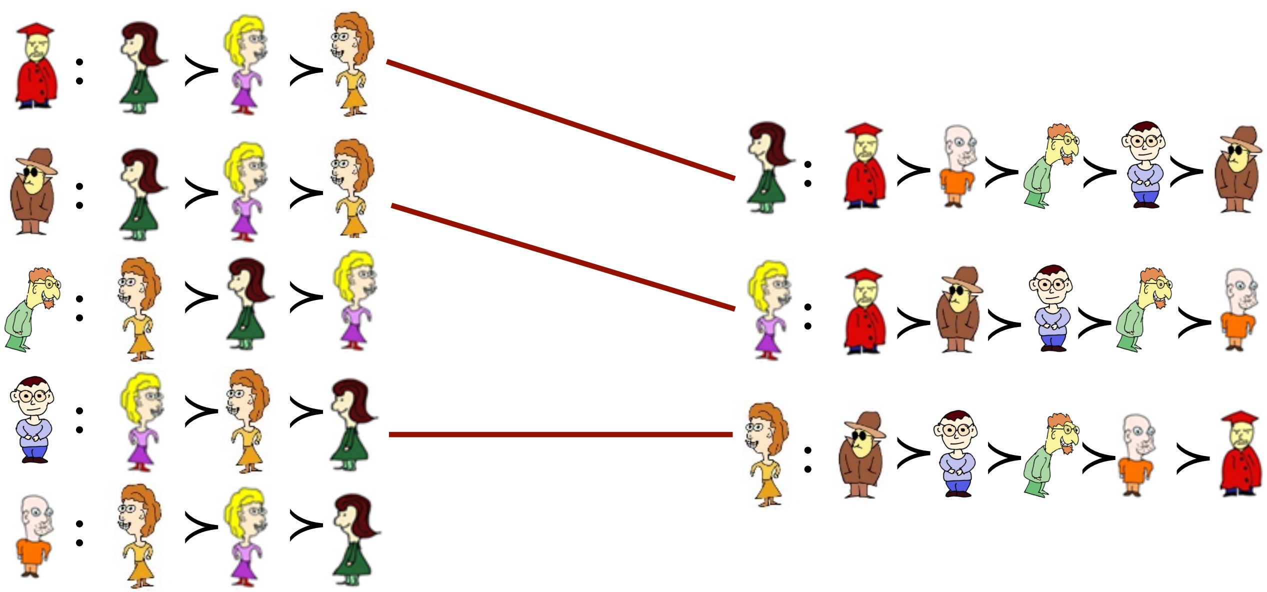
# Computational Social Choice

## Stable Matchings

**Piotr Skowron**
University of Warsaw



# The model

1. A set of $n$ men $U = \{u_1, u_2, \ldots, u_n\}$.
2. A set of $m$ women $W = \{w_1, w_2, \ldots, w_m\}$.

Each man $u_i$ has a preference relation $\succ_{u_i}$ over the set of women.

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**Goal:** Find a matching between men and women, such that each man is matched with at most one woman, and each woman is matched with at most one man.

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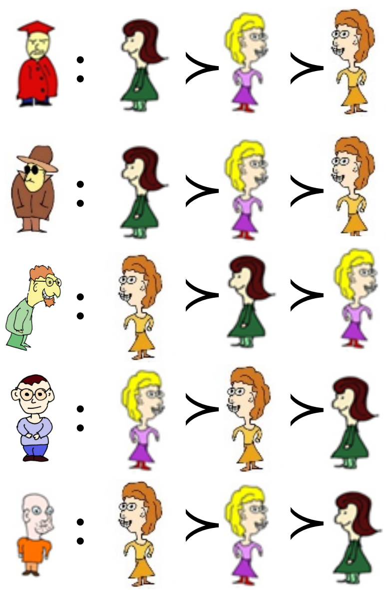

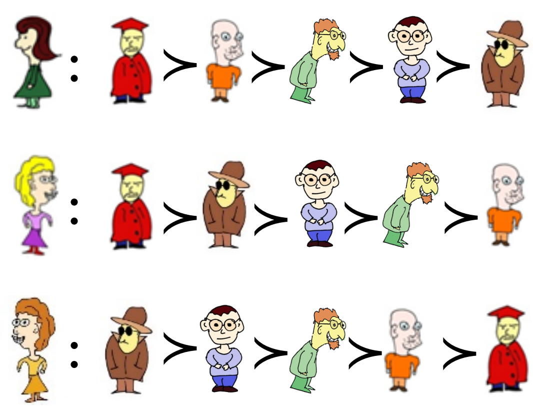

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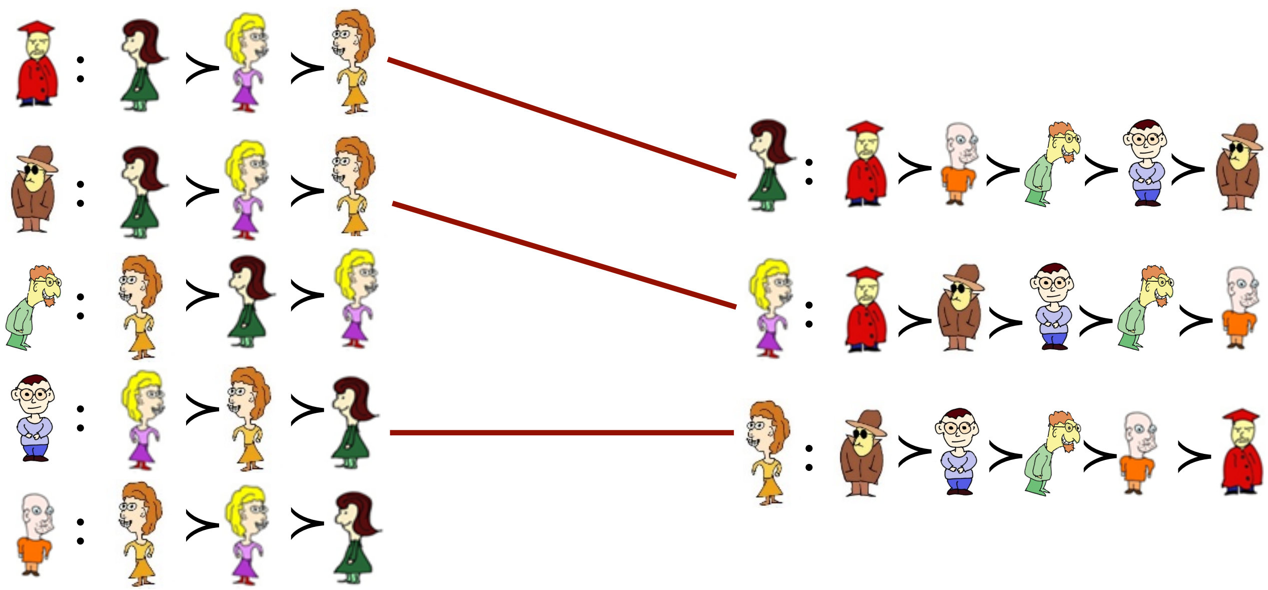

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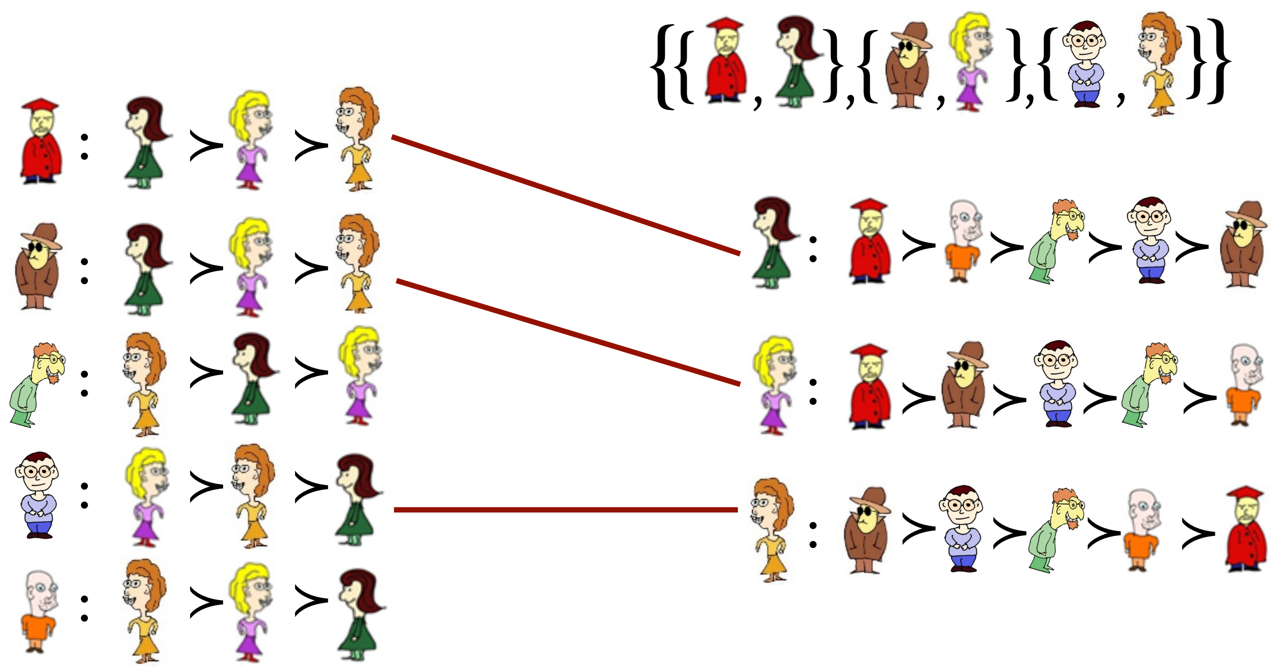

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**Stable Matching (CE)**: We say that a pair of a man and a woman $(u,w)$ blocks a matching $M$ if $u$ and $w$ are not matched and:

1. $u$ is unmatched or prefers $w$ to her partner $M(u)$ in the matching, and
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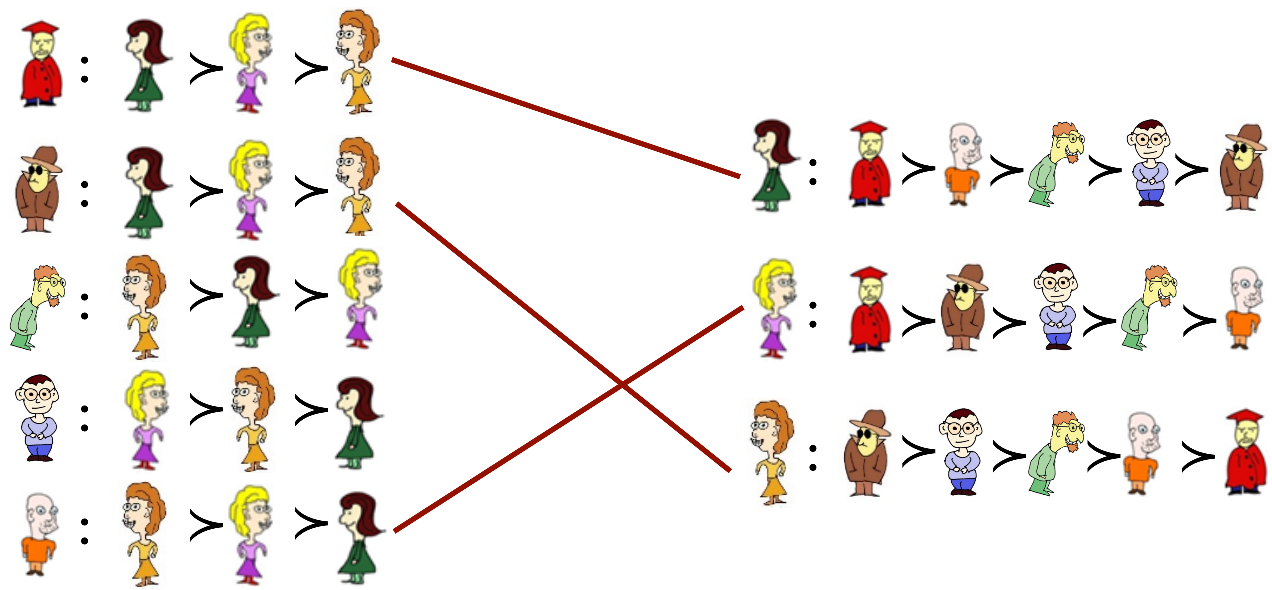

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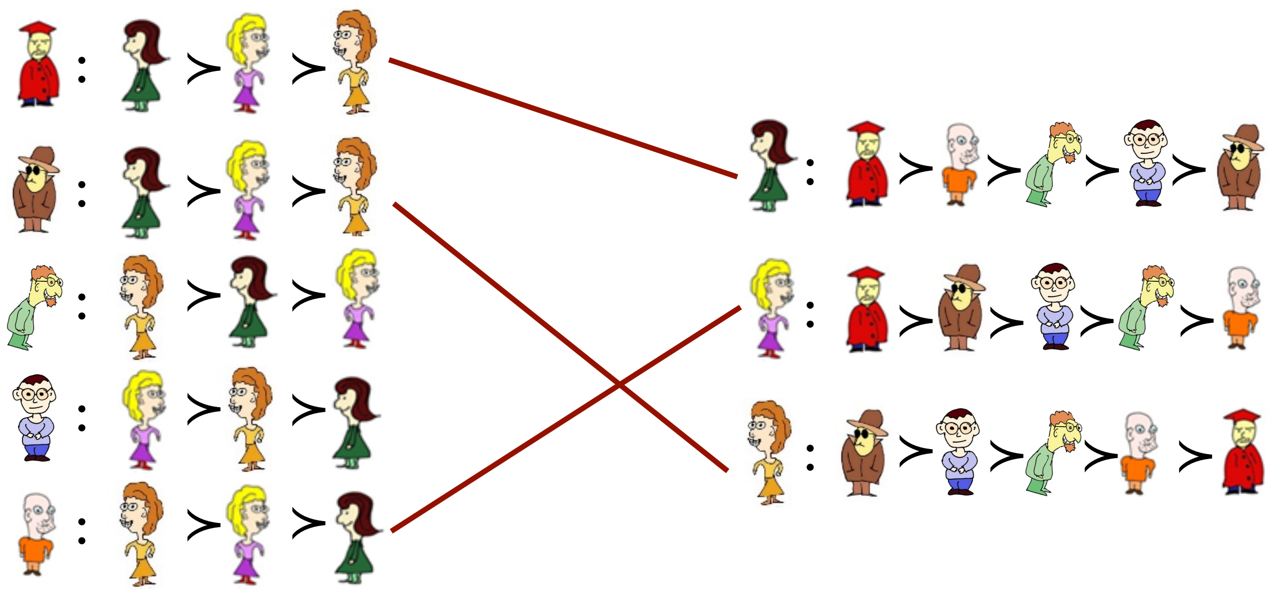
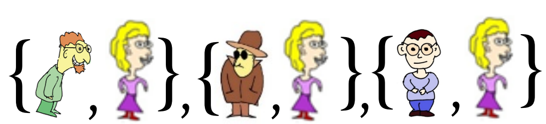


**Blocking pairs:** 

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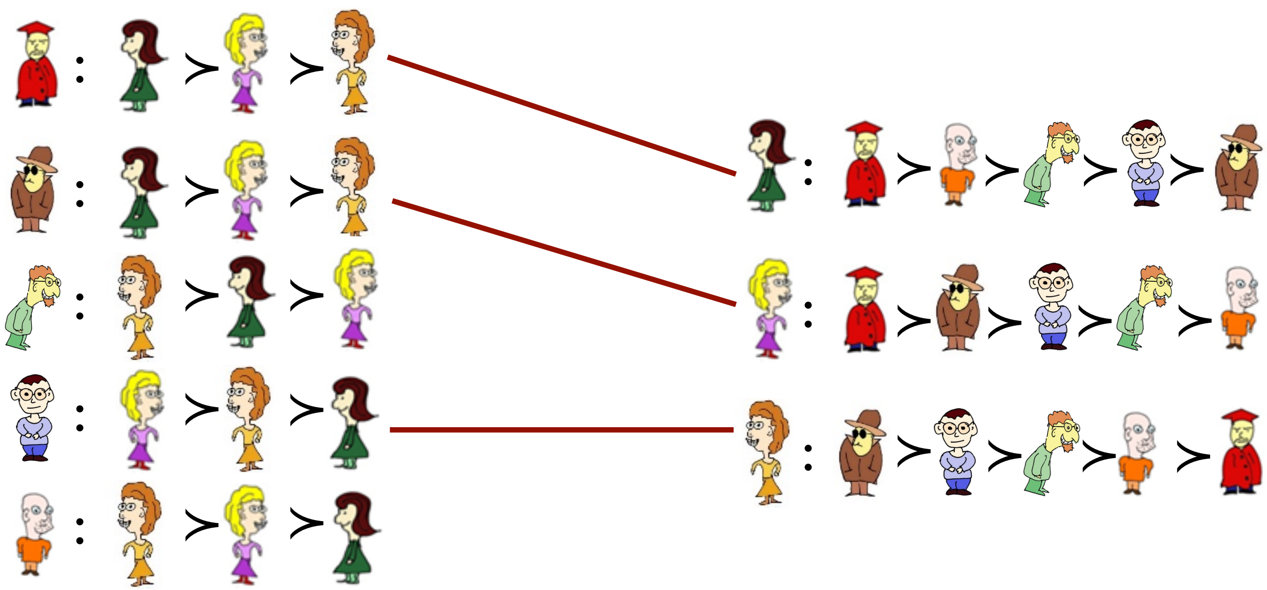

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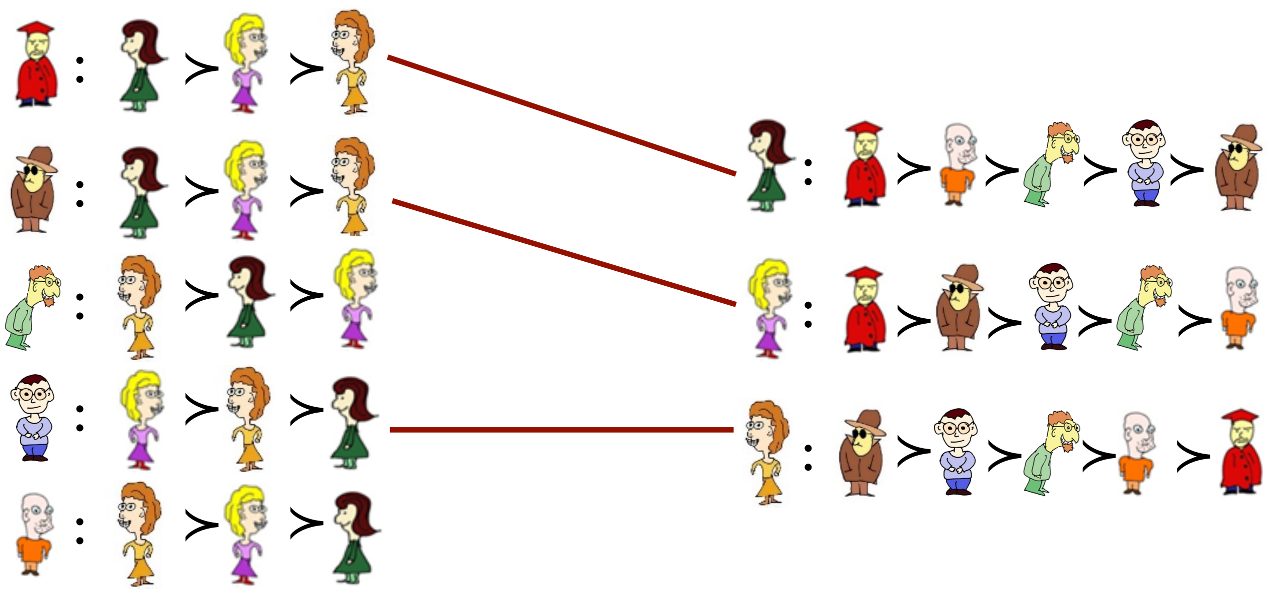

No blocking pairs. This matching is stable!

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**Does a stable matching always exists? If so, how one can find it?**

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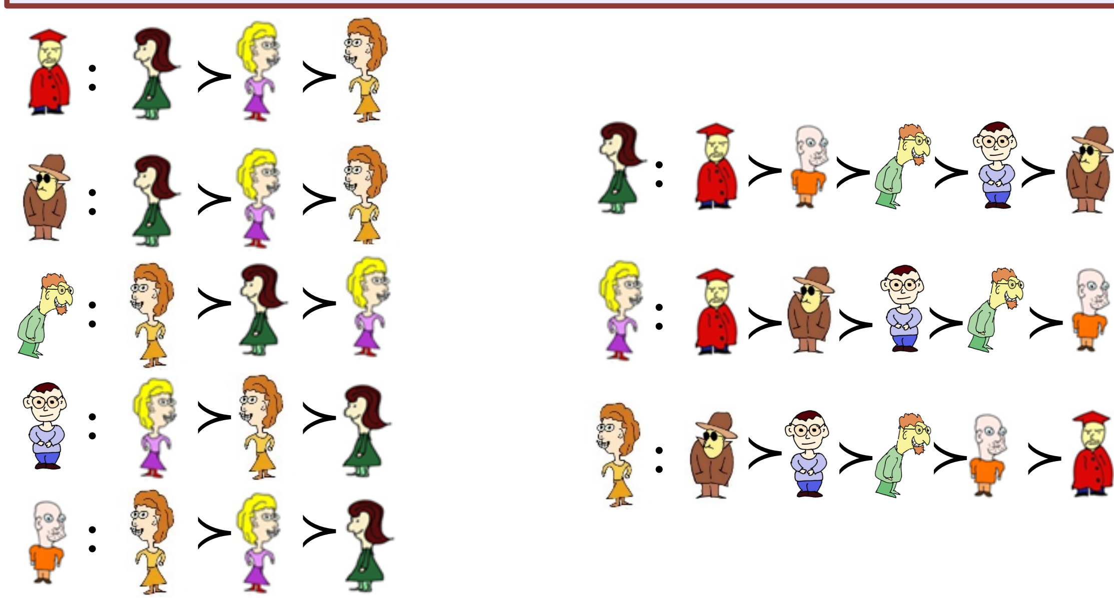

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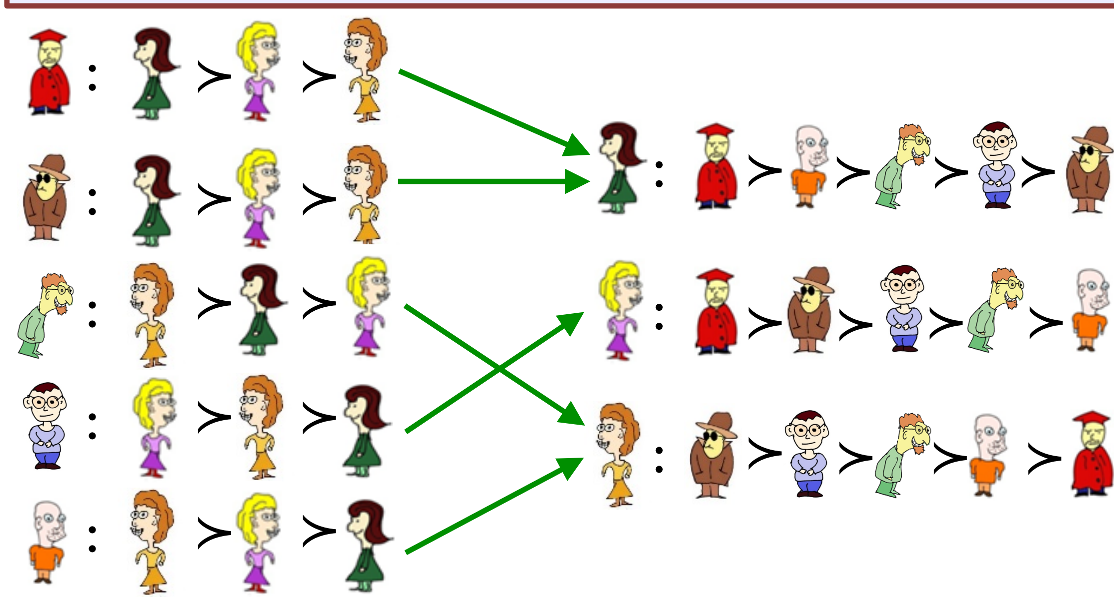

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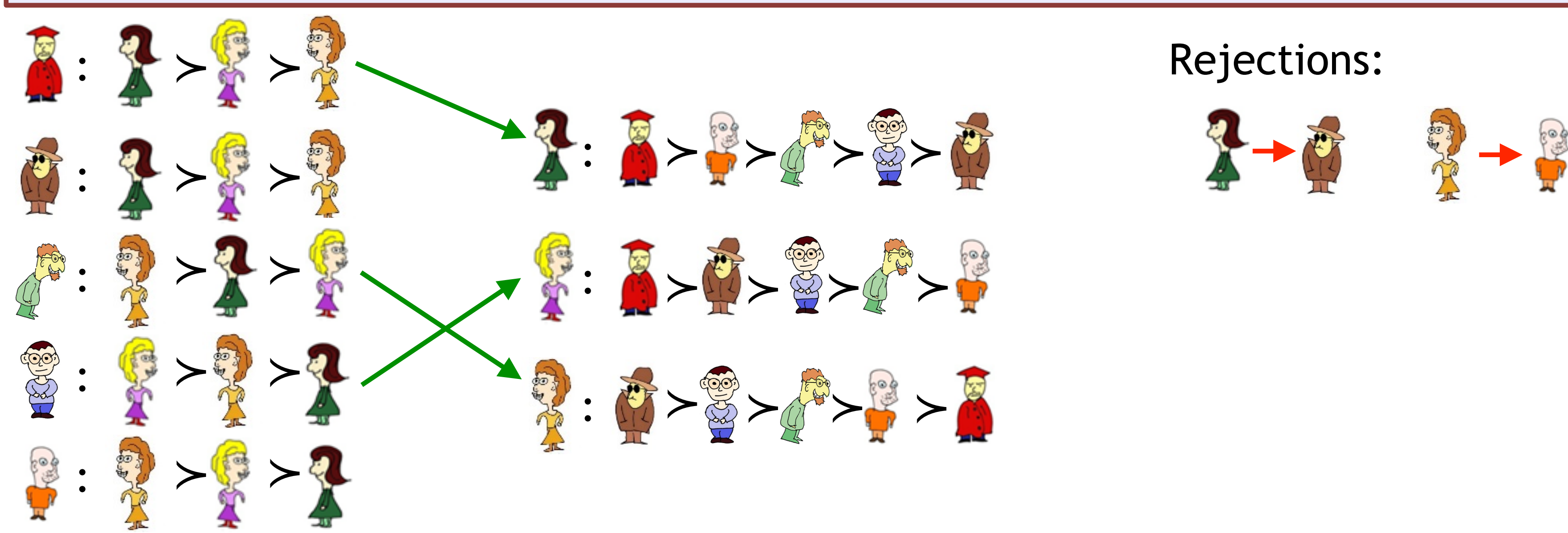

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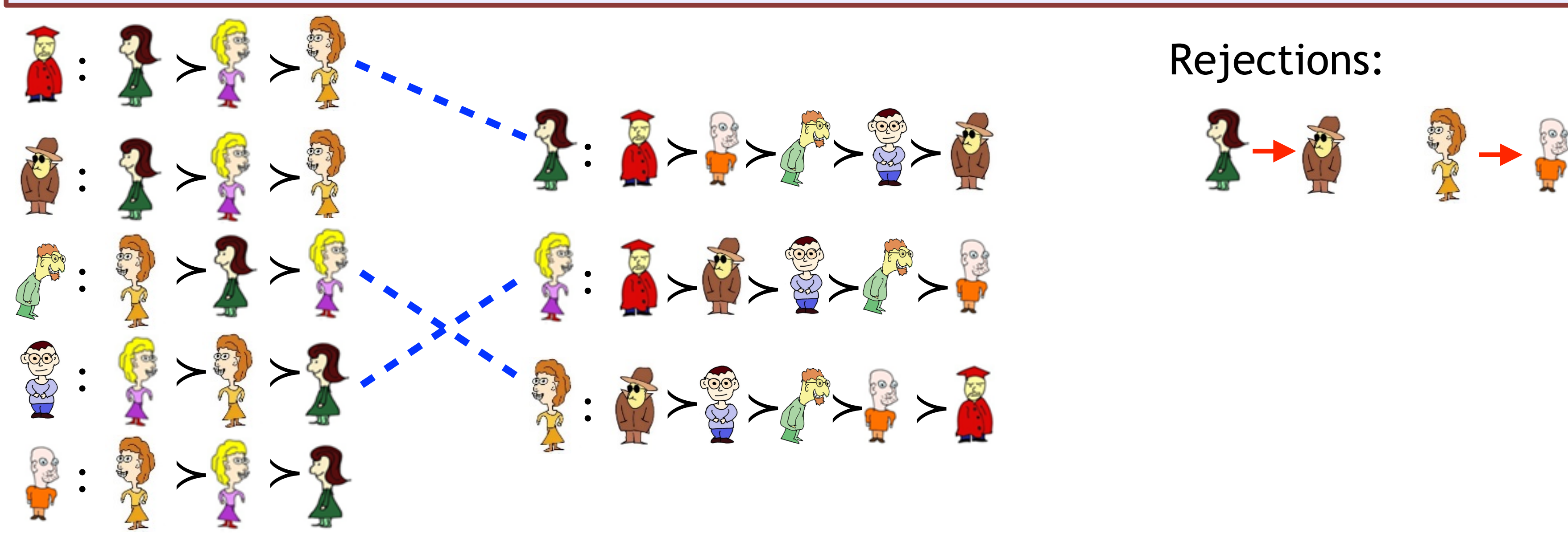

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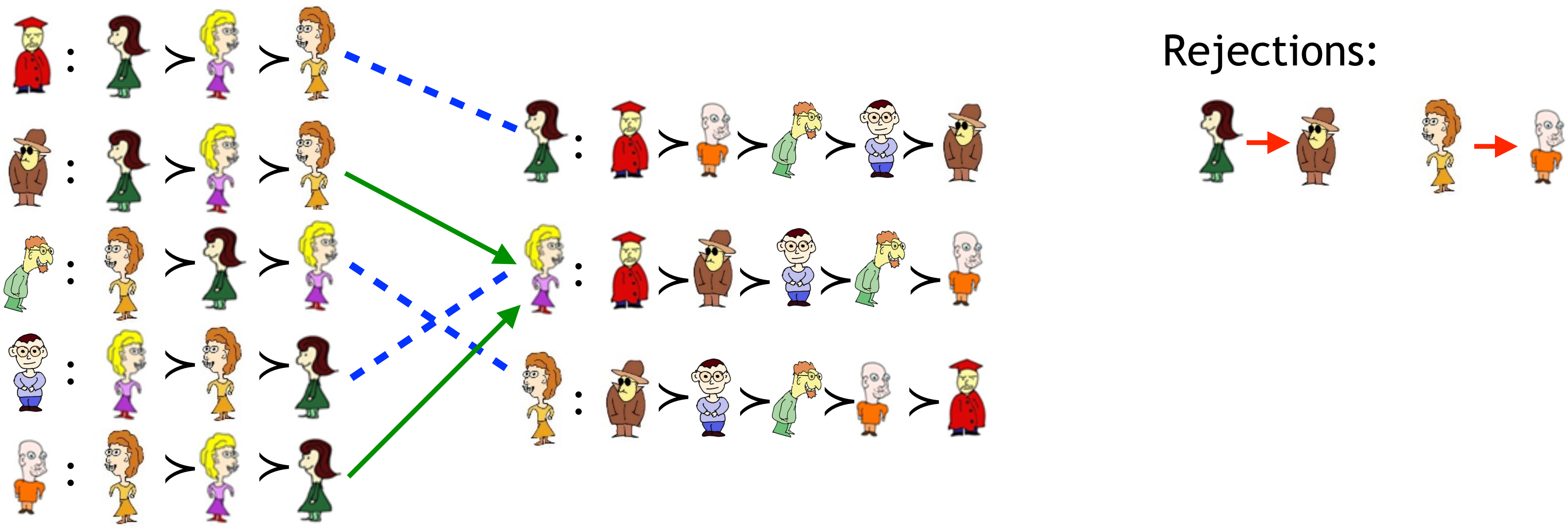

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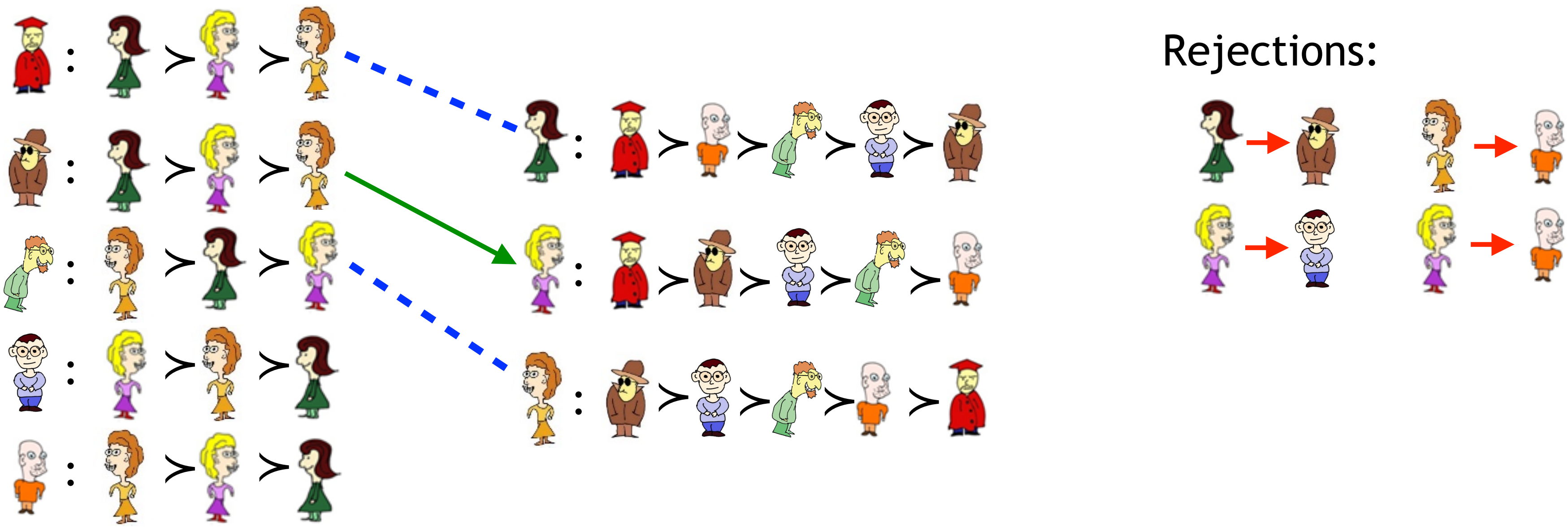

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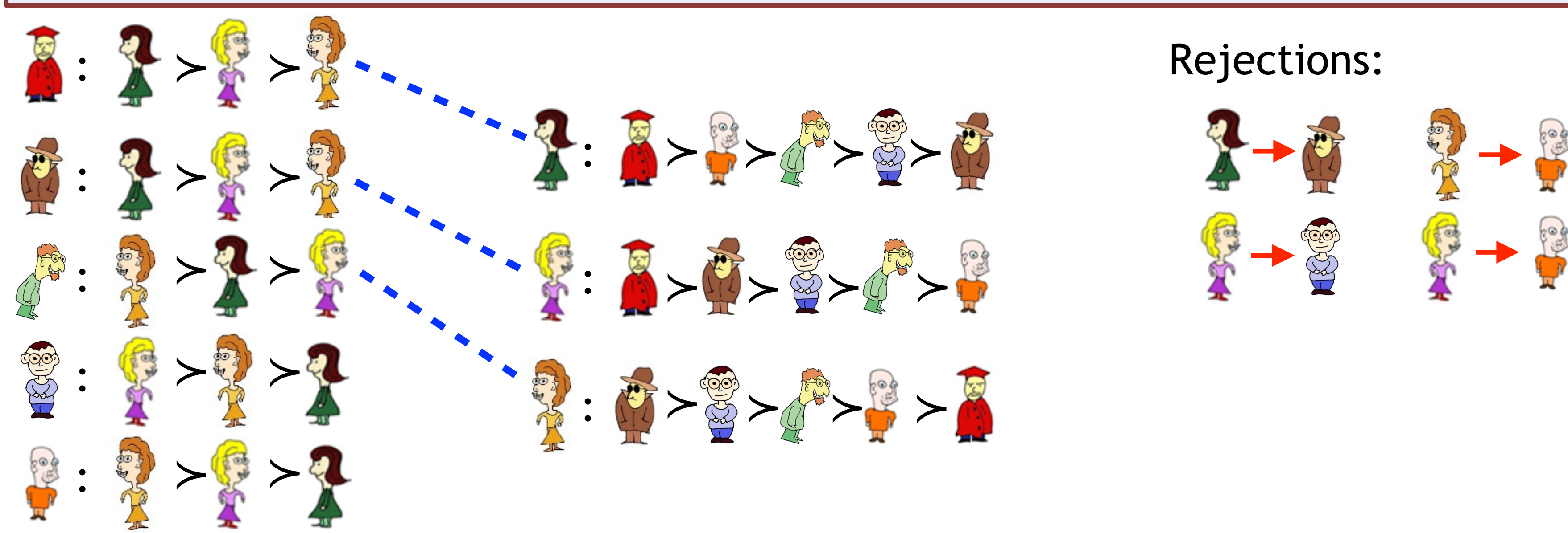

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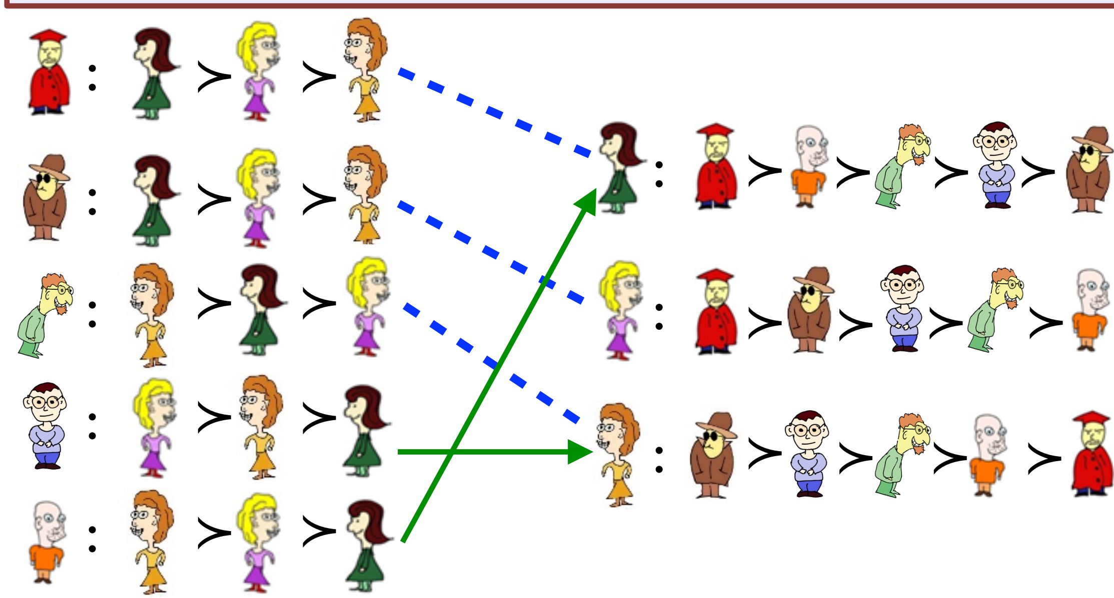

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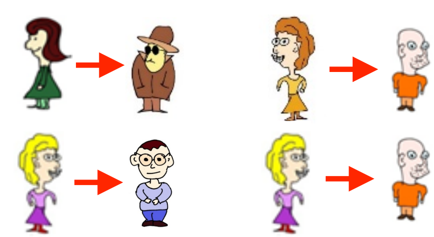

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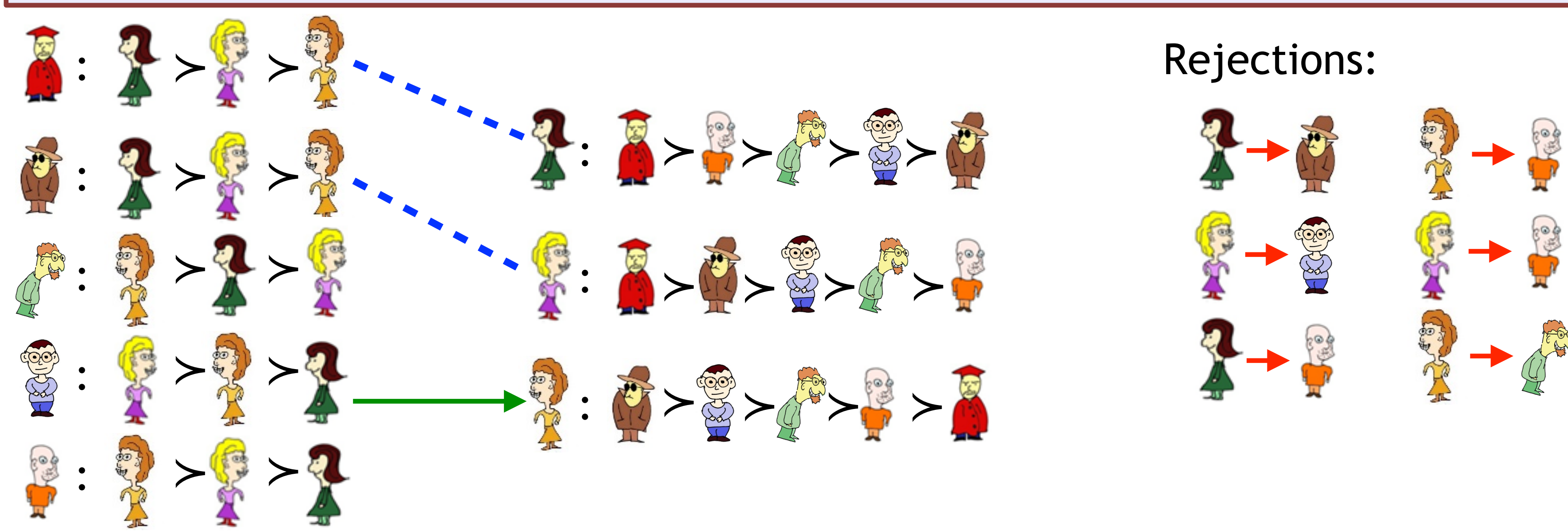

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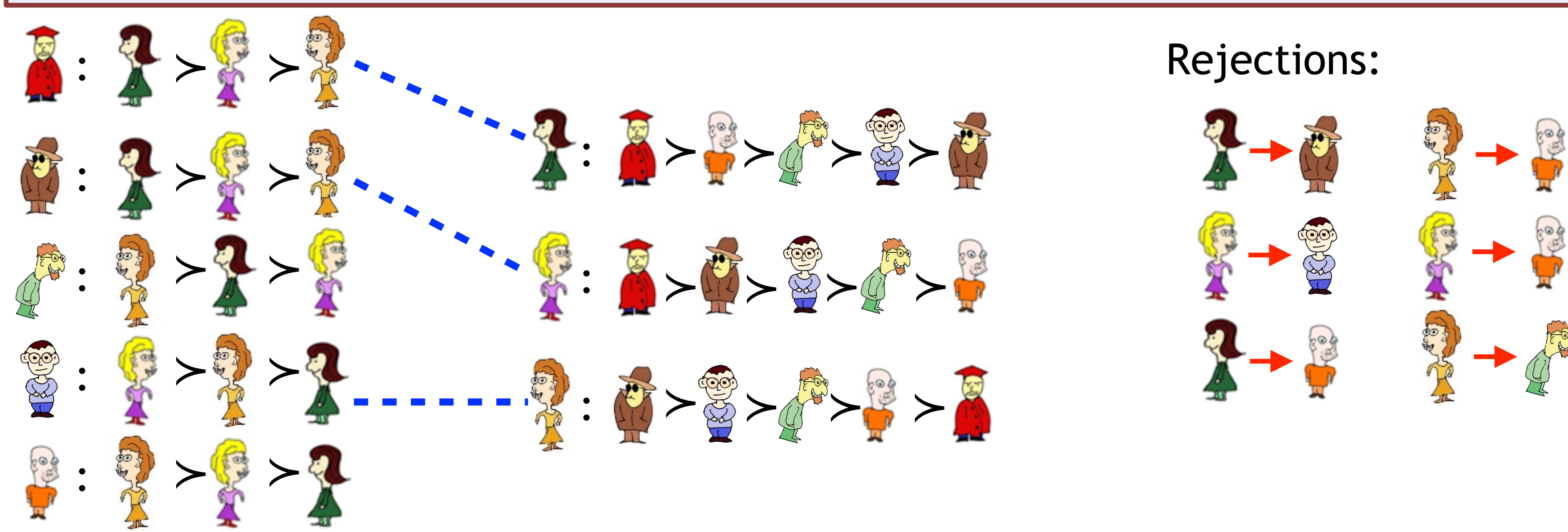

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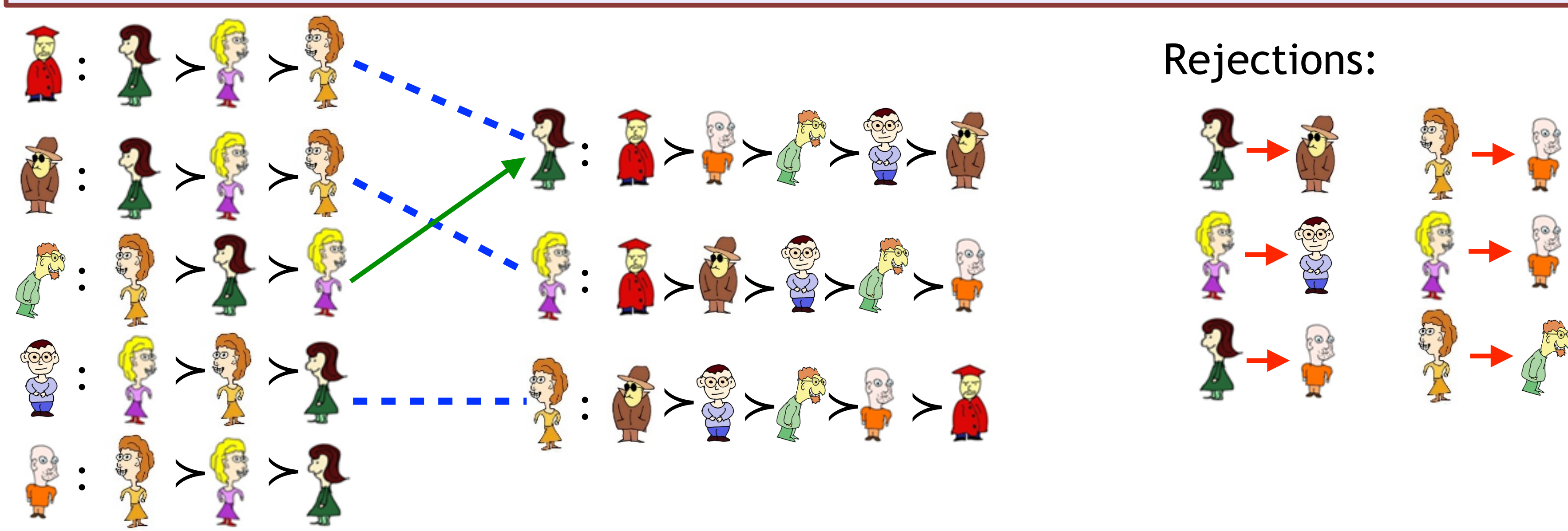

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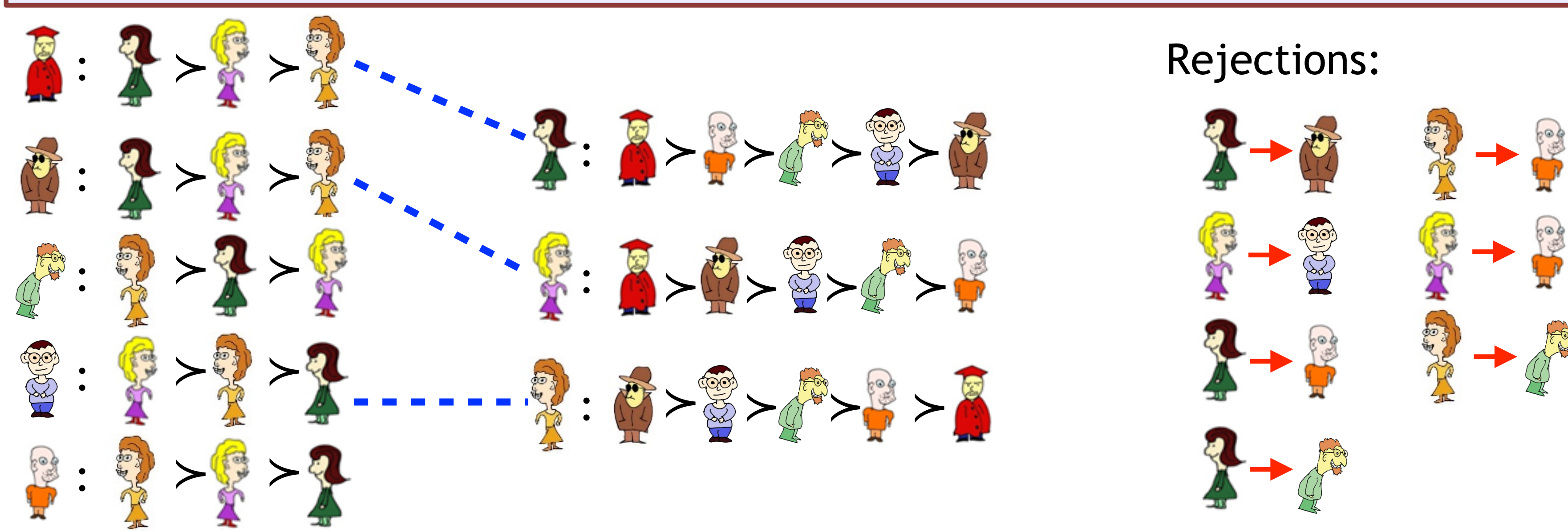

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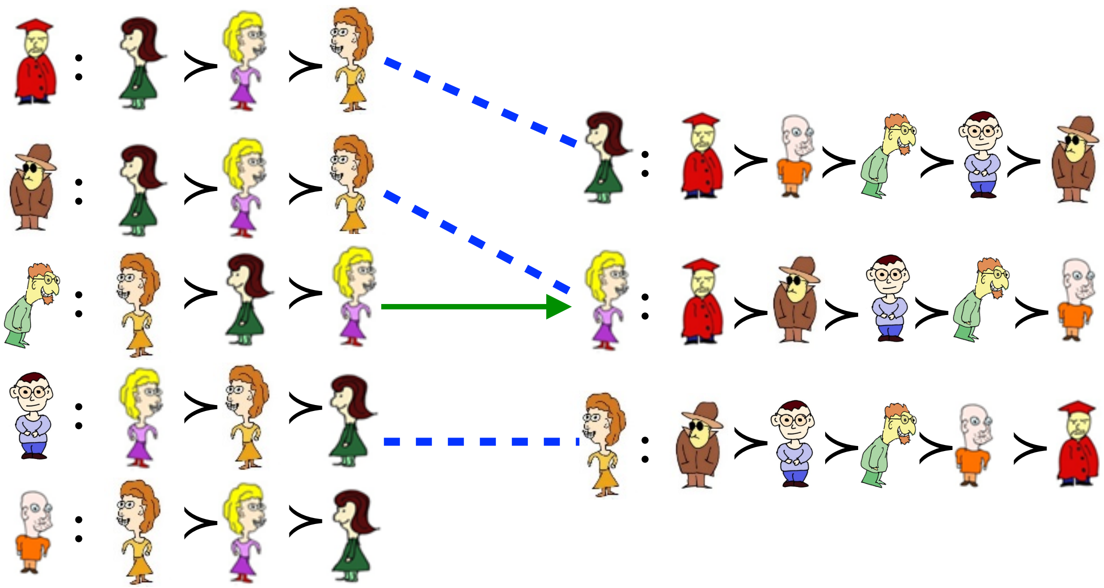

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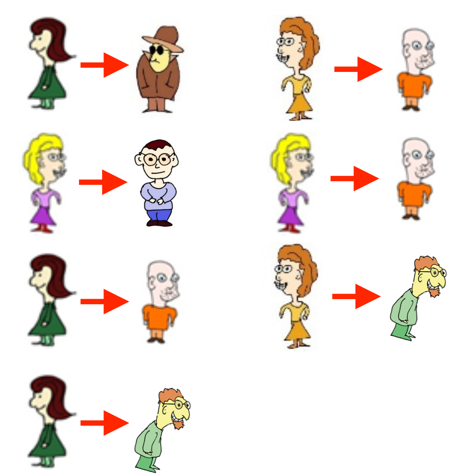

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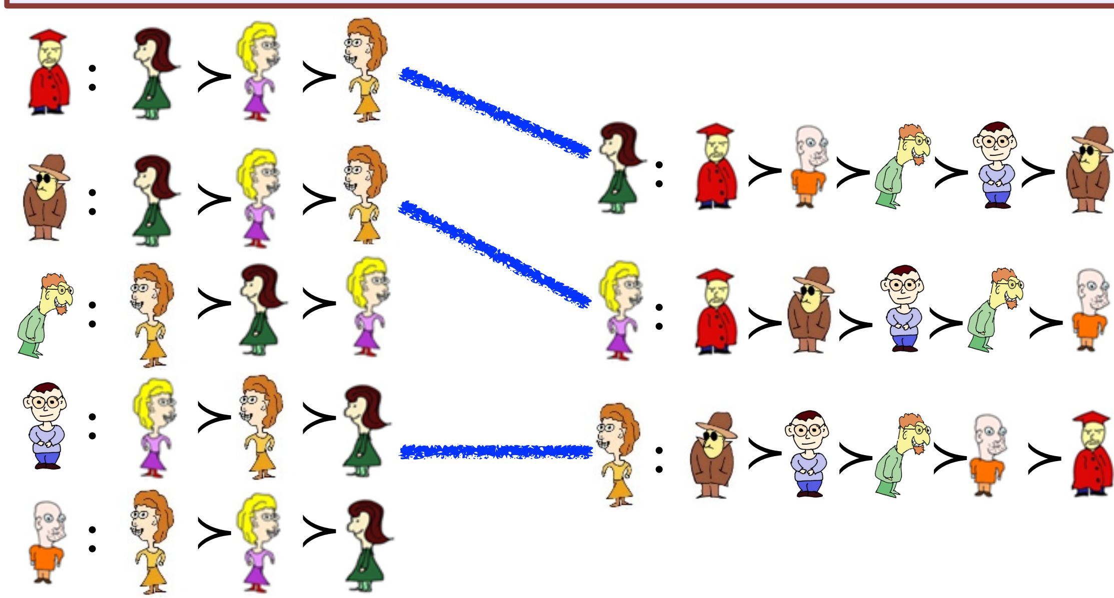

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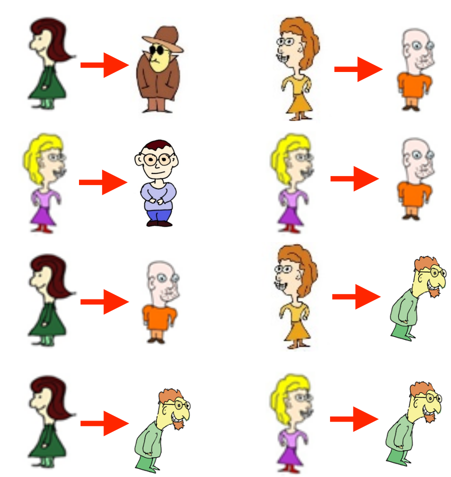

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$w$ breaks an engagement only when getting a better partner.

# Men-optimal and Women-optimal Stable Matchings

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Repeating the reasoning, we find sequences $u_1, u_2, \ldots, u_p$ and $w_1, w_2, \ldots, w_p$.

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$u_1$ $w_1$

$u_2$ $w_2$

$\vdots$ $\vdots$

$u_p$ $w_p$

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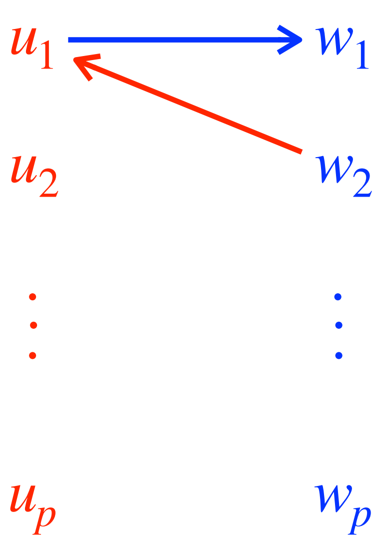


$u_1$ proposed to $w_1$

$w_2$ rejected $u_1$

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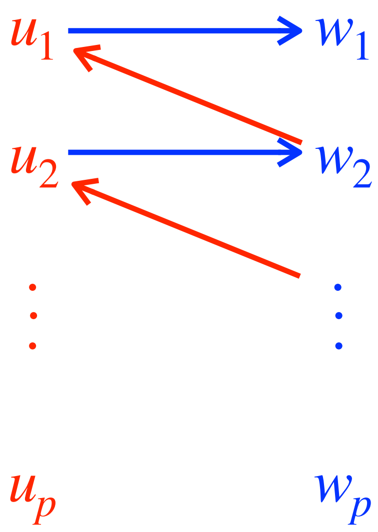


$u_1$ proposed to $w_1$ $w_2$ rejected $u_1$

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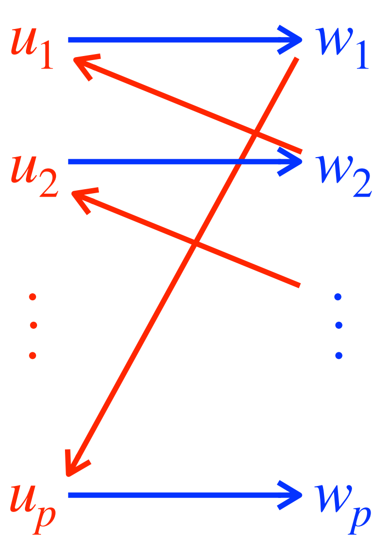


| | |
|---|---|
| $u_1$ proposed to $w_1$ | $w_2$ rejected $u_1$ |
| $u_2$ proposed to $w_2$ | $w_3$ rejected $u_2$ |
| … | … |
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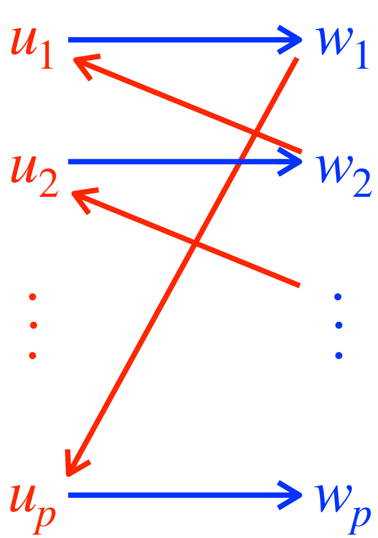


$u_1$ proposed to $w_1$ | $w_2$ rejected $u_1$

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… | …

$u_p$ proposed to $w_p$ | $w_1$ rejected $u_p$

$u_1$ proposed to $w_1$ after he was rejected by $w_2$

$w_2$ rejected $u_1$ after $u_2$ proposed to her.

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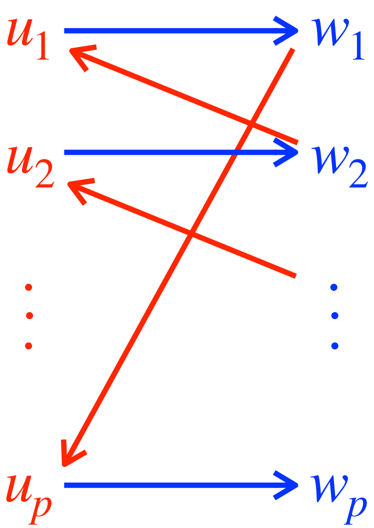


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… — …

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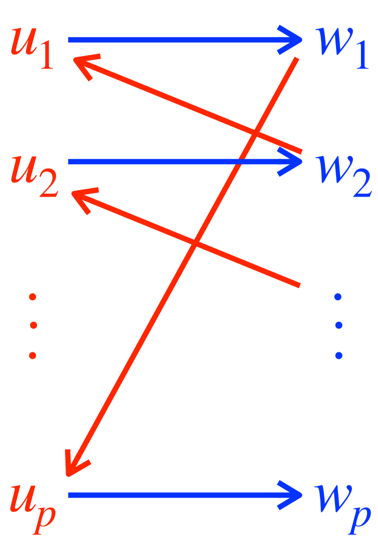


| | |
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| ... | ... |
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similarly, $u_2$ proposed to $w_2$ after $u_3$ proposed to $w_3$

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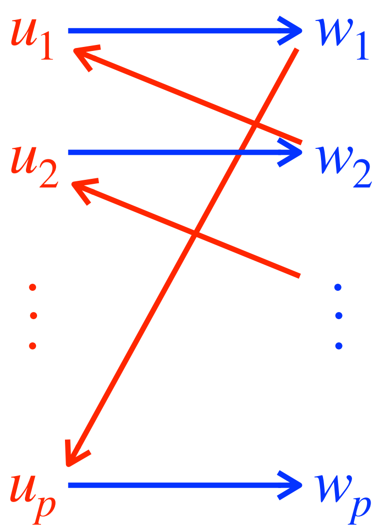


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similarly, $u_2$ proposed to $w_2$ after $u_3$ proposed to $w_3$

And so on, until we get a contradiction.

# Men-optimal and Women-optimal Stable Matchings

**Theorem**: In any stable matching no man can get a better partner than the one that he gets in the matching returned by the Gale-Shapley algorithm.

**How to get a woman optimal stable matching?**

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**How to get a woman optimal stable matching?**

**Women shall propose instead of men!**

# Applications of  Stable Matchings

1. Matching students to schools.

2. Matching residents to hospitals.

3. Assigning users to servers in a large distributed Internet service.

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| **1** | 4 | 6 | 2 | 5 | 3 |
| **2** | 6 | 3 | 5 | 1 | 4 |
| **3** | 4 | 5 | 1 | 6 | 2 |
| **4** | 2 | 6 | 5 | 1 | 3 |
| **5** | 4 | 2 | 3 | 6 | 1 |
| **6** | 5 | 1 | 4 | 2 | 3 |

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| **1** | 4 | 6 | 2 | 5 | 3 |
| **2** | 6 | 3 | 5 | 1 | 4 |
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| **4** | 2 | 6 | 5 | 1 | 3 |
| **5** | 4 | 2 | 3 | 6 | 1 |
| **6** | 5 | 1 | 4 | 2 | 3 |

**First phase:** proposals like in a Gale-Shapley algorithm. However, every agent holds keeps her own proposal and the proposal she receives.

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| **1** | 4 | 6 | 2 | 5 | 3 |
| **2** | 6 | 3 | 5 | 1 | 4 |
| **3** | 4 | 5 | 1 | 6 | 2 |
| **4** | 2 | 6 | 5 | 1 | 3 |
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1 proposes to 4

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| **1** | 4 | 6 | 2 | 5 | 3 |
| **2** | 6 | 3 | 5 | 1 | 4 |
| **3** | 4 | 5 | 1 | 6 | 2 |
| **4** | 2 | 6 | 5 | 1 | 3 |
| **5** | 4 | 2 | 3 | 6 | 1 |
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| **2** | 6 | 3 | 5 | 1 | 4 |
| **3** | 4 | 5 | 1 | 6 | 2 |
| **4** | 2 | 6 | 5 | 1 | 3 |
| **5** | 4 | 2 | 3 | 6 | 1 |
| **6** | 5 | 1 | 4 | 2 | 3 |

**First phase:** proposals like in a Gale-Shapley algorithm. However, every agent holds keeps her own proposal and the proposal she receives.

1 proposes to 4
6 proposes to 2

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| **1** | 4 | 6 | 2 | 5 | 3 |
| **2** | 6 | 3 | 5 | 1 | 4 |
| **3** | 4 | 5 | 1 | 6 | 2 |
| **4** | 2 | 6 | 5 | 1 | 3 |
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| **3** | 4 | 5 | 1 | 6 | 2 |
| **4** | 2 | 6 | 5 | 1 | 3 |
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| **6** | 5 | 1 | 4 | 2 | 3 |

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1 proposes to 4
6 proposes to 2
3 proposes to 4

# Stable Roommate: non-bipartite graphs

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|---|---|---|---|---|---|
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| **2** | 6 | 3 | 5 | 1 | 4 |
| **3** | 4 | 5 | 1 | 6 | 2 |
| **4** | 2 | 6 | 5 | 1 | 3 |
| **5** | 4 | 2 | 3 | 6 | 1 |
| **6** | 5 | 1 | 4 | 2 | 3 |

**First phase:** proposals like in a Gale-Shapley algorithm. However, every agent holds keeps her own proposal and the proposal she receives.

1 proposes to 4
6 proposes to 2
3 proposes to 4 (but gets rejected)

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| **1** | 4 | 6 | 2 | 5 | 3 |
| **2** | 6 | 3 | 5 | 1 | 4 |
| **3** | 4 | 5 | 1 | 6 | 2 |
| **4** | 2 | 6 | 5 | 1 | 3 |
| **5** | 4 | 2 | 3 | 6 | 1 |
| **6** | 5 | 1 | 4 | 2 | 3 |

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1 proposes to 4
6 proposes to 2
3 proposes to 4 (but gets rejected)
3 proposes to 5

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| **1** | 4 | 6 | 2 | 5 | 3 |
| **2** | 6 | 3 | 5 | 1 | 4 |
| **3** | 4 | 5 | 1 | 6 | 2 |
| **4** | 2 | 6 | 5 | 1 | 3 |
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6 proposes to 2
3 proposes to 4 (but gets rejected)
3 proposes to 5

# Stable Roommate: non-bipartite graphs

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**First phase:** proposals like in a Gale-Shapley algorithm. However, every agent holds keeps her own proposal and the proposal she receives.

1 proposes to 4
6 proposes to 2
3 proposes to 4 (but gets rejected)
3 proposes to 5
2 proposes to 4

# Stable Roommate: non-bipartite graphs

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|---|---|---|---|---|---|
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| **2** | 6 | 3 | 5 | 1 | 4 |
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1 proposes to 4
6 proposes to 2
3 proposes to 4 (but gets rejected)
3 proposes to 5
2 proposes to 4

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|---|---|---|---|---|---|
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6 proposes to 2
3 proposes to 4 (but gets rejected)
3 proposes to 5
2 proposes to 4

5 proposes to 4

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| **1** | 4 | 6 | 2 | 5 | 3 |
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1 proposes to 4
6 proposes to 2
3 proposes to 4 (but gets rejected)
3 proposes to 5
2 proposes to 4

5 proposes to 4 (4 rejects 1)

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| **1** | 4 | 6 | 2 | 5 | 3 |
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| **4** | 2 | 6 | 5 | 1 | 3 |
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| **6** | 5 | 1 | 4 | 2 | 3 |

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1 proposes to 4
6 proposes to 2
3 proposes to 4 (but gets rejected)
3 proposes to 5
2 proposes to 4

5 proposes to 4 (4 rejects 1)

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| **1** | 4 | 6 | 2 | 5 | 3 |
| **2** | 6 | 3 | 5 | 1 | 4 |
| **3** | 4 | 5 | 1 | 6 | 2 |
| **4** | 2 | 6 | 5 | 1 | 3 |
| **5** | 4 | 2 | 3 | 6 | 1 |
| **6** | 5 | 1 | 4 | 2 | 3 |

**First phase:** proposals like in a Gale-Shapley algorithm. However, every agent holds keeps her own proposal and the proposal she receives.

1 proposes to 4
6 proposes to 2
3 proposes to 4 (but gets rejected)
3 proposes to 5
2 proposes to 4

5 proposes to 4 (4 rejects 1)
1 proposes to 6

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| **1** | 4 | 6 | 2 | 5 | 3 |
| **2** | 6 | 3 | 5 | 1 | 4 |
| **3** | 4 | 5 | 1 | 6 | 2 |
| **4** | 2 | 6 | 5 | 1 | 3 |
| **5** | 4 | 2 | 3 | 6 | 1 |
| **6** | 5 | 1 | 4 | 2 | 3 |

**First phase:** proposals like in a Gale-Shapley algorithm. However, every agent holds keeps her own proposal and the proposal she receives.

1 proposes to 4
6 proposes to 2
3 proposes to 4 (but gets rejected)
3 proposes to 5
2 proposes to 4

5 proposes to 4 (4 rejects 1)
1 proposes to 6 (6 rejects 2)

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| **1** | 4 | 6 | 2 | 5 | 3 |
| **2** | 6 | 3 | 5 | 1 | 4 |
| **3** | 4 | 5 | 1 | 6 | 2 |
| **4** | 2 | 6 | 5 | 1 | 3 |
| **5** | 4 | 2 | 3 | 6 | 1 |
| **6** | 5 | 1 | 4 | 2 | 3 |

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1 proposes to 4
6 proposes to 2
3 proposes to 4 (but gets rejected)
3 proposes to 5
2 proposes to 4

5 proposes to 4 (4 rejects 1)
1 proposes to 6 (6 rejects 2)

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| **1** | 4 | 6 | 2 | 5 | 3 |
| **2** | 6 | 3 | 5 | 1 | 4 |
| **3** | 4 | 5 | 1 | 6 | 2 |
| **4** | 2 | 6 | 5 | 1 | 3 |
| **5** | 4 | 2 | 3 | 6 | 1 |
| **6** | 5 | 1 | 4 | 2 | 3 |

**First phase:** proposals like in a Gale-Shapley algorithm. However, every agent holds keeps her own proposal and the proposal she receives.

1 proposes to 4
6 proposes to 2
3 proposes to 4 (but gets rejected)
3 proposes to 5
2 proposes to 4

5 proposes to 4 (4 rejects 1)
1 proposes to 6 (6 rejects 2)
2 proposes to 3

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| **1** | 4 | 6 | 2 | 5 | 3 |
| **2** | 6 | 3 | 5 | 1 | 4 |
| **3** | 4 | 5 | 1 | 6 | 2 |
| **4** | 2 | 6 | 5 | 1 | 3 |
| **5** | 4 | 2 | 3 | 6 | 1 |
| **6** | 5 | 1 | 4 | 2 | 3 |

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1 proposes to 4
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3 proposes to 4 (but gets rejected)
3 proposes to 5
2 proposes to 4

5 proposes to 4 (4 rejects 1)
1 proposes to 6 (6 rejects 2)
2 proposes to 3

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| **1** | 4 | 6 | 2 | 5 | 3 |
| **2** | 6 | 3 | 5 | 1 | 4 |
| **3** | 4 | 5 | 1 | 6 | 2 |
| **4** | 2 | 6 | 5 | 1 | 3 |
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| **6** | 5 | 1 | 4 | 2 | 3 |

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1 proposes to 4
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3 proposes to 4 (but gets rejected)
3 proposes to 5
2 proposes to 4

5 proposes to 4 (4 rejects 1)
1 proposes to 6 (6 rejects 2)
2 proposes to 3
6 proposes to 5

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| **1** | 4 | 6 | 2 | 5 | 3 |
| **2** | 6 | 3 | 5 | 1 | 4 |
| **3** | 4 | 5 | 1 | 6 | 2 |
| **4** | 2 | 6 | 5 | 1 | 3 |
| **5** | 4 | 2 | 3 | 6 | 1 |
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1 proposes to 4
6 proposes to 2
3 proposes to 4 (but gets rejected)
3 proposes to 5
2 proposes to 4

5 proposes to 4 (4 rejects 1)
1 proposes to 6 (6 rejects 2)
2 proposes to 3
6 proposes to 5 (but gets rejected)

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| **1** | 4 | 6 | 2 | 5 | 3 |
| **2** | 6 | 3 | 5 | 1 | 4 |
| **3** | 4 | 5 | 1 | 6 | 2 |
| **4** | 2 | 6 | 5 | 1 | 3 |
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1 proposes to 4
6 proposes to 2
3 proposes to 4 (but gets rejected)
3 proposes to 5
2 proposes to 4
5 proposes to 4 (4 rejects 1)
1 proposes to 6 (6 rejects 2)
2 proposes to 3
6 proposes to 5 (but gets rejected)
6 proposes to 1

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| **1** | 4 | 6 | 2 | 5 | 3 |
| **2** | 6 | 3 | 5 | 1 | 4 |
| **3** | 4 | 5 | 1 | 6 | 2 |
| **4** | 2 | 6 | 5 | 1 | 3 |
| **5** | 4 | 2 | 3 | 6 | 1 |
| **6** | 5 | 1 | 4 | 2 | 3 |

**First phase:** proposals like in a Gale-Shapley algorithm. However, every agent holds keeps her own proposal and the proposal she receives.

1 proposes to 4
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3 proposes to 4 (but gets rejected)
3 proposes to 5
2 proposes to 4

5 proposes to 4 (4 rejects 1)
1 proposes to 6 (6 rejects 2)
2 proposes to 3
6 proposes to 5 (but gets rejected)
6 proposes to 1

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| 1 | 4 | 6 | 2 | 5 | 3 |
| 2 | 6 | 3 | 5 | 1 | 4 |
| 3 | 4 | 5 | 1 | 6 | 2 |
| 4 | 2 | 6 | 5 | 1 | 3 |
| 5 | 4 | 2 | 3 | 6 | 1 |
| 6 | 5 | 1 | 4 | 2 | 3 |

**First phase:** proposals like in a Gale-Shapley algorithm. However, every agent holds keeps her own proposal and the proposal she receives.

After this phase one person can be rejected by everyone.

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| **1** | 4 | 6 | 2 | 5 | 3 |
| **2** | 6 | 3 | 5 | 1 | 4 |
| **3** | 4 | 5 | 1 | 6 | 2 |
| **4** | 2 | 6 | 5 | 1 | 3 |
| **5** | 4 | 2 | 3 | 6 | 1 |
| **6** | 5 | 1 | 4 | 2 | 3 |

**First phase:** proposals like in a Gale-Shapley algorithm. However, every agent holds keeps her own proposal and the proposal she receives.

After this phase one person can be rejected by everyone.

(But then every other person holds a proposal. Why?)

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| 1 | 4 | 6 | 2 | 5 | 3 |
| 2 | 6 | 3 | 5 | 1 | 4 |
| 3 | 4 | 5 | 1 | 6 | 2 |
| 4 | 2 | 6 | 5 | 1 | 3 |
| 5 | 4 | 2 | 3 | 6 | 1 |
| 6 | 5 | 1 | 4 | 2 | 3 |

**First phase:** proposals like in a Gale-Shapley algorithm. However, every agent holds keeps her own proposal and the proposal she receives.

After this phase one person can be rejected by everyone.

(But then every other person holds a proposal. Why?)

**Because every person rejected her, and can only improve in the course Of the algorithm.**

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| **1** | 4 | 6 | 2 | 5 | 3 |
| **2** | 6 | 3 | 5 | 1 | 4 |
| **3** | 4 | 5 | 1 | 6 | 2 |
| **4** | 2 | 6 | 5 | 1 | 3 |
| **5** | 4 | 2 | 3 | 6 | 1 |
| **6** | 5 | 1 | 4 | 2 | 3 |

**Lemma:** In a stable matching no person can be matched to anyone who she rejected.

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| **1** | 4 | 6 | 2 | 5 | 3 |
| **2** | 6 | 3 | 5 | 1 | 4 |
| **3** | 4 | 5 | 1 | 6 | 2 |
| **4** | 2 | 6 | 5 | 1 | 3 |
| **5** | 4 | 2 | 3 | 6 | 1 |
| **6** | 5 | 1 | 4 | 2 | 3 |

**Lemma:** In a stable matching no person can be matched to anyone who she rejected.

For the sake of contradiction let $(x, y)$ be the first pair rejected that belongs to some stable matching $M$; $y$ rejected $x$.

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| **1** | 4 | 6 | 2 | 5 | 3 |
| **2** | 6 | 3 | 5 | 1 | 4 |
| **3** | 4 | 5 | 1 | 6 | 2 |
| **4** | 2 | 6 | 5 | 1 | 3 |
| **5** | 4 | 2 | 3 | 6 | 1 |
| **6** | 5 | 1 | 4 | 2 | 3 |

**Lemma:** In a stable matching no person can be matched to anyone who she rejected.

For the sake of contradiction let $(x, y)$ be the first pair rejected that belongs to some stable matching $M$; $y$ must have received a better proposal, from $z$.

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| **1** | 4 | 6 | 2 | 5 | 3 |
| **2** | 6 | 3 | 5 | 1 | 4 |
| **3** | 4 | 5 | 1 | 6 | 2 |
| **4** | 2 | 6 | 5 | 1 | 3 |
| **5** | 4 | 2 | 3 | 6 | 1 |
| **6** | 5 | 1 | 4 | 2 | 3 |

**Lemma:** In a stable matching no person can be matched to anyone who she rejected.

For the sake of contradiction let $(x, y)$ be the first pair rejected that belongs to some stable matching $M$; $y$ must have received a better proposal, from $z$. In $M$, $z$ must prefer her partner $w$ to $y$.

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| **1** | 4 | 6 | 2 | 5 | 3 |
| **2** | 6 | 3 | 5 | 1 | 4 |
| **3** | 4 | 5 | 1 | 6 | 2 |
| **4** | 2 | 6 | 5 | 1 | 3 |
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For the sake of contradiction let $(x, y)$ be the first pair rejected that belongs to some stable matching $M$; $y$ must have received a better proposal, from $z$. In $M$, $z$ must prefer her partner $w$ to $y$. Before $z$ proposed to $y$ she proposed to $w$.

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| 1 | 4 | 6 | 2 | 5 | 3 |
| 2 | 6 | 3 | 5 | 1 | 4 |
| 3 | 4 | 5 | 1 | 6 | 2 |
| 4 | 2 | 6 | 5 | 1 | 3 |
| 5 | 4 | 2 | 3 | 6 | 1 |
| 6 | 5 | 1 | 4 | 2 | 3 |

**Lemma:** In a stable matching no person can be matched to anyone who she rejected.

For the sake of contradiction let $(x, y)$ be the first pair rejected that belongs to some stable matching $M$; $y$ must have received a better proposal, from $z$. In $M$, $z$ must prefer her partner $w$ to $y$. Before $z$ proposed to $y$ she proposed to $w$. Rejection of $z$ by $w$ preceded rejection of $x$ by $y$.

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| **1** | 4 | 6 | 2 | 5 | 3 |
| **2** | 6 | 3 | 5 | 1 | 4 |
| **3** | 4 | 5 | 1 | 6 | 2 |
| **4** | 2 | 6 | 5 | 1 | 3 |
| **5** | 4 | 2 | 3 | 6 | 1 |
| **6** | 5 | 1 | 4 | 2 | 3 |

**Lemma:** In a stable matching no person can be matched to anyone who she rejected.

**Lemma:** If at any stage $x$ proposed to $y$ then:

1. $x$ cannot have a better partner than $y$, and
2. $y$ cannot have a worse partner than $x$.

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| **1** | 4 | 6 | 2 | 5 | 3 |
| **2** | 6 | 3 | 5 | 1 | 4 |
| **3** | 4 | 5 | 1 | 6 | 2 |
| **4** | 2 | 6 | 5 | 1 | 3 |
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| **6** | 5 | 1 | 4 | 2 | 3 |

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Indeed, $x$ was rejected by anyone who she prefers to $y$.

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| **1** | 4 | 6 | 2 | 5 | 3 |
| **2** | 6 | 3 | 5 | 1 | 4 |
| **3** | 4 | 5 | 1 | 6 | 2 |
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| **6** | 5 | 1 | 4 | 2 | 3 |

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**Lemma:** If at any stage $x$ proposed to $y$ then:

1. $x$ cannot have a better partner than $y$, and
2. $y$ cannot have a worse partner than $x$.

Indeed, $x$ was rejected by anyone who she prefers to $y$.
If $y$ would have a worse partner than $x$, say $z$, then $(x, y)$ would be blocking.

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| **1** | 4 | 6 | 2 | 5 | 3 |
| **2** | 6 | 3 | 5 | 1 | 4 |
| **3** | 4 | 5 | 1 | 6 | 2 |
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# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| **1** | | 6 | | | |
| **2** | | 3 | 5 | 1 | 4 |
| **3** | | 5 | 1 | 6 | 2 |
| **4** | 2 | 6 | 5 | | |
| **5** | 4 | 2 | 3 | | |
| **6** | | 1 | | | |

**Lemma:** In a stable matching no person can be matched to anyone who she rejected.

**Lemma:** If at any stage $x$ proposed to $y$ then:

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2. $y$ cannot have a worse partner than $x$.

**We can truncate preference lists!**

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| **1** | | 6 | | | |
| **2** | | 3 | 5 | 1 | 4 |
| **3** | | 5 | 1 | 6 | 2 |
| **4** | 2 | 6 | 5 | | |
| **5** | 4 | 2 | 3 | | |
| **6** | | 1 | | | |

**Lemma:** In a stable matching no person can be matched to anyone who she rejected.

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# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| 1 | | 6 | | | |
| 2 | | 3 | 5 | | 4 |
| 3 | | 5 | | | 2 |
| 4 | 2 | | 5 | | |
| 5 | 4 | 2 | 3 | | |
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**Lemma:** In a stable matching no person can be matched to anyone who she rejected.

**Lemma:** If at any stage $x$ proposed to $y$ then:

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# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| 1 | | 6 | | | |
| 2 | | 3 | 5 | | 4 |
| 3 | | 5 | | | 2 |
| 4 | 2 | | 5 | | |
| 5 | 4 | 2 | 3 | | |
| 6 | | 1 | | | |

**All-or-nothing cycle:** a sequence $(a_1, a_2, \ldots, a_r)$ such that:

- the second person in $a_i$'s list is the first person in $a_{i+1}$'s list,
- the second person in $a_r$'s list is the first person in $a_1$'s list.

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| 1 | | 6 | | | |
| 2 | | 3 | 5 | | 4 |
| 3 | | 5 | | | 2 |
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**How to find it?**

$q_i$: the second person in someone's (namely $a_i$'s) list.

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| 1 | | 6 | | | |
| 2 | | 3 | 5 | | 4 |
| 3 | | 5 | | | 2 |
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# Stable Roommate: non-bipartite graphs

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| 1 | | 6 | | | |
| 2 | | 3 | 5 | | 4 |
| 3 | | 5 | | | 2 |
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**How to find it?**

$q_i$: the second person in someone's (namely $a_i$'s) list.

$a_{i+1}$: the last person in $q_i$'s list.

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| 1 | | 6 | | | |
| 2 | | 3 | 5 | | 4 |
| 3 | | 5 | | | 2 |
| 4 | 2 | | 5 | | |
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# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| 1 | | 6 | | | |
| 2 | | 3 | 5 | | 4 |
| 3 | | 5 | | | 2 |
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# Stable Roommate: non-bipartite graphs

| | | | | | |
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| 1 | | 6 | | | |
| 2 | | 3 | 5 | | 4 |
| 3 | | 5 | | | 2 |
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| | | | | | |
|---|---|---|---|---|---|
| 1 | | 6 | | | |
| 2 | | 3 | 5 | | 4 |
| 3 | | 5 | | | 2 |
| 4 | 2 | | 5 | | |
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| 1 | | 6 | | | |
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| 3 | | 5 | | | 2 |
| 4 | 2 | | 5 | | |
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# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| 1 | | 6 | | | |
| 2 | | 3 | 5 | | 4 |
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# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| 1 | | 6 | | | |
| 2 | | 3 | 5 | | 4 |
| 3 | | 5 | | | 2 |
| 4 | 2 | | 5 | | |
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| 6 | | 1 | | | |

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# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| 1 | | 6 | | | |
| 2 | | 3 | 5 | | 4 |
| 3 | | 5 | | | 2 |
| 4 | 2 | | 5 | | |
| 5 | 4 | 2 | 3 | | |
| 6 | | 1 | | | |

**All-or-nothing cycle:** a sequence $(a_1, a_2, \ldots, a_r)$ such that:

- the second person in $a_i$'s list is the first person in $a_{i+1}$'s list,
- the second person in $a_r$'s list is the first person in $a_1$'s list.

**How to find it?**

$q_i$: the second person in someone's (namely $a_i$'s) list.

$a_{i+1}$: the last person in $q_i$'s list. (so $q_i$ is first in $a_{i+1}$'s list)

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| 1 | | 6 | | | |
| 2 | | 3 | 5 | | 4 |
| 3 | | 5 | | | 2 |
| 4 | 2 | | 5 | | |
| 5 | 4 | 2 | 3 | | |
| 6 | | 1 | | | |

**All-or-nothing cycle:** a sequence $(a_1, a_2, \ldots, a_r)$ such that:

- the second person in $a_i$'s list is the first person in $a_{i+1}$'s list,
- the second person in $a_r$'s list is the first person in $a_1$'s list.

Let $b_i$ denote the first person in $a_i$'s list.

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| 1 | | 6 | | | |
| 2 | | 3 | 5 | | 4 |
| 3 | | 5 | | | 2 |
| 4 | 2 | | 5 | | |
| 5 | 4 | 2 | 3 | | |
| 6 | | 1 | | | |

**Lemma:** If some $a_i$ gets her first choice in a stable-matching, then so all others on the cycle.

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| 1 | | 6 | | | |
| 2 | | 3 | 5 | | 4 |
| 3 | | 5 | | | 2 |
| 4 | 2 | | 5 | | |
| 5 | 4 | 2 | 3 | | |
| 6 | | 1 | | | |

**Lemma:** If some $a_i$ gets her first choice in a stable-matching, then so all others on the cycle.

If $a_i$ is matched to $b_i$ that she considers best, then $a_i$ is worst for $b_i$.

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| 1 | | 6 | | | |
| 2 | | 3 | 5 | | 4 |
| 3 | | 5 | | | 2 |
| 4 | 2 | | 5 | | |
| 5 | 4 | 2 | 3 | | |
| 6 | | 1 | | | |

**Lemma:** If some $a_i$ gets her first choice in a stable-matching, then so all others on the cycle.

If $a_i$ is matched to $b_i$ that she considers best, then $a_i$ is worst for $b_i$.
If $a_{i+1}$ is not matched to $b_{i+1}$ that she considers best, then she would get a candidate that is worse than $b_i$ (her second choice, matched to $a_i$).

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| 1 | | 6 | | | |
| 2 | | 3 | 5 | | 4 |
| 3 | | 5 | | | 2 |
| 4 | 2 | | 5 | | |
| 5 | 4 | 2 | 3 | | |
| 6 | | 1 | | | |

**Lemma:** If some $a_i$ gets her first choice in a stable-matching, then so all others on the cycle.

If $a_i$ is matched to $b_i$ that she considers best, then $a_i$ is worst for $b_i$.
If $a_{i+1}$ is not matched to $b_{i+1}$ that she considers best, then she would get a candidate that is worse than $b_i$ (her second choice, matched to $a_i$).
Thus, $a_{i+1}$ and $b_i$ would be a blocking pair.

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| 1 | | 6 | | | |
| 2 | | 3 | 5 | | 4 |
| 3 | | 5 | | | 2 |
| 4 | 2 | | 5 | | |
| 5 | 4 | 2 | 3 | | |
| 6 | | 1 | | | |

**Lemma:** If in the stable matching each person in the cycle gets her first choice, then after shifting the cycle, the matching would still be stable.

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| 1 | | 6 | | | |
| 2 | | 3 | 5 | | 4 |
| 3 | | 5 | | | 2 |
| 4 | 2 | | 5 | | |
| 5 | 4 | 2 | 3 | | |
| 6 | | 1 | | | |

**Lemma:** If in the stable matching each person in the cycle gets her first choice, then after shifting the cycle, the matching would still be stable.

Let $A = \{a_1, a_2, \ldots a_r\}$ and $B = \{b_1, b_2, \ldots b_r\}$.

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| 1 | | 6 | | | |
| 2 | | 3 | 5 | | 4 |
| 3 | | 5 | | | 2 |
| 4 | 2 | | 5 | | |
| 5 | 4 | 2 | 3 | | |
| 6 | | 1 | | | |

**Lemma:** If in the stable matching each person in the cycle gets her first choice, then after shifting the cycle, the matching would still be stable.

Let $A = \{a_1, a_2, \ldots a_r\}$ and $B = \{b_1, b_2, \ldots b_r\}$.

A new blocking pair $(x, y)$ such that $x$ preferred her partner to $y$, but received a worse candidate.

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| 1 | | 6 | | | |
| 2 | | 3 | 5 | | 4 |
| 3 | | 5 | | | 2 |
| 4 | 2 | | 5 | | |
| 5 | 4 | 2 | 3 | | |
| 6 | | 1 | | | |

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Let $A = \{a_1, a_2, \ldots a_r\}$ and $B = \{b_1, b_2, \ldots b_r\}$.

A new blocking pair $(x, y)$ such that $x$ preferred her partner to $y$, but received a worse candidate. Either $x = a_i$ or $x = b_i$ (other candidates have the same partners).

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| 1 | | 6 | | | |
| 2 | | 3 | 5 | | 4 |
| 3 | | 5 | | | 2 |
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| 5 | 4 | 2 | 3 | | |
| 6 | | 1 | | | |

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Let $A = \{a_1, a_2, \ldots a_r\}$ and $B = \{b_1, b_2, \ldots b_r\}$.
A new blocking pair $(x, y)$ such that $x$ preferred her partner to $y$, but received a worse candidate. Either $x = a_i$ or $x = b_i$ (other candidates have the same partners). The case $x = b_i$ is not possible, since $a_i$ gets a better candidate.

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| 1 | | 6 | | | |
| 2 | | 3 | 5 | | 4 |
| 3 | | 5 | | | 2 |
| 4 | 2 | | 5 | | |
| 5 | 4 | 2 | 3 | | |
| 6 | | 1 | | | |

**Lemma:** If in the stable matching each person in the cycle gets her first choice, then after shifting the cycle, the matching would still be stable.

Let $A = \{a_1, a_2, \ldots a_r\}$ and $B = \{b_1, b_2, \ldots b_r\}$.

A new blocking pair $(x, y)$ such that $x$ preferred her partner to $y$, but received a worse candidate; $x = a_i$.

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| 1 | | 6 | | | |
| 2 | | 3 | 5 | | 4 |
| 3 | | 5 | | | 2 |
| 4 | 2 | | 5 | | |
| 5 | 4 | 2 | 3 | | |
| 6 | | 1 | | | |

**Lemma:** If in the stable matching each person in the cycle gets her first choice, then after shifting the cycle, the matching would still be stable.

Let $A = \{a_1, a_2, \ldots a_r\}$ and $B = \{b_1, b_2, \ldots b_r\}$.

A new blocking pair $(x, y)$ such that $x$ preferred her partner to $y$, but received a worse candidate; $x = a_i$. Since $x$ gets her second choice we know $y$ was removed from the list of $x$ or $y = b_i$.

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| 1 | | 6 | | | |
| 2 | | 3 | 5 | | 4 |
| 3 | | 5 | | | 2 |
| 4 | 2 | | 5 | | |
| 5 | 4 | 2 | 3 | | |
| 6 | | 1 | | | |

**Lemma:** If in the stable matching each person in the cycle gets her first choice, then after shifting the cycle, the matching would still be stable.

Let $A = \{a_1, a_2, \ldots a_r\}$ and $B = \{b_1, b_2, \ldots b_r\}$.

A new blocking pair $(x, y)$ such that $x$ preferred her partner to $y$, but received a worse candidate; $x = a_i$. Since $x$ gets her second choice we know $y$ was removed from the list of $x$ or $y = b_i$. The latter is not possible since $b_i$ prefers her new partner.

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| 1 | | 6 | | | |
| 2 | | 3 | 5 | | 4 |
| 3 | | 5 | | | 2 |
| 4 | 2 | | 5 | | |
| 5 | 4 | 2 | 3 | | |
| 6 | | 1 | | | |

**Lemma:** If in the stable matching each person in the cycle gets her first choice, then after shifting the cycle, the matching would still be stable.

Let $A = \{a_1, a_2, \ldots a_r\}$ and $B = \{b_1, b_2, \ldots b_r\}$.

A new blocking pair $(x, y)$ such that $x$ preferred her partner to $y$, but received a worse candidate; $x = a_i$. Since $x$ gets her second choice we know $y$ was removed from the list of $x$.

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| 1 | | 6 | | | |
| 2 | | 3 | 5 | | 4 |
| 3 | | 5 | | | 2 |
| 4 | 2 | | 5 | | |
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Let $A = \{a_1, a_2, \ldots a_r\}$ and $B = \{b_1, b_2, \ldots b_r\}$.

A new blocking pair $(x, y)$ such that $x$ preferred her partner to $y$, but received a worse candidate; $x = a_i$. Since $x$ gets her second choice we know $y$ was removed from the list of $x$. If $x$ proposed to $y$ and got rejected then $x$ preferred $y$ to her matched candidate.

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| 1 | | 6 | | | |
| 2 | | 3 | 5 | | 4 |
| 3 | | 5 | | | 2 |
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**Lemma:** If in the stable matching each person in the cycle gets her first choice, then after shifting the cycle, the matching would still be stable.

Let $A = \{a_1, a_2, \ldots a_r\}$ and $B = \{b_1, b_2, \ldots b_r\}$.
A new blocking pair $(x, y)$ such that $x$ preferred her partner to $y$, but received a worse candidate; $x = a_i$. Since $x$ gets her second choice we know $y$ was removed from the list of $x$. If $x$ proposed to $y$ and got rejected then $x$ preferred $y$ to her matched candidate. So $y$ removed herself from the list of $x$ (got better).

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| 1 | | 6 | | | |
| 2 | | 3 | 5 | | 4 |
| 3 | | 5 | | | 2 |
| 4 | 2 | | 5 | | |
| 5 | 4 | 2 | 3 | | |
| 6 | | 1 | | | |

**Lemma:** If some $a_i$ gets her first choice in a stable-matching, then so all others on the cycle.

**Lemma:** If in the stable matching each person in the cycle gets her first choice, then after shifting the cycle, the matching would still be stable.

**The cycle can be eliminated!**

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| 1 | | 6 | | | |
| 2 | | 3 | 5 | | 4 |
| 3 | | 5 | | | 2 |
| 4 | 2 | | 5 | | |
| 5 | 4 | 2 | 3 | | |
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# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| **1** | | 6 | | | |
| **2** | | 3 | 5 | | 4 |
| **3** | | | | | 2 |
| **4** | | | 5 | | |
| **5** | 4 | 2 | 3 | | |
| **6** | | 1 | | | |

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# Stable Roommate: non-bipartite graphs

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**Lemma:** If some $a_i$ gets her first choice in a stable-matching, then so all others on the cycle.

**Lemma:** If in the stable matching each person in the cycle gets her first choice, then after shifting the cycle, the matching would still be stable.

**If some list is empty no stable matching exists.**

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| 1 | | 6 | | | |
| 2 | | 3 | 5 | | 4 |
| 3 | | | | | 2 |
| 4 | | | 5 | | |
| 5 | 4 | 2 | 3 | | |
| 6 | | 1 | | | |

**Lemma:** If some $a_i$ gets her first choice in a stable-matching, then so all others on the cycle.

**Lemma:** If in the stable matching each person in the cycle gets her first choice, then after shifting the cycle, the matching would still be stable.

**We continue the propose-reject algorithm.**

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| 1 | | 6 | | | |
| 2 | | 3 | 5 | | 4 |
| 3 | | | | | 2 |
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We continue the propose-reject algorithm.

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| **1** | | 6 | | | |
| **2** | | 3 | | | |
| **3** | | | | | 2 |
| **4** | | | 5 | | |
| **5** | 4 | 2 | 3 | | |
| **6** | | 1 | | | |

**Lemma:** If some $a_i$ gets her first choice in a stable-matching, then so all others on the cycle.

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# Stable Roommate: non-bipartite graphs

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|---|---|---|---|---|---|
| 1 | | 6 | | | |
| 2 | | 3 | | | |
| 3 | | | | | 2 |
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| 5 | 4 | 2 | 3 | | |
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**Lemma:** If some $a_i$ gets her first choice in a stable-matching, then so all others on the cycle.

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**We continue the propose-reject algorithm.**

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| 1 | | 6 | | | |
| 2 | | 3 | | | |
| 3 | | | | | 2 |
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| 5 | 4 | 2 | 3 | | |
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**Lemma:** If some $a_i$ gets her first choice in a stable-matching, then so all others on the cycle.

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**We continue the propose-reject algorithm.**

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| **1** | | 6 | | | |
| **2** | | 3 | | | |
| **3** | | | | | 2 |
| **4** | | | 5 | | |
| **5** | 4 | | | | |
| **6** | | 1 | | | |

**Lemma:** If some $a_i$ gets her first choice in a stable-matching, then so all others on the cycle.

**Lemma:** If in the stable matching each person in the cycle gets her first choice, then after shifting the cycle, the matching would still be stable.

**We continue the propose-reject algorithm.**

# Stable Roommate: non-bipartite graphs

| | | | | | |
|---|---|---|---|---|---|
| 1 | | 6 | | | |
| 2 | | 3 | | | |
| 3 | | | | | 2 |
| 4 | | | 5 | | |
| 5 | 4 | | | | |
| 6 | | 1 | | | |

**Lemma:** If some $a_i$ gets her first choice in a stable-matching, then so all others on the cycle.

**Lemma:** If in the stable matching each person in the cycle gets her first choice, then after shifting the cycle, the matching would still be stable.

**If each list contains a single element we have a stable matching.**

# Literature

https://en.wikipedia.org/wiki/Stable_marriage_problem

Robert W Irving . An efficient algorithm for the "stable roommates" problem. Journal of Algorithms Volume 6, Issue 4, December 1985, Pages 577-595.

**Further reading:**

Robert W. Irving and Paul Leather. The Complexity of Counting Stable Marriages. SIAM Journal on Computing, Volume 15, Issue 3 (1986).

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