Proportionality-Based Fairness in Social Choice

Dominik Peters

Piotr Skowron
Outline

Models:
1. Apportionment
2. Committee elections
3. Participatory budgeting

Why proportionality?
1. fairness towards groups of voters
2. equal voting power,
3. not ignoring minorities,
4. having all viewpoints present in a deliberative body.
Model: Apportionment

1. We have \( m \) political parties: \( P_1, P_2, \ldots, P_m \).
Model: Apportionment

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2. We have $n$ voters. Each voter votes for exactly one party.

   Let $n_i$ denote the number of votes cast on party $P_i$

   (of course, $\sum_{i=1}^{m} n_i = n$).
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   (of course, $\sum_{i=1}^{m} n_i = n$).

3. We have $k$ parliamentary seats and we need to distribute them among the parties. (In most cases we want to do it proportionally!)
Apportionment: two examples

number of seats: $k = 10$.

Example 1:

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<tbody>
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<td>10</td>
<td>20</td>
<td>20</td>
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</tr>
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Different apportionment methods will give different results!
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lower quota: party $P_i$ should at least $\left\lfloor k \cdot \frac{n_i}{n} \right\rfloor$ seats.
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<tr>
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lower quota: party \( P_i \) should at least \( \left\lfloor k \cdot \frac{n_i}{n} \right\rfloor \) seats.

upper quota: party \( P_i \) should at most \( \left\lceil k \cdot \frac{n_i}{n} \right\rceil \) seats.
The largest remainder method
(aka the Hamilton method or the Hare-Niemeyer method)

1. First, assign to each party its lower quota.
2. Next, sort the parties by the remainders $k \cdot \frac{n_i}{n} - \left\lfloor k \cdot \frac{n_i}{n} \right\rfloor$ and assign the remaining seats to the parties with the heist remainders.
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The largest remainder method satisfies lower and upper quota.
House monotonicity and Alabama paradox

**House monotonicity**: if we increase the number of seats $k$ then each party should get at least the same number of seats as before the increase.
House monotonicity and Alabama paradox

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**Alabama paradox:** the largest remainder method fails house monotonicity

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<tbody>
<tr>
<td>$k \cdot \frac{n_i}{n}$ for $k = 10$</td>
<td>4.286</td>
<td>4.286</td>
<td>1.429</td>
</tr>
<tr>
<td>#seats $k = 10$</td>
<td>4</td>
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<td>value $k \cdot \frac{n_i}{n}$ for $k = 11$</td>
<td>4.714</td>
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D’Hondt method
(aka the Jefferson method or the Hagenbach-Bischoff method)

In each iteration we assign one seat to one party. Let $s_i(r)$ denote the number of seats assigned to party $P_i$ until iteration $r$. In iteration $r$ we assign one seat to the party $P_i$ which maximises $\frac{n_i}{s_i(r) + 1}$. 
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<td>3</td>
<td>3.5</td>
<td>19.5</td>
<td>24</td>
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<td>#votes/3</td>
<td>2</td>
<td>2.33</td>
<td>13</td>
<td>16</td>
</tr>
<tr>
<td>#votes/4</td>
<td>1.5</td>
<td>1.75</td>
<td>9.75</td>
<td>12</td>
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<td>#votes/5</td>
<td>1.2</td>
<td>1.4</td>
<td>7.8</td>
<td>9.6</td>
</tr>
<tr>
<td>#votes/6</td>
<td>1</td>
<td>1.17</td>
<td>6.5</td>
<td>8.0</td>
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<tr>
<td>#votes/7</td>
<td>0.86</td>
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<td>5.57</td>
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<td>24</td>
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<td>5.57</td>
<td>6.86</td>
</tr>
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</table>

Number of seats: $k = 10$
**D’Hondt method**  
*(aka the Jefferson method or the Hagenbach-Bischoff method)*

In each iteration we assign one seat to one party. Let $s_i(r)$ denote the number of seats assigned to party $P_i$ until iteration $r$. In iteration $r$ we assign one seat to the party $P_i$ which maximises $\frac{n_i}{s_i(r) + 1}$.

<table>
<thead>
<tr>
<th></th>
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<th>Party 2</th>
<th>Party 3</th>
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<td>48</td>
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<td>24</td>
</tr>
<tr>
<td>#votes/3</td>
<td>2</td>
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<td>13</td>
<td>16</td>
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<td>9.75</td>
<td>12</td>
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<tr>
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<td>1.4</td>
<td>7.8</td>
<td>9.6</td>
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<td>1</td>
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<td>8.0</td>
</tr>
<tr>
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<tr>
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<td>6</td>
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</table>

**number of seats**: $k = 10$
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(aka the Jefferson method or the Hagenbach-Bischoff method)

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$$s_i(r) < \left\lfloor k \cdot \frac{n_i}{n} \right\rfloor \implies s_i(r) \leq \left\lfloor k \cdot \frac{n_i}{n} \right\rfloor - 1 \leq k \cdot \frac{n_i}{n} - 1.$$
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$s_j(r) = s_j(r + 1) - 1 > k \cdot \frac{n_j}{n} - 1.$
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$$s_j(r) = s_j(r + 1) - 1 > k \cdot \frac{n_j}{n} - 1.$$  

$$\frac{n_i}{s_i(r) + 1} \geq \frac{n_i}{k \cdot \frac{n_i}{n} - 1 + 1} = \frac{n}{k}.$$
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\]

\[
s_j(r) = s_j(r + 1) - 1 > k \cdot \frac{n_j}{n} - 1.
\]

\[
\frac{n_i}{s_i(r) + 1} \geq \frac{n_i}{k \cdot \frac{n_i}{n} - 1 + 1} = \frac{n}{k}.
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$\frac{n_j}{s_j(r) + 1} < \frac{n_j}{k \cdot \frac{n_j}{n} - 1 + 1} = \frac{n}{k}.$

Thus $P_i$ would be assigned the seat instead of $P_j$. 
Model: Approval-Based Elections

A committee of size $k$
A preference profile: an example

We have $n = 8$ voters, $m = 9$ candidates.
A preference profile: an example

We have $n = 8$ voters, $m = 9$ candidates.
How to define proportionality for more complex preferences?

$v_1$:  
$v_2$:  
$v_3$:  
$v_4$:  
$v_5$:  
$v_6$:  
$v_7$:  
$v_8$:  
How to define proportionality for more complex preferences?

$v_1$: 
$v_2$: 
$v_3$: 
$v_4$: 
$v_5$: 
$v_6$: 
$v_7$: 
$v_8$: 
Let’s move back in time to the end of the 19th century?
Let’s move back in time to the end of the 19th century?

Thorvald N. Thiele

Edvard Phragmén
Proportional Approval Voting (Thiele)

Assume voter $v$ approves $t$ members of a committee $W$. Then $v$ gives to $W$ the following number of points:

$$\sum_{i=1}^{t} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{t}$$
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E.g., consider a committee

Points per voter:

$v_1$:
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$v_1$: $1 + \frac{1}{2}$
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E.g., consider a committee

Points per voter:

$v_1: 1 + \frac{1}{2}$

$v_2: 1 + \frac{1}{2}$
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$$

E.g., consider a committee

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$v_2$: $1 + \frac{1}{2}$

$v_3$: $1 + \frac{1}{2} + \frac{1}{3}$

$v_4$: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$

$v_5$: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$

$v_6$: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$

$v_7$: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$

$v_8$: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$
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E.g., consider a committee

Points per voter:

\( v_1: 1 + \frac{1}{2} \)

\( v_2: 1 + \frac{1}{2} \)

\( v_3: 1 + \frac{1}{2} + \frac{1}{3} \)

\( v_4: 1 + \frac{1}{2} \)

\( v_5: 1 + \frac{1}{2} \)

\( v_6: 0 \)

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E.g., consider a committee

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- \( v_2: 1 + \frac{1}{2} \)
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E.g., consider a committee

Points per voter:

- $v_1$: $1 + \frac{1}{2}$
- $v_3$: $1 + \frac{1}{2} + \frac{1}{3}$
- $v_5$: $1 + \frac{1}{2}$
- $v_7$: $0$
- $v_2$: $1 + \frac{1}{2}$
- $v_4$: $1 + \frac{1}{2}$
- $v_6$: $0$
- $v_8$: $1$
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Sum of points = \( 8 + \frac{5}{6} \)
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\]

E.g., consider a committee

Committee with the highest score wins the election.

Point

\( v_1 \):
\( v_3 \): 1 + \frac{1}{2} + \frac{1}{3}
\( v_4 \): 1 + \frac{1}{2}
\( v_5 \): 1 + \frac{1}{2}
\( v_6 \): 0
\( v_7 \): 0
\( v_8 \): 1

Sum of points = 8 + \( \frac{5}{6} \)
Proportional Approval Voting is welfarist

The welfare vector of a committee $W$ is defined as:

$$( |A_1 \cap W|, |A_2 \cap W|, \ldots, |A_n \cap W| )$$

where:

- $A_i$ is the set of candidates approved by voter $i$
- $( |A_i \cap W| )$ is the number of representatives of $i$
Proportional Approval Voting is welfarist

The welfare vector of a committee $W$ is defined as:

$$\left( |A_1 \cap W|, |A_2 \cap W|, \ldots, |A_n \cap W| \right)$$

where:

- $A_i$ is the set of candidates approved by voter $i$

$$\left( |A_i \cap W| \text{ is the number of representatives of } i \right)$$

A rule is welfarist if the decision which committee to elect can be made solely based on welfare vectors of the committees.
Phragmén’s Rule

- Voters earn money with the constant speed ($1 per time unit).
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- In the first moment when there is a group of voters $S$ who all have $n$ dollars in total and who all approve a not-yet selected candidate $c$, do:
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- Voters earn money with the constant speed ($1 per time unit).
- In the first moment when there is a group of voters $S$ who all have $n$ dollars in total and who all approve a not-yet selected candidate $c$, do:
  1. Add $c$ to the committee.
  2. Make voters from $S$ pay for $c$ (resetting their budget to 0).
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$$k = 12$$
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\[ k = 12 \]

\[
\begin{array}{cccc}
  c_4 & c_5 & c_6 \\
  c_3 & c_{13} & c_{14} & c_{15} \\
  c_2 & c_{10} & c_{11} & c_{12} \\
  c_1 & c_7 & c_8 & c_9 \\
\end{array}
\]

\[ t_0 = 0 \]

\[
\begin{array}{cccccccc}
  v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\
\end{array}
\]
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<table>
<thead>
<tr>
<th>$c_4$</th>
<th>$c_5$</th>
<th>$c_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_3$</td>
<td>$c_{13}$</td>
<td>$c_{14}$</td>
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<tr>
<td>$c_2$</td>
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</tr>
<tr>
<td>$c_1$</td>
<td>$c_7$</td>
<td>$c_8$</td>
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</table>

$t_2 = 4$
$t_1 = 2$
$t_0 = 0$
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Phragmén’s Rule

• Voters earn money with the constant speed ($1 per time unit).

• In the first moment when there is a group of voters $S$ who all have $n$ dollars in total and who all approve a not-yet selected candidate $c$, do:
  1. Add $c$ to the committee.
  2. Make voters from $S$ pay for $c$ (resetting their budget to 0).
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PAV versus Phragmén’s Rule
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\[ k = 12 \]

Phragmén’s Rule

PAV
PAV versus Phragmén’s Rule

\[ k = 12 \]

Phragmén’s Rule

- Proportionality with respect to power

PAV

- Proportionality with respect to welfare
PAV versus Phragmén’s Rule

\[ k = 12 \]

Phragmén’s Rule

Proportionality with respect to power

Priceability

PAV

Proportionality with respect to welfare

Extended justified representation
How to reason about proportionality?

First approach: Axioms for Cohesive Groups
How to define proportionality for more complex preferences?

\[ \nu_1: \]
\[ \nu_2: \]
\[ \nu_3: \]
\[ \nu_4: \]
\[ \nu_5: \]
\[ \nu_6: \]
\[ \nu_7: \]
\[ \nu_8: \]
How to define proportionality for more complex preferences?
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For \( k = 4 \) these voters should approve (on average) 2 candidates in the selected committee.
How to define proportionality for more complex preferences?

\[ v_1: \]

\[ v_2: \]

\[ v_3: \]

\[ v_4: \]

\[ v_5: \]

\[ v_6: \]

\[ v_7: \]

\[ v_8: \]
How to define proportionality for more complex preferences?

For $k = 4$ these voters should approve (on average) 1 candidate in the selected committee.
How to define proportionality for more complex preferences?

**Definition:** Each group with at least $\ell n / k$ voters who approve at least $\ell$ same candidates should have on average at least $\ell$ representatives in the elected committee.
How to define proportionality for more complex preferences?

**Definition:** Each group with at least \( \ell n/k \) voters who approve at least \( \ell \) same candidates should have on average at least \( \ell \) representatives in the elected committee.

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Does there exist a system which satisfies this property?
How to define proportionality for more complex preferences?

**Definition:** Each group with at least $\ell n/k$ voters who approve at least $\ell$ same candidates should have on average at least $\ell$ representatives in the elected committee.

Does there exist a system which satisfies this property?

$v_1: \{a, d\}$  
$v_2: \{a\}$  
$v_3: \{a\}$  
$v_4: \{a, b\}$  
$v_5: \{b\}$  
$v_6: \{b\}$  
$v_7: \{b, c\}$  
$v_8: \{c\}$  
$v_9: \{c\}$  
$v_{10}: \{c, d\}$  
$v_{11}: \{d\}$  
$v_{12}: \{d\}$  

$n = 12$  
$k = 3$
How to define proportionality for more complex preferences?

**Definition:** Each group with at least $\ell n/k$ voters who approve at least $\ell$ same candidates should have on average at least $\ell - 1$ representatives in the elected committee.

But PAV satisfies a slightly weaker property!
How to define proportionality for more complex preferences?

**Definition:** Each group with at least $\ell n/k$ voters who approve at least $\ell$ same candidates should have on average at least $\ell - 1$ representatives in the elected committee.

But PAV satisfies a slightly weaker property!

Phragmén’s Rule would satisfy it only if we replaced $\ell - 1$ with $(\ell - 1)/2$. 
A few formal definitions

$A_i$: the set of candidates approved by voter $i$

An $\ell$-cohesive group: a group of voters $S \subseteq N$ is cohesive if

1. $|S| \geq \ell \cdot n/k$, and
2. $\left| \bigcap_{i \in S} A_i \right| \geq \ell$. 
A few formal definitions

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**Proportionality degree**: an outcome $W$ has the proportionality degree of $f(\cdot)$ if for each $\ell$-cohesive group of voters $S$ it holds that:

$$\frac{1}{\ell} \cdot \sum_{i \in S} |A_i \cap W| \geq f(\ell)$$
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Extended Justified Representation (EJR): an outcome $W$ satisfies extended justified representation if for each $\ell$-cohesive group of voters $S$ it holds that:

there exists $i \in S$ such that $|A_i \cap W| \geq \ell$
A few formal definitions

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**Proportional Justified Representation (PJR):** an outcome $W$ satisfies proportional justified representation if for each $\ell$-cohesive group of voters $S$ it holds that:

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A few formal definitions

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\[ \bigcup_{i \in S} |A_i \cap W| \geq \ell \]

Justified Representation (JR): an outcome \( W \) satisfies justified representation if for each 1-cohesive group of voters \( S \) it holds that:

there is a voter \( i \in S \) such that \( |A_i \cap W| \geq \ell \)
Relation between axioms

proportionality
degree of $\frac{\ell - 1}{2}$

EJR

PJR

JR

lower quota
Relation between axioms

Phragmén’s Rule

proportionality
degree of \( \frac{\ell - 1}{2} \)

EJR

PJR

JR

lower quota
Relation between axioms

- Proportionality
- Degree of \( \frac{\ell - 1}{2} \)
- EJR
- PJR
- JR
- Phragmén’s Rule
- PAV
- Lower quota
Another Notion of Proportionality

Fair distribution of power

(failed by PAV)
Priceability
Priceability

A price system is a pair $ps = (p, \{p_i\}_{i \in [n]})$, where $p > 0$ is a price, and for each voter $i \in [n]$, there is a payment function $p_i : C \to [0,1]$ such that:

1. A voter can only pay for candidates she approves of.
2. A voter can spend at most one dollar.
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We say that a price system \( ps = (p, \{ p_i \}_{i \in [n]} ) \) supports a committee \( W \) if the following hold:

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1. For each elected candidate, the sum of the payments to this candidate equals the price $p$.

2. No candidate outside of the committee gets any payment.

3. There exists no unelected candidate whose supporters, in total, have a remaining unspent budget of more than $p$. 
Priceability: Example

\[ k = 12 \]

The price is \( p = 0.5 \).

1. \( v_1 \) pays \( \frac{1}{6} \) for \( c_1, c_2 \) and \( c_3 \) and \( \frac{1}{2} \) for \( c_4 \).
2. \( v_2 \) pays \( \frac{1}{6} \) for \( c_1, c_2 \) and \( c_3 \) and \( \frac{1}{2} \) for \( c_5 \).
3. \( v_3 \) pays \( \frac{1}{6} \) for \( c_1, c_2 \) and \( c_3 \) and \( \frac{1}{2} \) for \( c_6 \).
4. \( v_4 \) pays \( \frac{1}{2} \) for \( c_7 \) and \( c_{10} \).
5. \( v_5 \) pays \( \frac{1}{2} \) for \( c_8 \) and \( c_{11} \).
6. \( v_6 \) pays \( \frac{1}{2} \) for \( c_9 \) and \( c_{12} \).
Relation between axioms

proportionality
degree of $\frac{\ell - 1}{2}$

EJR

PJR

JR

lower quota
Relation between axioms

Priceability

Lower quota

Proportionality

Degree of \( \frac{\ell - 1}{2} \)

EJR

PJR

JR
Relation between axioms

 relation between axioms

 Phragmén’s Rule

 proportionality

degree of \( \frac{\ell - 1}{2} \)

 lower quota

 priceability

 EJR

 PJR

 JR

 PAV
Relation between axioms

(Pareto optimality)

PAV

proportionality

degree of $\frac{\ell - 1}{2}$

(EJR

PJR

JRx

Phragmén’s Rule

priceability

lower quota

(House monotonicity)
No welfarist rule can be priceable
Core
Core: Definition

We say that a committee $W$ is in the core if there exists no group of voters $S$ and a subset of candidates $T$ such that:

1. $\frac{|T|}{k} \leq \frac{|S|}{n}$, and

2. Each voter in $S$ prefers $T$ to $W$. 


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$k = 12$

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$v_1$ $v_2$ $v_3$ $v_4$ $v_5$ $v_6$
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\[
\begin{array}{cccc}
\text{c_4} & \text{c_5} & \text{c_6} \\
\text{c_3} & \text{c_{13}} & \text{c_{14}} & \text{c_{15}} \\
\text{c_2} & \text{c_{10}} & \text{c_{11}} & \text{c_{12}} \\
\text{c_1} & \text{c_7} & \text{c_8} & \text{c_9} \\
\hline
\text{v_1} & \text{v_2} & \text{v_3} & \text{v_4} & \text{v_5} & \text{v_6}
\end{array}
\]

$k = 12$

\[
\begin{array}{cccc}
\text{c_4} & \text{c_5} & \text{c_6} \\
\text{c_3} & \text{c_{13}} & \text{c_{14}} & \text{c_{15}} \\
\text{c_2} & \text{c_{10}} & \text{c_{11}} & \text{c_{12}} \\
\text{c_1} & \text{c_7} & \text{c_8} & \text{c_9} \\
\hline
\text{v_1} & \text{v_2} & \text{v_3} & \text{v_4} & \text{v_5} & \text{v_6}
\end{array}
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\[ k = 12 \]  

Not in the core!
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$k = 12$

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Core contradicts the Pigou-Dalton principle!

Not in the core!

Theorem: PAV gives the best possible Approximation of the core subject to Satisfying the Pigou-Dalton principle!
Relation between axioms

- The core
- EJR
- JR

Proportionality: $\ell - 1$\newline
Degree of: $\frac{\ell - 1}{2}$\newline
Priceability

Lower quota
Relation between axioms

stable priceability

the core

FJR

core subject to price ability with equal payments

EJR

perfect representation

proportionality degree of $\ell - 1 \over 2$

priceability

lower quota

PJR

JR
A lot of open questions around core!

1. Does there always exist a committee in the core? *(no welfarist rule is in the core)*
A lot of open questions around core!

1. Does there always exist a committee in the core? (no welfarist rule is in the core)

2. How close we can get to the core?
   A. Core relaxation by randomisation. [Cheng et al., ACM-EC-2019]
   B. Core relaxation by approximation. [Jiang et al., STOC-2020], [Fain et al., ACM-EC-2018], [Munagala et al., SODA-2022], [Peters and Skowron, ACM-EC-2020]
How to reason about proportionality?

Another approach: Axiomatic Extensions of Apportionment Methods
Some basic axiomatic properties: Symmetry
Some basic axiomatic properties: Symmetry

\( \nu_1: \)

\( \nu_2: \)

\( \nu_3: \)

\( \nu_4: \)

\( \nu_5: \)

\( \nu_6: \)

\( \nu_7: \)

\( \nu_8: \)
Some basic axiomatic properties: Symmetry

$v_1$:  
v_2:  
v_3:  
v_4:  
v_5:  
v_6:  
v_7:  
v_8:  

$v_1$:  
v_2:  
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v_7:  
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$\rightarrow$
Some basic axiomatic properties: Symmetry

\[ v_1: \quad v_2: \quad v_3: \quad v_4: \quad v_5: \quad v_6: \quad v_7: \quad v_8: \]
Some basic axiomatic properties: Consistency
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\[ v_1: \]
\[ v_2: \]
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\[ v_4: \]
\[ v_5: \]
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\[ v_7: \]
\[ v_8: \]
Some basic axiomatic properties: Consistency

\[ \begin{align*}
\nu_1: & \\
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\nu_3: & \\
\nu_4: & \\
\nu_5: & \\
\nu_6: & \\
\nu_7: & \\
\nu_8: & \\
\end{align*} \]

\[ \begin{align*}
\nu_1: & \\
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\nu_3: & \\
\nu_4: & \\
\nu_5: & \\
\nu_6: & \\
\nu_7: & \\
\nu_8: & \\
\end{align*} \]
Some basic axiomatic properties: Continuity
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<th>$v_1$:</th>
<th>$v_2$:</th>
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$\Rightarrow$

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Some basic axiomatic properties: Continuity

$E_1$: $v_1$: $v_2$: $v_3$: $v_4$: $v_5$: $v_6$: $v_7$: $v_8$: $v_9$: $v_{10}$

$E_2$: $v_1$: $v_2$: $v_3$: $v_4$: $v_5$: $v_6$: $v_7$: $v_8$: $v_9$: $v_{10}$

Then, there exists (possibly very large) value $z$ such that:

$z \cdot E_1 + E_2$
Theorem: Proportional Approval Voting is the only ABC ranking rule that satisfies symmetry, consistency, continuity and D’Hondt proportionality.

Axiomatic Characterisations

**Theorem:** Proportional Approval Voting satisfies symmetry, consistency, continuity and D’Hondt proportionality.

Axiomatic Characterisations

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More information about ABC voting