better than at $x + \epsilon$ or $x - \epsilon$ was pretty high, so the simulation was not dynamic enough.

It was also difficult to adjust the value of epsilon—high values were leading to omitting local maximums of the winners density in the graph and low values resulted in falling into local minimums and not being able to skip them in order to reach higher values as shown in Figure 2.1.



(a) $\epsilon = 0.1$



(b) $\epsilon = 0.9$

Figure 2.1: Example plot of utility function U with distinguished positions of candidates c_1 and c_2. (a) Choosing too low value of ϵ prevents candidate c_1 from moving from local maximum in order to reach the global one. (b) Too high value of ϵ causes missing higher values of utility U and staying in a local minimum by candidate c_2.

Consequently, let us consider a set of epsilons, instead of a single one. Now, the utility is checked not only for a location e.g. 0.2 away from the current candidate's position, but for locations $\ell \in \{0.1, 0.2, 0.3, 0.4, 0.5\}$ away. That way, the model facilitates looking at a bigger picture.

Distributions

To simulate a series of elections that complies with the model, we need to define one set of candidates and p sets of voters. As already mentioned, in the 1-dimensional Euclidean preferences model, voters and candidates are represented by positions in 1D space. For the experiments, let us randomly select coordinates of election participants. However, people's preferences are not, in general, distributed uniformly in space. On the contrary, we can usually observe focal points that aggregate people with similar views. In order to represent this behaviour, we will use normal distribution to randomly select positions on the number line.

For the purpose of this analysis, we assume that distributions of voters and candidates are identical within one simulation (i.e. the execution of fixed number of election rounds with candidates strategical shifting their positions)—the probability that a voter is at position x is the same as the probability of a candidate taking this position. Similar approach to election visualisation was taken e.g. by Elkind, Faliszewski, et al. (2017).

The experiments will be conducted with the following distributions:

1. **Gaussian** with $mean = 0$ and $scale = 0.25$. The highest density of election participants

is around point 0. It represents like-minded society with similar preferences (Figure 2.2).

2. **2-Gaussian** with $mean = -1$, $scale = 0.25$ and $mean = 1$, $scale = 0.25$. There are two focal points of preferences—the society is divided equally into two factions with minor common part (Figure 2.3).

3. **2-Gaussian** with $mean = -0.25$, $scale = 0.25$ and $mean = 0.25$, $scale = 0.25$. There are two focal points of preferences—the society is divided equally into two factions, but there is also visible common part (Figure 2.4).

4. **2-Gaussian** with $mean = -0.25$, $scale = 0.25$ scaled by 0.3 and $mean = 0.25$, $scale = 0.25$ scaled by 0.7. There are two focal points of preferences with one being stronger than the other. Common part of those two plots is clear (Figure 2.5). The asymmetric Gaussian model is commonly used for election visualization and voting rules analysis (e.g. Godziszewski et al. (2021) considered this distribution when generating histograms of winner density in a 2D space for approval-based voting rules).

5. **2-Gaussian** with $mean = -1$, $scale = 0.25$ scaled by 0.3 and $mean = 1$, $scale = 0.25$ scaled by 0.7. There are two focal points of preferences but one of them is chosen more frequently than the other. Common part of those two fractions is minor (Figure 2.6).

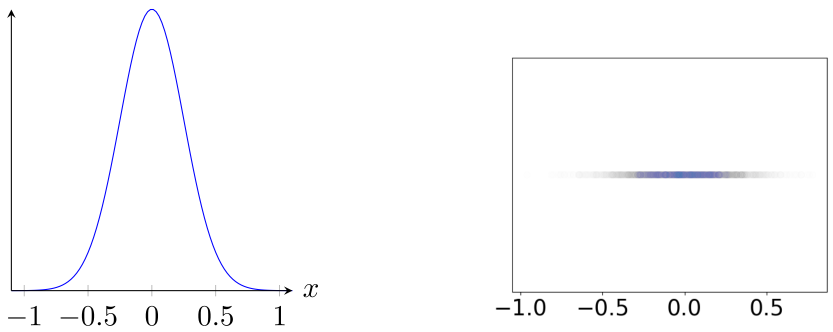


Figure 2.2: Graph of normal function with $mean = 0$, $scale = 0.25$ and sample of 2000 points randomly chosen using this distribution.

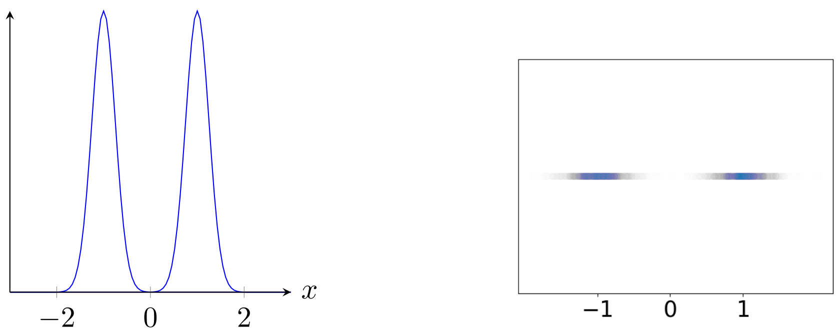


Figure 2.3: Graph of two normal functions: with $mean = -1$, $scale = 0.25$ and $mean = 1$, $scale = 0.25$ and sample of 2000 points randomly chosen using this distribution.

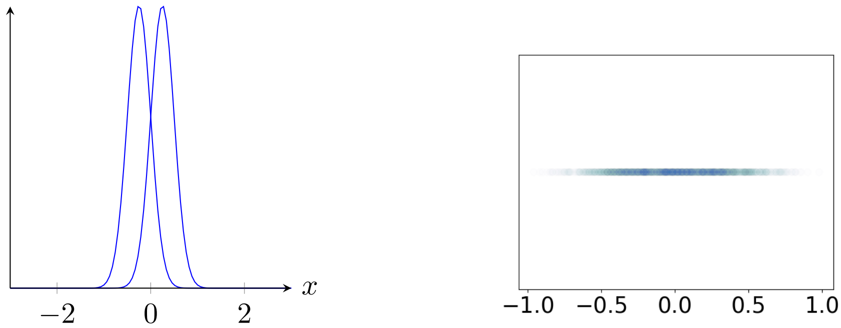


Figure 2.4: Graph of two normal functions: with *mean* $= -0.25$, *scale* $= 0.25$ and *mean* $= 0.25$, *scale* $= 0.25$ and sample of 2000 points randomly chosen using this distribution.

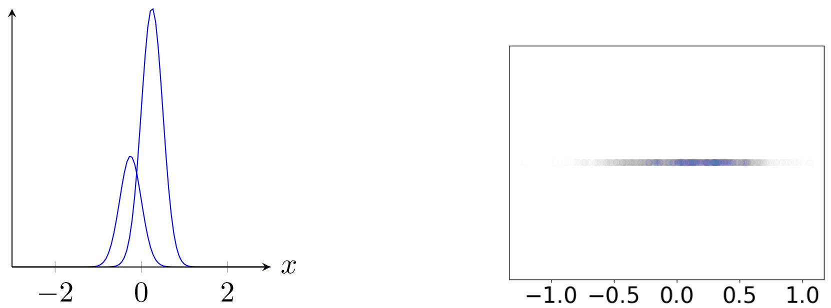


Figure 2.5: Graph of two normal functions: with *mean* $= -0.25$, *scale* $= 0.25$ scaled by 0.3 and *mean* $= 0.25$, *scale* $= 0.25$ scaled by 0.7 and sample of 2000 points randomly chosen using this distribution.

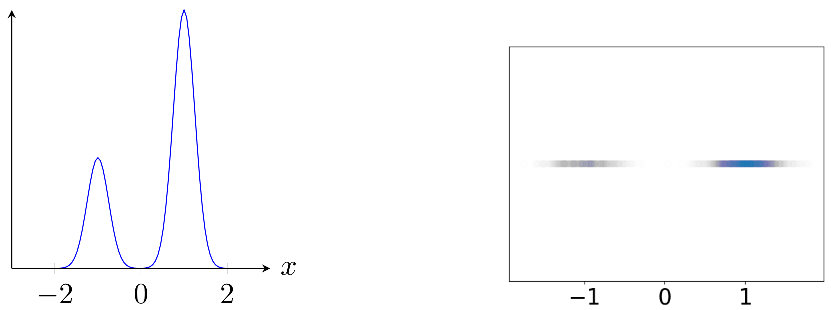
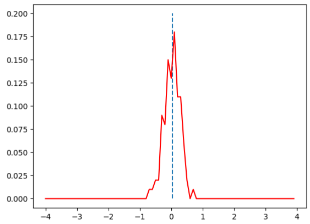


Figure 2.6: Graph of two normal functions: with *mean* $= -1$, *scale* $= 0.25$ scaled by 0.3 and *mean* $= 1$, *scale* $= 0.25$ scaled by 0.7 and sample of 2000 points randomly chosen using this distribution.

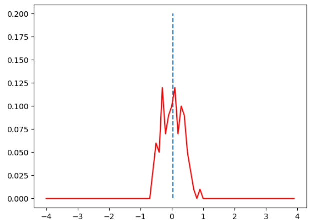
2.2. Experimental setup

2.2.1. Hardware

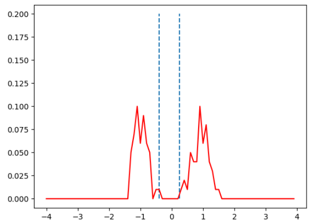
All of the experiments were conducted on a device with Intel® Core™ i5-7200U CPU @ 2.50GHz $\times$ 4 processor and NVIDIA GeForce 940MX graphic card.



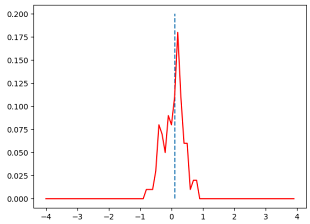
(a) 1-Gaussian 100% $loc = 0$



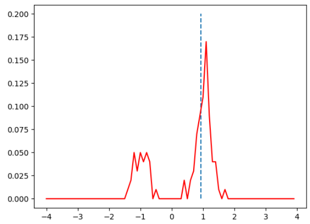
(b) 2-Gaussian 50% $loc = -0.25$, 50% $loc = 0.25$



(c) 2-Gaussian 50% $loc = -1$, 50% $loc = 1$



(d) 2-Gaussian 30% $loc = -0.25$, 70% $loc = 0.25$



(e) 2-Gaussian 30% $loc = -1$, 70% $loc = 1$

Figure 2.7: Example probe of 100 election participant randomly chosen using different distributions. Blue dashed lines point locations of two median values from the set of coordinates—number of participants is even, so in order to ensure that median belongs to the set, we consider two medians instead of a single value.

2.2.2. Default values of variables

In order to get representative results, the analysis had to be conducted on averaged results from multiple executions of Algorithm 1 with different input data.

Thus, unless explicitly redefined for the needs of a specific test case, let:

- `K = 50` be the number of independent executions of Algorithm 1 with different sets of candidates and voters,
- `k = 1000` be the number of iterations per single algorithm execution i.e. maximum number of candidates' position shifts,
- `m = 100` be the size of set of candidates and each set of voters,
- `p = 100` be the number of sets of voters per single algorithm execution, that is the number of elections held in each iteration with the same set of candidates and different sets of voters.

2.2.3. Execution time

While running the algorithm once with thus defined variables could be time-consuming, especially for voting methods such as Borda or Schulze, the real issue appeared when we wanted to repeat the execution with fresh data (different set of candidates and sets of voters) K times. Thus, it has become apparent that there is a need to include some optimization mechanism in the algorithm implementation.

Code optimization

As implementation of the algorithm consists of loops and computations on arrays, it became clear that a mechanism for parallelising could be crucial in the efforts to accelerate execution time. A tool, that we decided to test was Numba.

Numba Numba is a compiler that is used for optimizing Python code. It uses LLVM to translate Python syntax into machine code. Numba also supports code parallelising for CPUs and GPUs as it has an option to target Nvidia CUDA GPUs. Python library that works best with Numba is NumPy. Complier also performs well with loops.

The most fundamental way to use Numba is to apply `@jit` decorator to functions that we want to compile to machine code. Forcing the `nopython` mode, which does not grant access to Python C API is the most efficient way to use the compiler but puts more limitations on function implementation and used types.

Numba compiler is an open-source project available on github.

Adapting experiment implementation to Numba specification included changing as many used types as possible to NumPy implementation and extracting parts of code that could be labeled with `numba.jit()` decorator and run in `nopython` mode.

Results, that we were able to achieve after the addition of Numba were really satisfactory. Execution time comparison between single algorithm loop iteration with and without `@jit` decorator is presented in Table 2.1 and Table 2.2.

Without Numba (μs)	**With Numba (μs)**	**Ratio**
4027	14	0.0035

Table 2.1: Execution time comparison of calculating distances between voters and candidates in a 1D Euclidean preferences model with and without the usage of Python Numba compiler.

Voting rule	**Without Numba (μs)**	**With Numba (μs)**	**Ratio**
Plurality	159	21	0.13
Borda	7298	396	0.054
Schulze	667697	121459	0.18
Harmonic weights	4005	298	0.074
Reversed harmonic weights	4546	302	0.066
Condorcet	462597	6296	0.014
IRV	28100	18376	0.65

Table 2.2: Comparison of execution times of voting rule algorithms between implementations with and without Python Numba usage.

Computing Condorcet winner

The weakest point execution time-wise, even after Numba optimization, was by far the Schulze method. Single iteration of this voting rule algorithm with github implementation was 172 times slower than second worst—positional voting rule with reversed harmonic weights and 5785 times slower than the fastest one—Plurality rule.

However, as previously mentioned, Schulze method is Condorcet. What is more, in a one dimensional space of Euclidean preferences, a Condorcet winner always exists, due to

Schulze method (μs)	**Condorcet winner (μs)**	**Ratio**
121459	6296	0.052

Table 2.3: Execution time comparison on one loop iteration between Schulze method and finding Condorcet winner.

the transitive property which states that if candidate c_1 would win a pair-wise election with candidate c_2 and c_2 would win with candidate c_3, then c_1 would also defeat c_3. This property has been proven by Black (1948).

Consequently, as Schulze method always returns a Condorcet winner if it exists and in a 1D space it always does, discussed voting rule returns the Condorcet winner for every election in our model. Thus, instead of determining who the winner is using the Schulze algorithm, we could use the Condorcet winner definition directly.

A Condorcet winner is a candidate that would win in a pair-wise election with any other opponent. Knowing that such a candidate always exists, finding a Condorcet winner may be limited to a simple loop as presented in Algorithm 2

Algorithm 2 Finding a Condorcet winner in a 1D Euclidean preference model

Ensure: C—set of candidates, $wouldWin(c1, c2)$—function returning *true* if $c1$ would win two-candidate election with $c2$, i.e. it checks if more voters are closer in a one dimensional space of preferences to $c1$ than $c2$

Require: $winner = 0$

for all $C \leftarrow c$ **do**
  if $wouldWin(c, winner)$ **then**
    $winner = c$
  end if
end for
return $winner$

The execution time difference between finding the winner using Schulze method and directly from the Condorcet winner definition is shown in Table 2.3.

2.3. Results

The main goal of our experiments was to observe whether the model with strategic candidates moves towards ε-local-equilibrium—a state in which candidates' movements are almost nonexistent. To be able to explore this issue, we will be analysing the following three metrics:

1. **Candidates' mobility**—average position shifts in candidates' locations in between iterations, averaged over K executions.

   To be more precise, let $f_i : \{1, ..., k\} \to \mathbb{R}_+$, $i \in \{1, ..., K\}$ be the functions describing average position shifts in between subsequent iterations, made by candidates during p election instances in i-th execution of the algorithm. Namely, $f_i(x)$ defines difference in candidates' own positions compared to state at iteration $i - 1$. The value of mobility metric at $x \in \{1, ..., k\}$ equals to $\frac{\sum_{i \in \{1,...,K\}} f_i(x)}{K}$.

   This metric allows us to see if the candidates' mobility is actually decreasing, so whether the candidates lead towards balance.

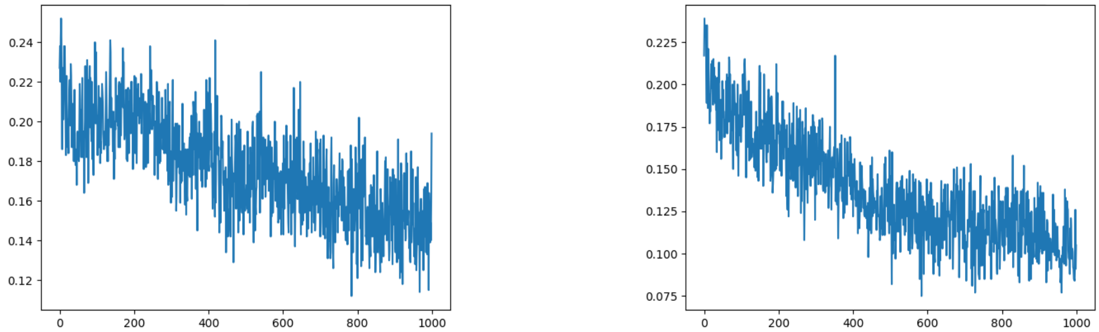


Figure 2.8: Candidates mobility on experiment with 1000 iterations for two example distributions—1-Gaussian and asymmetric 2-Gaussian.

2. **Averaged min/max count**—minimum/maximum number of candidates in given location interval during final 500 iterations, averaged over K executions. For every execution of the algorithm, we take into consideration the maximum and minimum number of candidates, that all took position in between $z/10$ and $(z+1)/10$ for some $z \in \mathbb{Z}$ at the same time in any of the last 500 iterations—$min_i : \{z/10|z \in \mathbb{Z}\} \to \{0, ..., m\}$, $max_i : \{z/10|z \in \mathbb{Z}\} \to \{0, ..., m\}$, $i \in \{1, ..., K\}$. The value of min metric per coordinate equals to $\frac{\sum_{i\in\{1,...,K\}} min_i(x)}{K}$ and max metric—$\frac{\sum_{i\in\{1,...,K\}} max_i(x)}{K}$.

   These values also show us whether the candidates reach equilibrium which results in the same values of averaged min and max counts—convergence of min and max graphs is equivalent to the lack of mobility.

3. **Global min/max count**—minimum/maximum number of candidates in given location interval during final 500 iterations in any of K algorithm executions. With $min_i : \{z/10|z \in \mathbb{Z}\} \to \{0, ..., m\}$, $max_i : \{z/10|z \in \mathbb{Z}\} \to \{0, ..., m\}$, $i \in \{1, ..., K\}$ defined as above, the value of global min metric per coordinate x equals to the minimum over $min_i(x)$ and global max metric—maximum over $max_i(x)$.

   The final metric provides us a picture on final distribution of candidates, so positions that they are leading towards. We may be able to notice if irrespective of input data, randomly selected according to some distribution, candidates move toward the same coordinates, or if it differs, depending on the case.

Plurality

The first voting rule to examine was Plurality. As shown in Figure 2.8, both graphs of candidates' mobility for two different distributions do not reach the value of 0—candidates are still moving even after 1000 election rounds. Nevertheless, we can also see that despite the fact that balance is not reached, position shifts are becoming less significant. They drop from 0.24 to 0.12 for 1-Gaussian distribution and to 0.1 in case of 2-Gaussian one. What is more, we can make one more observation regarding 2-Gaussian distribution, namely, even though the graph of mobility is not reaching zero, it is visibly flattening out. To examine this behaviour further, we will run experiments for a larger number of iterations, i.e. `k = 10 000`. Results of discussed tests are shown in Table 2.4 and Table 2.5.

Mobility We can see that candidates' mobility is dropping as time passes, irrespective of the initial distribution of election participants. The flattening-out effect visible for 1000 iterations is even more apparent in the extended test. At some point, shifts in positions seems to oscillate around the same values, which depends on the distribution.

It is obvious that candidates' mobility significantly drops compered to first iterations. However, it does not reach 0 and looking at the shape of the graph, we may predict that it would not tend towards this value, even if the experiment was extended even more. This would mean, that the model with strategic candidates with Plurality voting rule does not reach ε-local-equilibrium, but leaves the participants at constant movement.

In order to check how this incessant mobility appears in a single experiment instance, the project repository includes implementation of GIF generation, which shows distribution of candidates as iterations pass. What we can observe in this simulation is an pendulum effect. Namely, after a certain number of iteration, the majority of candidates seem to take one position to then all move by some ε value and come back once again to their initial location. The shape of min/max candidate counts graphs with a valley shape seem to confirm this behaviour. It is not clear how to explain this phenomenon, but we might make a hypothesis that strong competition is not conducive to winning under the Plurality rule. Maybe candidates, after observing the density of winners in one location decide to move there, but at some point satiety ensues, the competition becomes simply too large and votes need to be distributed among too many candidates for them to receive high enough score to win the election.

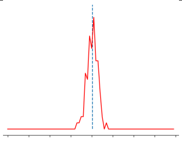
Distribution Even despite the fact that balance is not achieved, we can still observe the general shape of candidates' distribution after 9500 iterations. Number of candidates at given position is changing, but the set of positions that are taken stays the same, which correlates with the pendulum effect visible during the GIF animation. Hence, we can see that, in general, final distribution of candidates do not coincide with initial one—in most cases more extreme locations tend to be taken. Candidates do not stay in positions occupied by the largest number of voters, but they oscillate around nearby coordinates to the right and left of the means of appropriate normal distributions.

An interesting observation can be made after looking at the graphs of global minimum and maximum number of candidates per location. Competitors, in fact, take positions only in close approximation to the means of adequate distributions (at a distance of about 0.5 which is the maximal possible value of position shift), which is indicated by the red plot of maximums. However, the graph of minimums is a constant function equal to 0. What it means, is that we cannot define an approximation of final distribution that would be universal for every input data. Instead of this, we can only indicate some intervals of possible final positions, but an exact coordinate can host both 0 as well as even 100 candidates for 1-Gaussian distribution and 50 for symmetric or 80 for asymmetric 2-Gaussian ones.

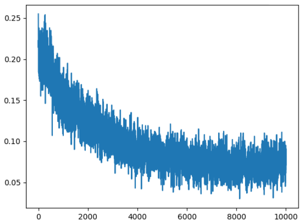
Borda

The next voting rule to analyze is Borda count. Even though the experiments were conducted with the same parameters and distributions of election participants, we can see in the graphs in Table 2.6 and Table!2.7 that the results are different compared to the Plurality rule. The initial hypothesis that our Euclidean preferences model with strategic candidates does not reach equilibrium is not confirmed in this case. Let us distinguish key points that arise from experiments with Borda method.

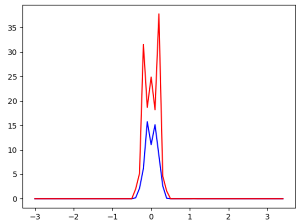
Mobility	Averaged max/min	Global min/max



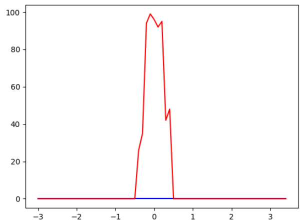
Initial distribution: **100%**: mean = 0, scale = 0.25



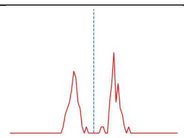
(a)



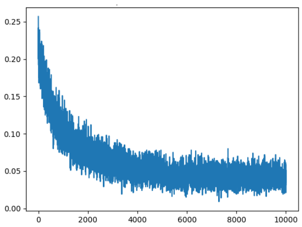
(b)



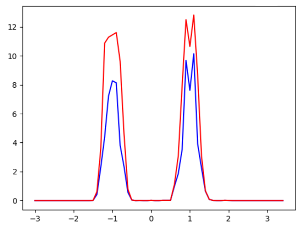
(c)



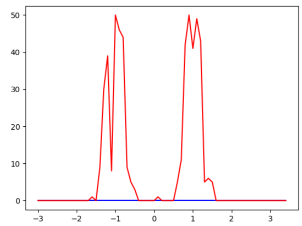
Initial distribution: **50%**: mean = -1, scale = 0.25, **50%**: mean = 1, scale = 0.25



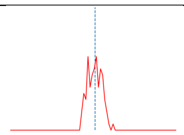
(d)



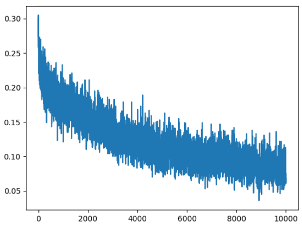
(e)



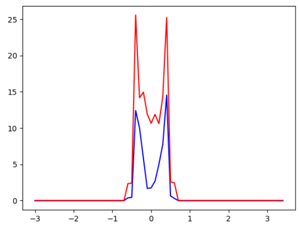
(f)



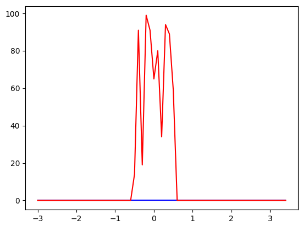
Initial distribution: **50%**: mean = -0.25, scale = 0.25, **50%**: mean = 0.25, scale = 0.25



(g)



(h)



(i)

Table 2.4: Graphs representing metrics used for analysis of one dimensional model with Euclidean preferences and strategic candidates. Results are obtained from an experiment instance with Plurality voting rule and symmetric distributions of election participants.

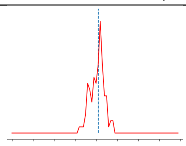
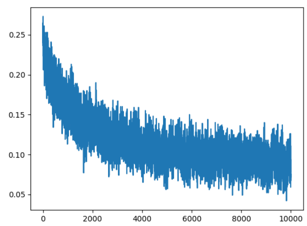
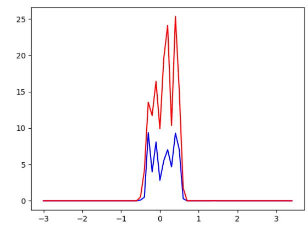
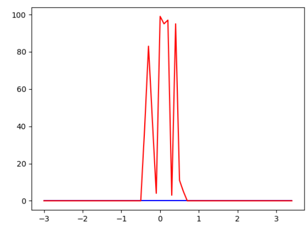
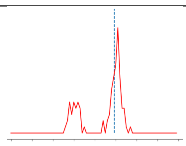
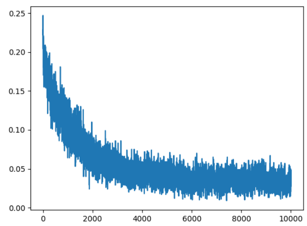
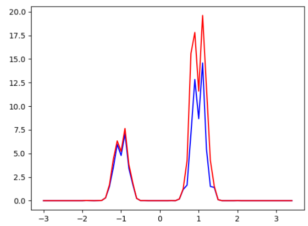
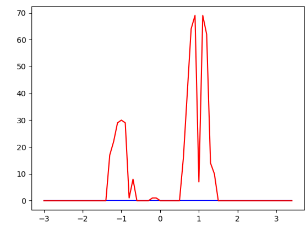
Mobility	Averaged max/min	Global min/max
		
	Initial distribution: **30%**: mean = -0.25, scale = 0.25, **70%**: mean = 0.25, scale = 0.25	
		
(a)	(b)	(c)
		
	Initial distribution: **30%**: mean = -1, scale = 0.25, **70%**: mean = 1, scale = 0.25	
		
(d)	(e)	(f) a

Table 2.5: Graphs representing metrics used for analysis of one dimensional model with Euclidean preferences and strategic candidates. Results are obtained from an experiment instance with Plurality voting rule and asymmetric distributions of election participants.

Mobility Irrespective of the distribution of candidates and voters, candidates mobility initially oscillates around 0.25 and is significantly dropping in the first 500 iterations, in order to finally reach zero and stay in this state till the end of the experiment. What follows from this observation, is that there may exist a state of ε-local-equilibrium for our model with Borda voting rule i.e. a distribution which is optimal for every candidate, given that they can change their position only by an offset taken from a fixed set of possibilities.

What it more, whereas candidates in the Plurality rule case were still mobile after 10000 iterations, the movement under Borda method stops after around 500 rounds.

Distribution As candidates' mobility reaches 0, plots of averaged minimum and maximum number of candidates throughout last 500 iterations are identical. What is more, the graphs of global min/max counts are also almost overlapping. Thus, they can be treated as a good representation of final distribution of candidates—distribution which is the state of equilibrium for Borda.

The first visible thing is that, in general, all candidates move towards the same positions. In the Hotelling-Downs model, which we modified, it was mentioned about the concept of median voter as a position which was sought. Let us compare the location of median voters presented in Figure 2.7 with peaks in the graphs of candidate counts per coordinate.

For a 1-Gaussian distribution shown in Figure 2.11b, we can see that candidates move toward point 0, which is also the median of normal function used to randomly select candidates and voters at the beginning of the experiment. Almost 100% of candidates end at this position. Remaining candidates stay on the sidelines probably simply because fixed set of possible offsets is not versatile enough to provide an advantage of moving.

For other distribution, this looks quite similar. In general, the majority of candidates end in the position of median voter, but we can also see a slight border effect, where some of the candidates are too far away from the median and stay on the edges. Thus, we can observe the following:

- For symmetric distribution of normal functions with close means (Figure 2.11i), we can observe a peak at 0.
- For more distant means (Figure 2.11f), there are two maximums shifted toward 0 in relation to means of normal functions, just like in case of median values. However, in this case the exact locations of those peaks are not clearly defined, as there is a divergence between global min and global max metrics. Nevertheless, the coordinates of median voters also depend on the input data and are not fixed in this case.
- For asymmetric distributions (Figure 2.12c and Figure 2.12f), the hypothesis once again confirms—candidates move towards the mean of dominant distribution slightly shifted towards the mean of the second Gauss function where also a statistical median voter is.

In terms of shapes of final distributions, we can also see some discrepancies with the graphs generated by the Plurality rule. The main one is visible for a 2-Gaussian distribution with means close to each other (Figure 2.11i). For Plurality rule, the initial shape of candidates distribution with two peaks at -0.25 and 0.25 was kept even after 10000 iterations. At the same time, Borda count, considering the fact that those two graphs of normal function overlap around 0, presents a situation, where the majority of candidates converge at this position.

Harmonic weights

Now, let us analyze the next positional voting rule with scoring function defined by harmonic weights. At first glance at graphs gathered in Table 2.8 and Table 2.9, it seems to be in between the Plurality and Borda rules results, which is is line with the shapes of scoring function graphs presented in Figure 1.1.

In this case, similar to the Plurality rule, default experiment setting with 1000 iterations was not enough for candidates to reach balance. Thus, we increased the value of this parameter to 10000.

Mobility Even after extending the experiment, candidates are still mobile after 10000 iterations. Nevertheless, scale of movement is significantly smaller than in case of Plurality rule. Whereas Plurality results indicate average position shift on the level of even 0.15 after 10 000 iterations, in case of harmonic weights this value does not exceed 0.1 or 0.05, depending on the distribution. Consequently, we can see some discrepancies in the graphs of average minimum and maximum number of candidates per location in the last 1000 iterations, but in general, those two metrics converge.

Distribution Analyzing the global min/max metric, we can once again observe an effect similar to the Plurality method. Although the graph of maximums has a more predictable shape with not as many height changes, the graph of minimums is again an almost constant, equal to 0 function. This indicates, that the final distribution universal for every input data does not exist. Nevertheless, it is worth taking a look at the values of the red plot.

- In Figure 2.13c, we can see that the maximal value of 100, which is a total number of candidates is reached for a few coordinates surrounding point 0. What follows from this observation is that all candidates, in fact, tend to move towards a single coordinate, just like it was in case of Borda count. However, what is different is that this location is not clearly specified, but it can differ depending on the initial data.
- For other distribution, a similar effect can be noticed, where we can distinguish a plateau of maximum values rather than a single peak. The value at this plateau is proportional to initial distribution of election participants—if e.g. 70% of candidates is selected from distribution with mean at 1, 70% of candidates will also end around this mean value.
- The most surprising and worth discussing instance is the one with 1-Gaussian distribution shown in Figure 2.13b. Here, even though global max count of candidates reaches the value of 100, the averaged metric indicates that, for most cases, candidates prefer to take positions slightly shifted in relation to the mean. Competitors, instead of moving towards the mean location, abandon this position.

  Explanation for this phenomenon is not straightforward, but it is also reflected in case of 2-Gaussian distribution with symmetric, more distant means (Figure 2.13e), which for other voting rules also tends to behave like two down-scaled and to a large degree independent experiments of 1-Gaussian distribution kind.

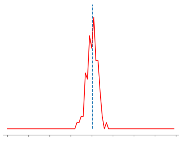
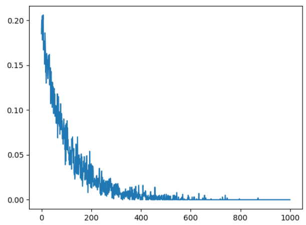
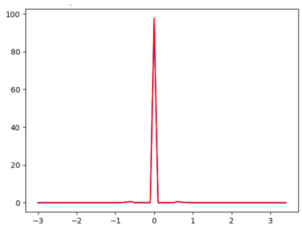
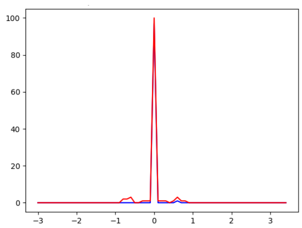
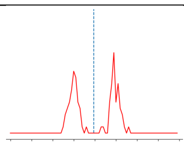
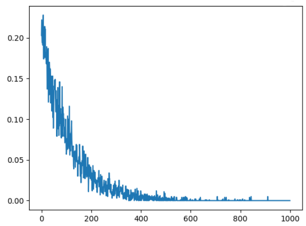
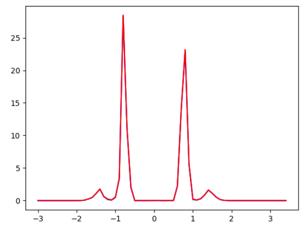
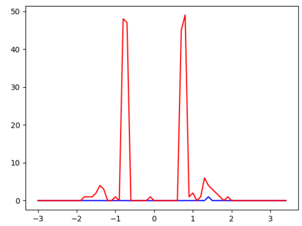
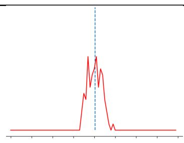
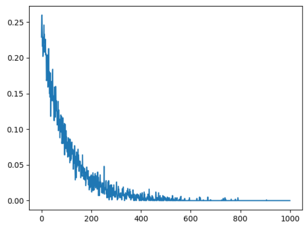
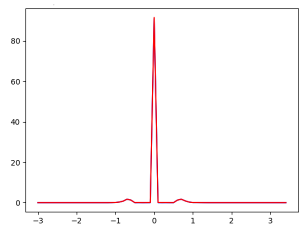
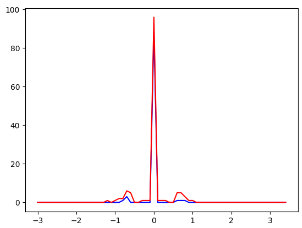
Mobility	Averaged max/min	Global min/max
		
	Initial distribution: **100%**: mean = 0, scale = 0.25	
		
(a)	(b)	(c)
		
	Initial distribution: **50%**: mean = -1, scale = 0.25, **50%**: mean = 1, scale = 0.25	
		
(d)	(e)	(f)
		
	Initial distribution: **50%**: mean = -0.25, scale = 0.25, **50%**: mean = 0.25, scale = 0.25	
		
(g)	(h)	(i)

Table 2.6: Graphs representing metrics used for analysis of one dimensional model with Euclidean preferences and strategic candidates. Results are obtained from an experiment instance with Borda voting rule and symmetric distributions of election participants.

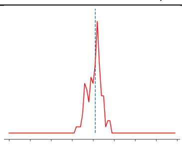
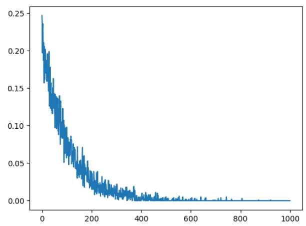
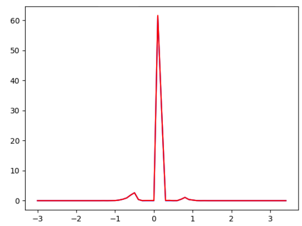
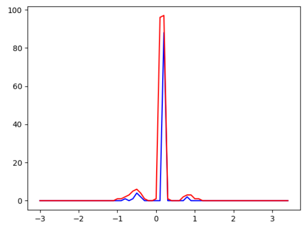
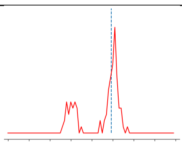
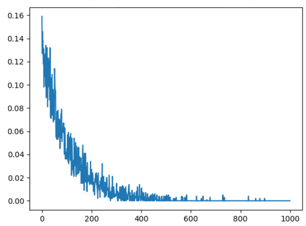
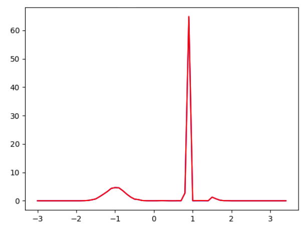
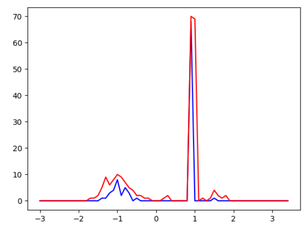
Mobility	Averaged max/min	Global min/max
		
Initial distribution: **30%**: mean = -0.25, scale = 0.25, **70%**: mean = 0.25, scale = 0.25		
		
(a)	(b)	(c)
		
Initial distribution: **30%**: mean = -1, scale = 0.25, **70%**: mean = 1, scale = 0.25		
		
(d)	(e)	(f)

Table 2.7: Graphs representing metrics used for analysis of one dimensional model with Euclidean preferences and strategic candidates. Results are obtained from an experiment instance with Borda voting rule and asymmetric distributions of election participants.

As mentioned above, for positional voting with harmonic weights as a scoring function, the median voter effect is not maintained. Candidates seem to move towards means of normal functions used to generate election participants (so to positions where the majority of voters are located), instead of median values of voters' sets. In this manner, we can once again see some similarities to the Plurality method.

Reversed harmonic weights

Results obtained in the experiment with reversed harmonic weights as a scoring function are very similar to those of the Borda rule.

Here, we may point out a certain property. The graph of harmonic function looks similar to the Plurality scoring function—it drops rapidly at the beginning and becomes constant in case of Plurality and almost constant in case of harmonic weights. This correspondence is reflected in the comparison of results between those two methods. In the same manner, we may compare reversed harmonic weights to Borda count. Both functions grant some positive score to the highest ranked candidates and 0 to least preferred one, which could explain the similarities of the results.

Mobility Similar to the Borda rule, with reversed harmonic weights candidates' mobility is dropping to 0 in the first ~ 500 iterations. The, equilibrium is reached and position shifts can no longer be observed. Graphs of averaged local counts per location confirms it, as for every coordinate, graphs of minimum and maximum number of candidates converge.

Distribution Graphs of global max and min candidate counts are almost identical only in case of 1-Gaussian distribution. In other cases, the min count function is equal to 0 for the vast majority of coordinates. Nevertheless, the plateau effect can be observed once again, where the value of the maximum number of candidates (proportional to the distribution) is reached for a couple positions around particular coordinates. It means, that the candidates tend towards one location, but its exact value may differ. Consequently, the global max count plot can be used to analyze final distribution of candidates. which is model's state of ε-local-equilibrium.

Similarly to Borda, the majority of candidates seem to move towards the median voter location leaving a few competitors on the sidelines.

- For 1-Gaussian (Figure 2.15c) and symmetric, close 2-Gaussian (Figure 2.15i) distributions practically every candidate end in point 0.
- For asymmetric distributions (Figure 2.16c and Figure 2.16f), it is the mean of dominant normal function slightly shifted towards zero that candidates move towards.

We can see that the border effect is magnified for every experiment instance with reversed harmonic weights, compared to the Borda ones. However, the most visible difference between those two methods appears in case of 2-Gaussian distribution with means at -1 and 1 (Figure 2.16f). In this test case, local maximums in the number of candidates per location around functions' means, in the opposite to 0 direction can reach even the value of 10 on each side. This would suggest, that discussed scoring function either makes it more difficult for distant candidates to move towards more favorable coordinates or

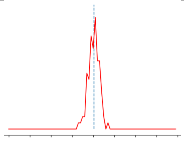
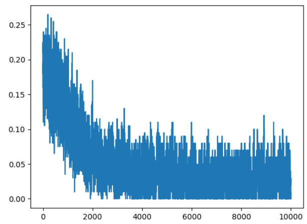
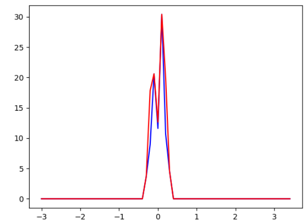
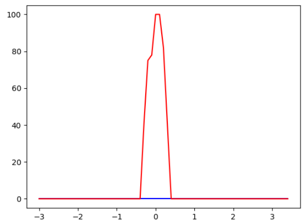
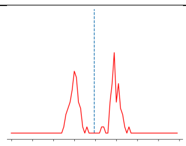
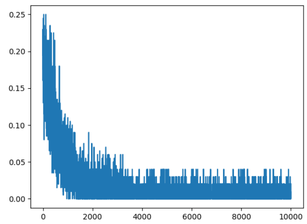
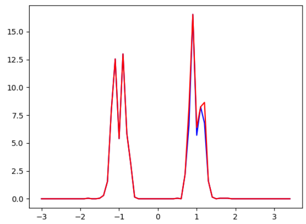
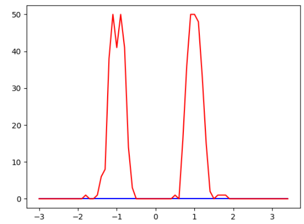
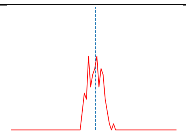
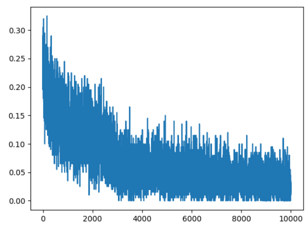
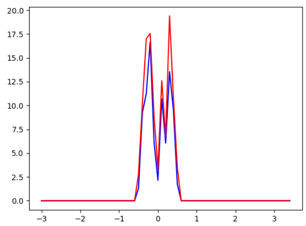
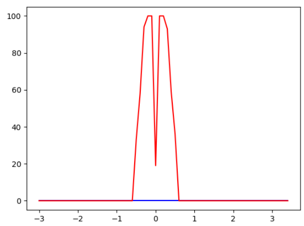
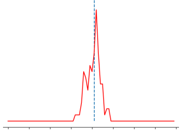
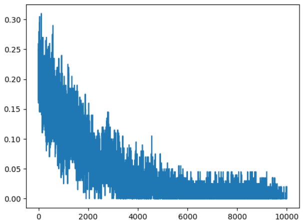
Mobility	Averaged max/min	Global min/max
	 Initial distribution: **100%**: mean = 0, scale = 0.25	
 (a)	 (b)	 (c)
	 Initial distribution: **50%**: mean = -1, scale = 0.25, **50%**: mean = 1, scale = 0.25	
 (d)	 (e)	 (f)
	 Initial distribution: **50%**: mean = -0.25, scale = 0.25, **50%**: mean = 0.25, scale = 0.25	
 (g)	 (h)	 (i)

Table 2.8: Graphs representing metrics used for analysis of one dimensional model with Euclidean preferences and strategic candidates. Results are obtained from an experiment instance with positional voting rule using harmonic weights as a scoring function and symmetric distributions of election participants.

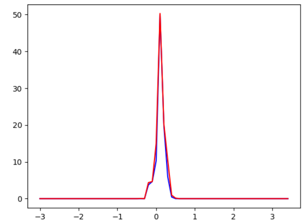
Mobility	Averaged max/min	Global min/max



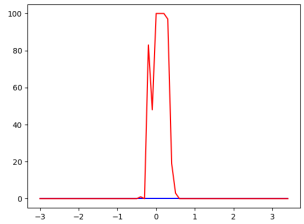
Initial distribution: **30%**: mean = -0.25, scale = 0.25, **70%**: mean = 0.25, scale = 0.25



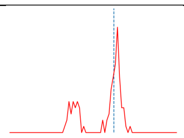
(a)



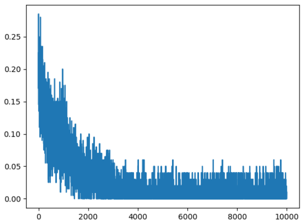
(b)



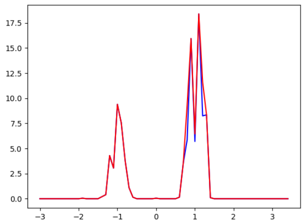
(c)



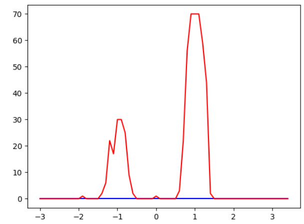
Initial distribution: **30%**: mean = -1, scale = 0.25, **70%**: mean = 1, scale = 0.25



(d)



(e)



(f)

Table 2.9: Graphs representing metrics used for analysis of one dimensional model with Euclidean preferences and strategic candidates. Results are obtained from an experiment instance with positional voting rule using harmonic weights as a scoring function and asymmetric distributions of election participants.

gives a better chance to candidates staying on the sidelines of distribution graphs due to weaker competition in this area.

The fact of existence of discussed border effect and reasons behind it are also described for multiwinner rules by Elkind, Faliszewski, et al. (2017).

Now that we have analyzed all of the positional voting rules, let us summarise the key discoveries.

- Experiments conducted with Plurality rule and reversed harmonic weights do not reach equilibrium even after extending the number of iterations, whereas for remaining methods candidates seem to become stable from a certain point.

- Even for voting rules that reach balance, we cannot distinguish an exact distribution that candidates move towards. However, with Borda count and reversed harmonic weights as scoring functions, we can roughly predict final distribution of candidates, as the graphs of global minimum and maximum number of candidates per location are almost overlapping.

  Remaining rules without the equilibrium also leave us with some general shape of candidates' final distribution, but due to constant movement it is far less precise. For almost every coordinate with global max count metric value greater than 0, min count is equal to zero. It means, that results obtained from test cases on Plurality rule and harmonic weights scoring function allow us to determined where the candidates could be after a certain number of iteration, but exact position cannot be specified.

- There exists some correspondence between graphs of scoring functions and experiment results. As mentioned before, there are similarities in metric values got from Borda and reversed harmonic weights experiments, as well as between Plurality and harmonic weights. At the same time, we can also see the resemblance in shapes of scoring function graphs between methods in those pairs.

  It is also worth mentioning, that functions which become constant or almost constant as iterations passes do not reach equilibrium, whereas those that clearly differentiate scoring in between less preferred candidates become stable relatively quickly. This observation for sure deserves further exploration.

Condorcet

The next stage of our research included analysis of Condorcet methods. Results are presented in Table 2.12 and Table 2.13.

Mobility Candidates, in general, reach their final positions about around 600 iterations. Later on, we can notice only single cases of position shifts. Nevertheless, it is safe to say that there exists a state of ε-local-equilibrium for algorithm selecting the Condorcet winner, which is also confirmed by the look of averaged max and min count graphs which do converge.

What seems to be different in the graph of mobility, compared to Borda rule which also reaches the value of 0, is that the amplitudes of magnitude of position shifts between subsequent iterations in the first stage of experiment are visibly larger. Whereas for mentioned positional voting rule candidates were moving on average by similar offset, in case of Condorcet method this value can really differ depending on the iteration.

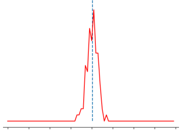
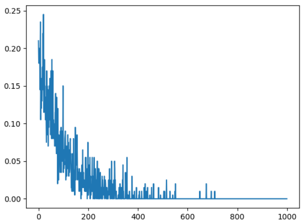
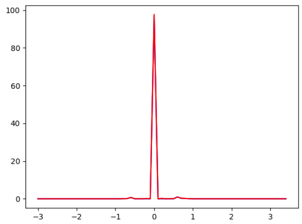
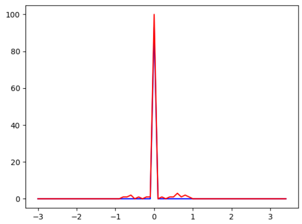
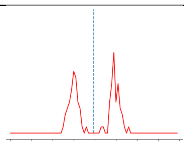
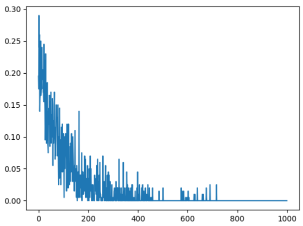
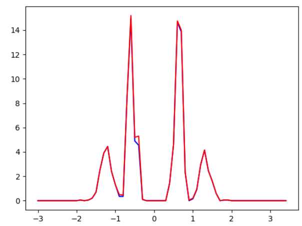
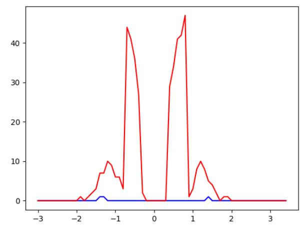
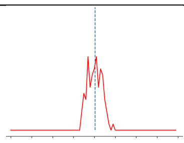
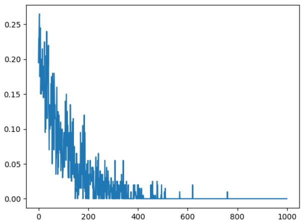
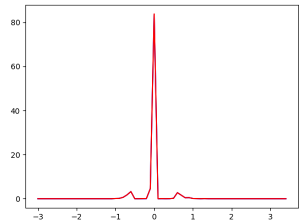
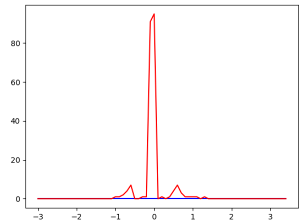
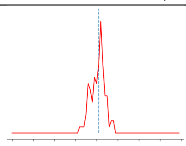
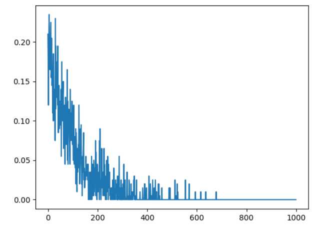
Mobility	Averaged max/min	Global min/max
		
	Initial distribution: **100%**: mean = 0, scale = 0.25	
 (a)	 (b)	 (c)
		
	Initial distribution: **50%**: mean = -1, scale = 0.25, **50%**: mean = 1, scale = 0.25	
 (d)	 (e)	 (f)
		
	Initial distribution: **50%**: mean = -0.25, scale = 0.25, **50%**: mean = 0.25, scale = 0.25	
 (g)	 (h)	 (i)

Table 2.10: Graphs representing metrics used for analysis of one dimensional model with Euclidean preferences and strategic candidates. Results are obtained from an experiment instance with positional voting rule using reversed harmonic weights as a scoring function and symmetric distributions of election participants.

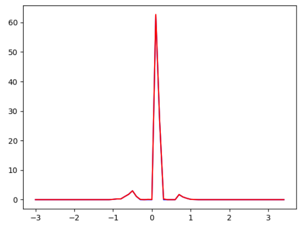
Mobility	**Averaged max/min**	**Global min/max**



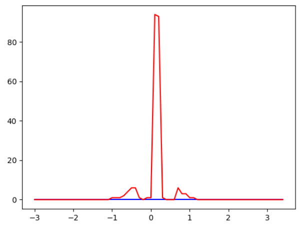
Initial distribution: **30%**: mean = -0.25, scale = 0.25, **70%**: mean = 0.25, scale = 0.25



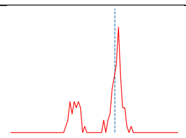
(a)



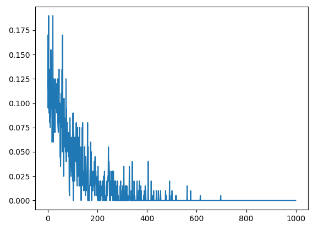
(b)



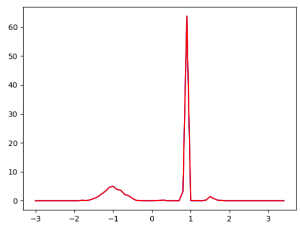
(c)



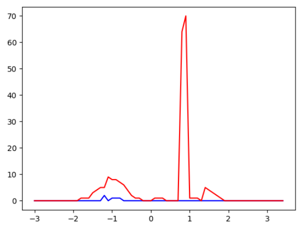
Initial distribution: **30%**: mean = -1, scale = 0.25, **70%**: mean = 1, scale = 0.25



(d)



(e)



(f)

Table 2.11: Graphs representing metrics used for analysis of one dimensional model with Euclidean preferences and strategic candidates. Results are obtained from an experiment instance with positional voting rule using reversed harmonic weights as a scoring function and asymmetric distributions of election participants.

Distribution Similarly to Borda count and reversed harmonic weights, we can see that the graphs of global minimum and maximum number of candidates per location are a good approximation of one another. What follows from this fact is that as long as election participants are randomly selected from a certain distribution, they tend to move towards approximately the same coordinates no matter what the exact input data is. The one exception is Figure 2.17f, where the global min count metric is equal to 0 where max count metric reaches its heights values. The reason for it may once again lie in the fact, the as far as 2-Gaussian distribution with means at 1 and -1 is concerned, the location of two median values can really differ depending on the input data.

As far as the median voters are concerned, we can once again show a similarity to the Hotelling-Down model. That is because the peaks in the graphs of global min/max are exactly at or close to the location of median voters. Almost every candidate end at those position. Nevertheless, we can observe the border effect in this experiment instance, as well.

IRV

Finally, let us focus on the Instant-runoff rule. Just to recall, we could define IRV rule as a series of Plurality rule round with candidate elimination in between them. Thus, it will be interesting to see if those two rules behave in a similar manner, or maybe this extension results in more favourable behaviour of candidates.

Mobility Already the first metric to analyze, which is candidates' mobility, demonstrates differences between Plurality and IRV. Namely, our model with discussed voting rule seem to reach ε-local-equilibrium. Although 1000 iteration was not enough, after extending test length to 2000 it became visible that the value of 0 is, in fact, reached and, in most cases, kept.

The only exception is a symmetric 2-Gaussian distribution with means at -0.25 and 0.25 (Figure 2.20a). Here, after 2000 iteration candidates are still moving, however the graphs of averaged min and max counts are almost identical, so we may assume that balance is just a matter of time. To confirm this hypothesis, we can take a look at even more extended experiment instance with this distribution presented in Figure 2.19.

Distribution Even though the mobility metrics shows similarities to Borda or Condorcet methods rather than Plurality one, it looks opposite in case of the final distribution of candidates which is also the state of equilibrium in our model. Results, in term of this metric, resemble those obtained using Plurality rule. We cannot distinguish a single coordinate that candidates move towards irrespective of input data compatible with chosen distribution. Instead of this, we can show a set of possible positions which candidate will with a high probability end in.

The graph of global min count is a function almost always equal to 0. We can also again see a plateau instead of a single peak in the graph of global max count. Nevertheless, those intervals of positions that candidates move towards are much more regular and compact compered to Plurality results. We can say that candidates move towards the means of normal function used to randomly choose election participants or somewhere around those values—it can be where the median voter is located, but it is not true for every test case.

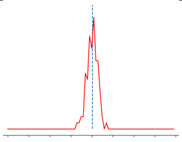
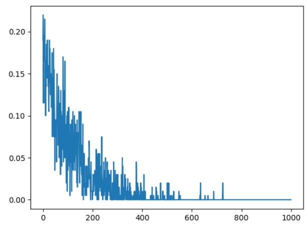
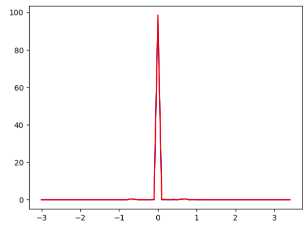
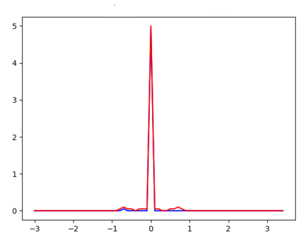
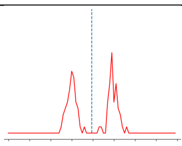
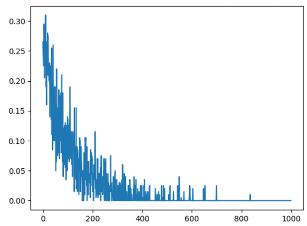
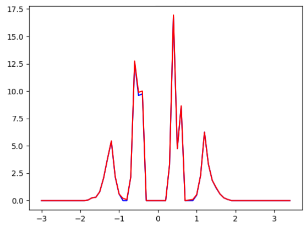
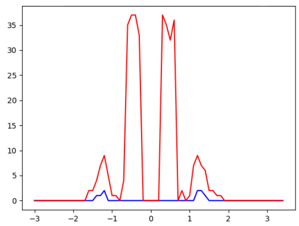
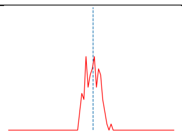
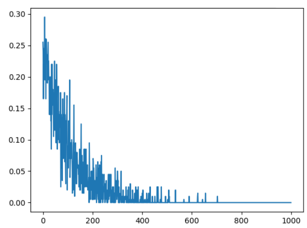
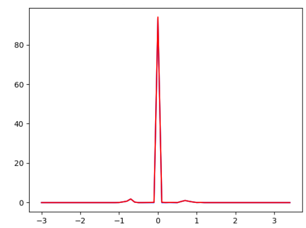
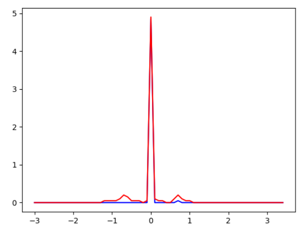
Mobility	Averaged max/min	Global min/max
		
	Initial distribution: **100%**: mean = -0.25, scale = 0.25	
		
(a)	(b)	(c)
		
	Initial distribution: **50%**: mean = -1, scale = 0.25, **50%**: mean = 1, scale = 0.25	
		
(d)	(e)	(f)
		
	Initial distribution: **50%**: mean = -0.25, scale = 0.25, **50%**: mean = 0.25, scale = 0.25	
		
(g)	(h)	(i)

Table 2.12: Graphs representing metrics used for analysis of one dimensional model with Euclidean preferences and strategic candidates. Results are obtained from an experiment instance selecting a Condorcet winner and symmetric distributions of election participants.

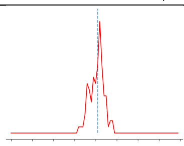
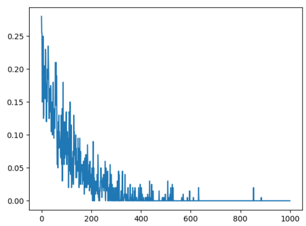
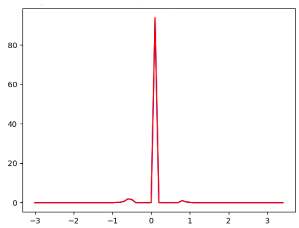
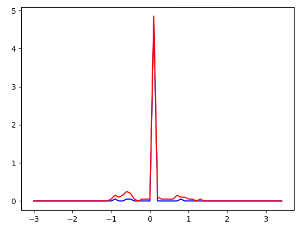
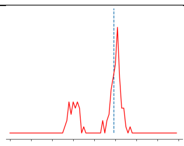
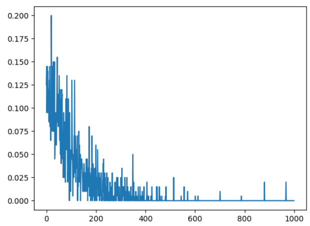
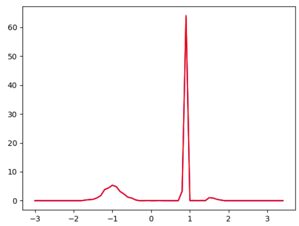
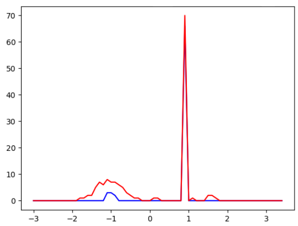
Mobility	Averaged max/min	Global min/max
		
	Initial distribution: **30%**: mean = -0.25, scale = 0.25, **70%**: mean = 0.25, scale = 0.25	
		
(a)	(b)	(c)
		
	Initial distribution: **30%**: mean = -1, scale = 0.25, **70%**: mean = 1, scale = 0.25	
		
(d)	(e)	(f)

Table 2.13: Graphs representing metrics used for analysis of one dimensional model with Euclidean preferences and strategic candidates. Results are obtained from an experiment instance selecting a Condorcet winner and asymmetric distributions of election participants.

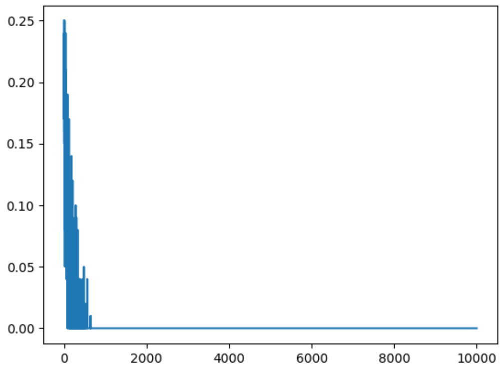


Figure 2.19: Candidates' mobility during an experiment instance with IRV voting rule on a symmetric 2-Gaussian distributions with means at -0.25 and 0.25.

The number of candidates around a specific coordinate is proportional to the size of candidate set chosen from corresponding distribution. Just as it was said:

- for 1-Gaussian distribution (Figure 2.20c), all candidates move towards a single coordinate at or around 0,
- for symmetric, distant 2-Gaussian distribution (Figure 2.20f) candidates splits in half and also locate themselves around function means,
- in case of asymmetric, close distribution (Figure 2.21c), it is also 100% of candidates that end in between points 0 and 0.25,
- finally, asymmetric, distant 2-Gaussian distribution (Figure 2.21f) leaves 30% of candidates around -1 and 70% around 1.

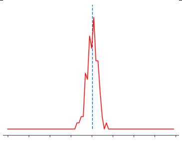
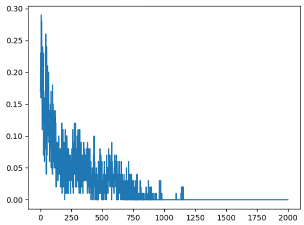
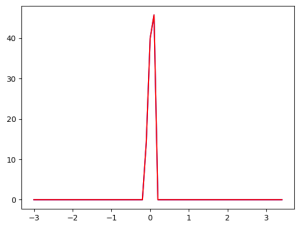
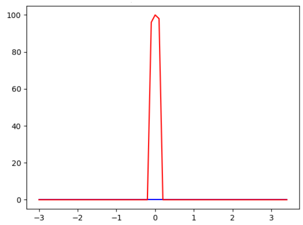
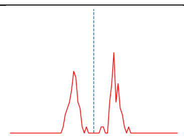
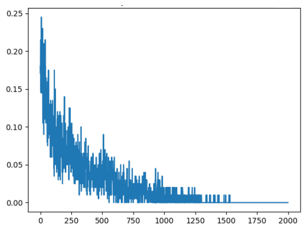
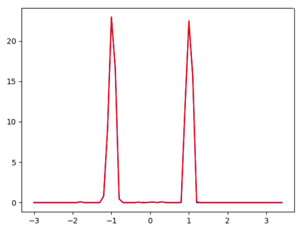
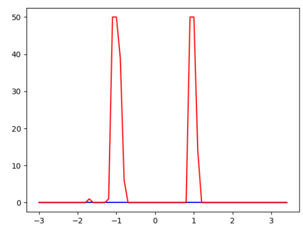
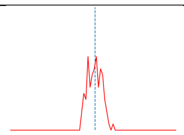
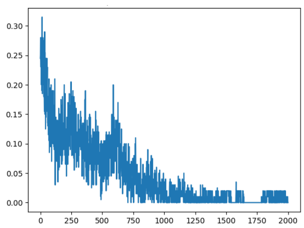
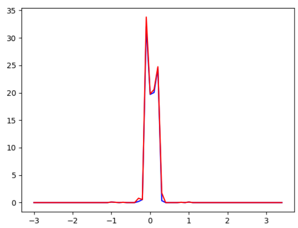
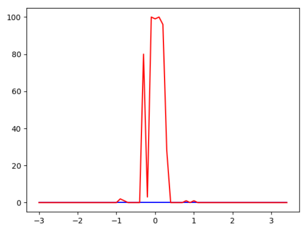
Mobility	Averaged max/min	Global min/max
		
	Initial distribution: **100%**: mean = 0, scale = 0.25	
 (a)	 (b)	 (c)
		
	Initial distribution: **50%**: mean = -1, scale = 0.25, **50%**: mean = 1, scale = 0.25	
 (d)	 (e)	 (f)
		
	Initial distribution: **50%**: mean = -0.25, scale = 0.25, **50%**: mean = 0.25, scale = 0.25	
 (g)	 (h)	 (i)

Table 2.14: Graphs representing metrics used for analysis of one dimensional model with Euclidean preferences and strategic candidates. Results are obtained from an experiment instance with IRV voting rule and symmetric distributions of election participants.

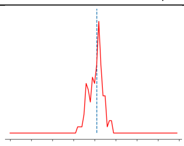
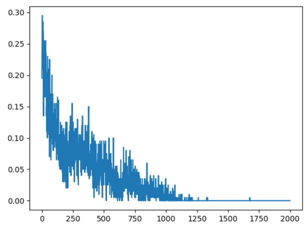
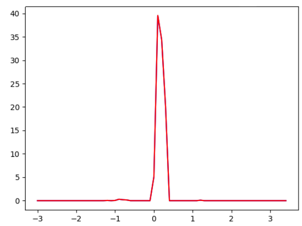
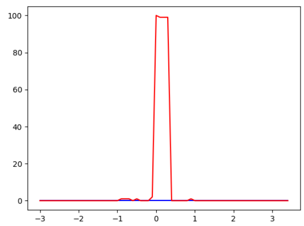
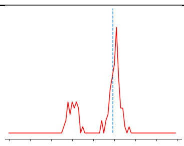
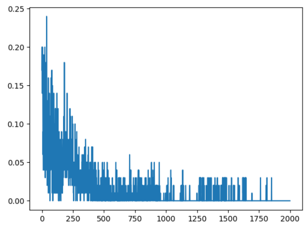
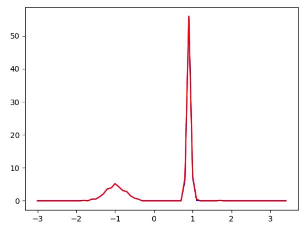
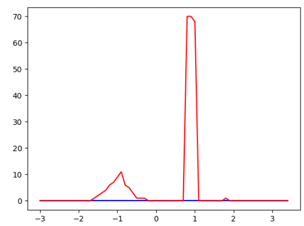
Mobility	Averaged max/min	Global min/max
		
	Initial distribution: **30%**: mean = -0.25, scale = 0.25, **70%**: mean = 0.25, scale = 0.25	
		
(a)	(b)	(c)
		
	Initial distribution: **30%**: mean = -1, scale = 0.25, **70%**: mean = 1, scale = 0.25	
		
(d)	(e)	(f)

Table 2.15: Graphs representing metrics used for analysis of one dimensional model with Euclidean preferences and strategic candidates. Results are obtained from an experiment instance with IRV voting rule and asymmetric distributions of election participants.

Chapter 3

Conclusions and related work

After simulating elections with strategic candidates in 1D-Euclidean preferences model, we can clearly see that the conclusions differ depending on the voting rule. We shall consider the results in two aspects—whether the equilibrium of candidates exists and, if it does, what the equilibrium distribution looks like.

Analysis of the candidates' mobility after certain number of algorithm iterations indicates that the equilibrium exists only for a subset of presented voting rules. Our results suggest that Borda, reversed harmonic scoring function, IRV and Condorcet rules guarantee convergence to equilibrium. Candidates under mentioned methods decrease their mobility in the first iterations of our algorithm so that it finally reaches the value of 0 and there are no more shifts in the positions of candidates. The rule with harmonic scoring function indicates mobility which does not tend to zero even after 10000 iterations, however the range of movement becomes so limited, that the distribution of candidates barely changes. On the other hand, Plurality rule does not seem to guarantee any degree of equilibrium, as candidates' position shifts remain on a significant level. Moreover, in the model of strategic candidates with an option to quit or enter election, Plurality also does not reach equilibrium, as shown by Polukarov et al. (2015).

How candidates' distributions shape under model's equilibrium is also dependent on the voting rule. Two tendencies can be observed: candidates either remain around the means of corresponding normal distributions (harmonic scoring function, IRV) or move towards the locations of median voters (Borda, reversed harmonic scoring function, Condorcet winner).

As mentioned before, the importance of median values in the mathematical model of elections has been already noticed by Downs, but it was also further discussed e.g. by Davis et al. (1968). Their work focuses on two candidate elections—first, under normal or multinormal distribution and finally under any distribution. It is mathematically shown, that a candidate who takes a median position in a space of voters' preferences will always win over a candidate who does not. What is interesting, the median loses its significance when voting takes place on more than one issue at the time.

Our analysis confirms the importance of median values for half of discussed voting rules. We can say, that those three methods encourage the candidates to adopt the compromise by not taking the positions where the majority of voters are, but slightly adjusting them in order to also please (to some extent) voters at the other extreme. Borda, reversed harmonic scoring function and finding the Condorcet winner are the rules which aim to reach the consensus, in contrast to harmonic scoring function and IRV rules more focused on pleasing the majorities and Plurality which, resulting from the lack of balance, could also favour taking more extreme positions.

Finally, let us once again focus on experimental cases with a 2-Gaussian distribution where 30% of voters and candidates are randomly chosen from a normal distribution with mean at -1 and 70%—with mean at 1, as an interesting tendency can be observed there. For a harmonic scoring function, we can see that the candidates at the end of the experiment position themselves rather proportionally to the number of voters i.e. after analysing the graphs of global minimum and maximum per location, we can distinguish a coordinate around point -1 where 30% of candidates end in and a coordinate around point 1 with the other 70% of contenders.

With the exception of Plurality rule, where no convergence is reached, remaining voting rules show a different property. We can still highlight a focal point around 1 with 70% of candidates, however when it comes to the mean of the less dominant normal distribution, global maximums around -1 do not exceed the value of 15. That would mean, that the candidates initially chosen from the smaller normal distribution do not tend towards one coordinate. On the contrary, they try to escape the mean value of -1, however the limited set of possible position shifts makes it impossible for them to reach the locations around the other mean value. Consequently, the candidates around -1 are distributed over the wider range of coordinates.

Although the complete convergence is not reached in case of harmonic scoring function, the range of movements after 10000 iterations is so insignificant that it can be treated as a good approximation of equilibrium. Thus, with the following two observations that in case of harmonic scoring function candidates tend towards the mean values of appropriate normal functions and that the proportions of used distributions are kept, it is interesting to notice, that there exists one voting rule where the final spectrum of the political scene looks really similar to the spectrum of the electorate and, at the same time, the stability is relatively good.

As we can see, who will be announced as the election winner can really depend on the voting rule. Some of the decision functions tend to please the median voter, whereas the others favour majorities and disregard minorities. Thanks to our analysis, we were able to observe where it is best for the candidates to locate their beliefs in order to win the election under different voting rules. Knowing how the people have historically voted has proved to be helpful while trying to maximise one's chances to win for the majority of cases.

Results obtained in our work can come helpful both for the candidates while determining their election strategy, as well as for those who decide about the election rules to ensure that they are non-discriminatory. Proposed model may be further extended e.g. to a two-dimensional domain. The algorithm can be also applied for different distributions, for instance based on real life, historic election data, that can be obtained from PrefLib—a library of preference data, containing datasets even from presidential election.

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