Computational Social Choice

Stable Matchings

Piotr Skowron
University of Warsaw
The model

1. A set of $n$ men $U = \{u_1, u_2, \ldots, u_n\}$.
2. A set of $m$ women $W = \{w_1, w_2, \ldots, w_m\}$.

Each man $u_i$ has a preference relation $\succ_{u_i}$ over the set of women. Each woman $w_i$ has a preference relation $\succ_{w_i}$ over the set of men.
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Each man \( u_i \) has a preference relation \( >_{u_i} \) over the set of women.

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**Goal:** Find a matching between men and women, such that each man is matched with at most one woman, and each woman is matched with at most one man.
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Stable Matchings

**Stable Matching (CE):** We say that a pair of a man and a woman \((u,w)\) blocks a matching \(M\) if \(u\) and \(w\) are not matched and:

1. \(u\) is unmatched or prefers \(w\) to her partner \(M(u)\) in the matching, and
2. \(w\) is unmatched or prefers \(u\) to her partner \(M(w)\) in the matching.

We say that a matching \(M\) is stable if there exists no pair that blocks \(M\).
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**Blocking pairs:** \(\{\text{man}, \text{woman}\}, \{\text{man}, \text{woman}\}, \{\text{man}, \text{woman}\}\)
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No blocking pairs. This matching is stable!
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Does a stable matching always exists? If so, how one can find it?
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**Gale-Shapley Algorithm:**

1. In the first round each man proposes to his favourite woman. A woman that gets one or multiple proposals picks the man she prefers most, makes a temporary engagement with this man, and rejects all other man.

2. In each subsequent round each unengaged man makes a proposal to his most preferred woman among those who did not reject him. A woman who gets one or multiple proposals picks the one that she prefers most. If she prefers this man to her temporary engaged partner, she breaks this engagement, and makes a temporary engagement with the currently best man among those who proposed. She rejects all proposed man except the one she is engaged to.

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Proof: Consider a matching \( M \) returned by the Gale-Shapley algorithm. Towards a contradiction assume there exists a blocking pair \( \{ u, w \} \).
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Proof: Consider a matching $M$ returned by the Gale-Shapley algorithm.
Towards a contradiction assume there exists a blocking pair $\{u, w\}$. $u$ must have been rejected by $w$ (either because $u$ does not have a partner or because he has a partner who is less preferred).
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Thus, $w$, when rejecting $u$ was engaged to someone she prefers to $u$. $w$ breaks an engagement only when getting a better partner.
Men-optimal and Women-optimal
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**Theorem:** In any stable matching no man can get a better partner than the one that he gets in the matching returned by the Gale-Shapley algorithm.
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**Proof**: Consider a matching $M$ returned by the Gale-Shapley algorithm. Assume that in a matching $M'$ there is a man $u_1$ such that:

$$w_2 = M'(u_1) >_{u_1} M(u_1) = w_1.$$
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In the Gale-Shapley algorithm $w_2$ must have rejected $u_1$ in favour of $u_2$, who proposed to $w_2$:

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Since $M'$ is stable, it must be the case that: $w_3 = M'(u_2) >_{u_2} w_2$. 

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Repeating the reasoning, we find sequences $u_1, u_2, \ldots, u_p$ and $w_1, w_2, \ldots, w_p$. 
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\[
\begin{array}{cc}
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  u_2 & w_2 \\
  \vdots & \vdots \\
  u_p & w_p \\
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Proof: Repeating the reasoning, we find sequences $u_1, u_2, \ldots, u_p$ and $w_1, w_2, \ldots, w_p$.

$u_1$ proposed to $w_1$, $w_2$ rejected $u_1$
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$u_1$ proposed to $w_1$

$u_2$ rejected $w_1$

$u_1$ rejected $u_2$

$u_2$ proposed to $w_2$

$u_2$ rejected $w_2$

$u_3$ rejected $u_2$

$u_3$ proposed to $w_3$

$u_3$ rejected $u_3$

$u_4$ rejected $u_3$

$u_4$ proposed to $w_4$

$u_4$ rejected $u_4$

$u_5$ rejected $u_4$

$u_5$ proposed to $w_5$

$u_5$ rejected $u_5$

$u_6$ rejected $u_5$

$u_6$ proposed to $w_6$

$u_6$ rejected $u_6$

$u_7$ rejected $u_6$

$u_7$ proposed to $w_7$

$u_7$ rejected $u_7$

$u_8$ rejected $u_7$

$u_8$ proposed to $w_8$

$u_8$ rejected $u_8$
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- $u_1$ proposed to $w_1$
- $u_2$ proposed to $w_2$
- $\vdots$
- $u_p$ proposed to $w_p$
- $w_2$ rejected $u_1$
- $w_3$ rejected $u_2$
- $\vdots$
- $w_1$ rejected $u_p$

$u_1$ proposed to $w_1$ after he was rejected by $w_2$

$w_2$ rejected $u_1$ after $u_2$ proposed to her.
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- $u_1$ proposed to $w_1$
- $w_2$ rejected $u_1$
- $u_2$ proposed to $w_2$
- $w_3$ rejected $u_2$
- $\cdots$
- $u_p$ proposed to $w_p$
- $w_1$ rejected $u_p$
- $u_1$ proposed to $w_1$ after he was rejected by $w_2$
- $w_2$ rejected $u_1$ after $u_2$ proposed to her. Thus,
- $u_1$ proposed to $w_1$ after $u_2$ proposed to $w_2$
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- $u_1$ proposed to $w_1$ after he was rejected by $w_2$.
- $w_2$ rejected $u_1$ after $u_2$ proposed to her. Thus,
- $u_1$ proposed to $w_1$ after $u_2$ proposed to $w_2$.

Similarly, $u_2$ proposed to $w_2$ after $u_3$ proposed to $w_3$. 

<table>
<thead>
<tr>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_p$</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>proposed to</td>
<td>proposed to</td>
<td>proposed to</td>
<td>rejected by</td>
<td>rejected by</td>
<td>rejected by</td>
</tr>
<tr>
<td>$w_1$</td>
<td>$w_2$</td>
<td>$w_p$</td>
<td>$w_2$</td>
<td>$w_3$</td>
<td>$w_1$</td>
</tr>
</tbody>
</table>
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Proof: Repeating the reasoning, we find sequences $u_1, u_2, \ldots, u_p$ and $w_1, w_2, \ldots, w_p$

\[ u_1 \text{ proposed to } w_1 \quad w_2 \text{ rejected } u_1 \]

\[ u_2 \text{ proposed to } w_2 \quad w_3 \text{ rejected } u_2 \]

\[ \vdots \quad \vdots \]

\[ u_p \text{ proposed to } w_p \quad w_1 \text{ rejected } u_p \]

\[ u_1 \text{ proposed to } w_1 \text{ after he was rejected by } w_2 \]

\[ w_2 \text{ rejected } u_1 \text{ after } u_2 \text{ proposed to her. Thus,} \]

\[ u_1 \text{ proposed to } w_1 \text{ after } u_2 \text{ proposed to } w_2 \]

similarly, $u_2$ proposed to $w_2$ after $u_3$ proposed to $w_3$

And so on, until we get a contradiction.
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How to get a woman optimal stable matching?
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How to get a woman optimal stable matching?

Women shall propose instead of men!
Applications of Stable Matchings

1. Matching students to schools.

2. Matching residents to hospitals.

3. Assigning users to servers in a large distributed Internet service.