Computational Social Choice

Approval-Based Committee Elections

Piotr Skowron
University of Warsaw

\[ v_1: \quad v_2: \quad v_3: \quad v_4: \quad v_5: \]

A committee of size \( k \)
Model: Approval-Based Elections

A committee of size $k$
A preference profile: an example

We have \( n = 8 \) voters, \( m = 9 \) candidates.

\[ \nu_1: \]

\[ \nu_2: \]

\[ \nu_3: \]

\[ \nu_4: \]

\[ \nu_5: \]

\[ \nu_6: \]

\[ \nu_7: \]

\[ \nu_8: \]
A preference profile: an example

We have $n = 8$ voters, $m = 9$ candidates.
A preference profile: an example

We have $n = 8$ voters, $m = 9$ candidates.

$v_1$: $c_1 \ c_2 \ c_3 \ c_4$
$v_2$: $c_1 \ c_2 \ c_3 \ c_4$
$v_3$: $c_1 \ c_2 \ c_3 \ c_4$
$v_4$: $c_1 \ c_2 \ c_3 \ c_4$
$v_5$: $c_5 \ c_6 \ c_7$
$v_6$: $c_5 \ c_6 \ c_7$
$v_7$: $c_8 \ c_9$
$v_8$: $c_8 \ c_9$
A preference profile: an example

Assume the committee size to be elected is $k = 4$. 
A preference profile: an example

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Which committee should be selected?
A preference profile: an example

Assume the committee size to be elected is $k = 4$.

Which committee should be selected?

Everything depends on the context!
Context: selecting finalists of a competition
Let’s start with an example!

We have $n = 8$ voters, $m = 9$ candidates, and our goal is to elect a committee of size $k = 4$

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$v_2$:  
$v_3$:  
$v_4$:  
$v_5$:  
$v_6$:  
$v_7$:  
$v_8$:  

Which committee should be selected?

Everything depends on the context!
Let’s start with an example!

We have $n = 8$ voters, $m = 9$ candidates, and our goal is to elect a committee of size $k = 4$

Which committee should be selected?

Multi-winner Approval Voting selects $k$ candidates who are approved by most voters

Everything depends on the context!
Context: electing a representative body
Back to the example!

Assume the committee size to be elected is $k = 4$.

Which committee should be selected?

*In this context the committee should be proportional.*
Which committee should be selected?

In this context the committee should be proportional.

But what does it mean and how could we achieve that?

Assume the committee size to be elected is \( k = 4 \).
Proportionality on the example of party-list systems.

Each voter casts one vote for a single party. Our goal is to select a committee of size $k = 4$:

- **Party 1** gets 40 votes.
- **Party 2** gets 20 votes.
- **Party 3** gets 20 votes.

How should the parliament look like?
Proportionality on the example of party-list systems.

Each voter casts one vote for a single party. Our goal is to select a committee of size $k = 4$:

- Party 1 gets 40 votes.
- Party 2 gets 20 votes.
- Party 3 gets 20 votes.

How should the parliament look like?

- Party 1 should get 2 seats.
- Party 2 should get 1 seat.
- Party 3 should get 1 seat.
Back to the example!

Assume the committee size to be elected is $k = 4$.

Which committee should be selected?
Back to the example!

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Which committee should be selected?
Back to the example!

Assume the committee size to be elected is $k = 4$.

Which committee should be selected?
How to define proportionality for more complex preferences?

\[
\nu_1:
\]

\[
\nu_2:
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\nu_3:
\]

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\nu_4:
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\nu_5:
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\nu_6:
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\nu_7:
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\nu_8:
\]
How to define proportionality for more complex preferences?

$v_1$:

$v_2$:

$v_3$:

$v_4$:

$v_5$:

$v_6$:

$v_7$:

$v_8$:
Let’s move back in time to the end of the 19th century?
Let’s move back in time to the end of the 19th century?

Thorvald N. Thiele

Edvard Phragmén
Proportional Approval Voting (Thiele)

Assume voter $v$ approves $t$ members of a committee $W$. Then $v$ gives to $W$ the following number of points:

$$\sum_{i=1}^{t} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{t}$$
Proportional Approval Voting (Thiele)

Assume voter \( v \) approves \( t \) members of a committee \( W \). Then \( v \) gives to \( W \) the following number of points:

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E.g., consider a committee

Points per voter:

$v_1$:
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E.g., consider a committee

Points per voter:

\( v_1: 1 + \frac{1}{2} \)

\( v_2: 1 + \frac{1}{2} \)
Proportional Approval Voting (Thiele)

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E.g., consider a committee

Points per voter:

- $v_1$: $1 + \frac{1}{2}$
- $v_2$: $1 + \frac{1}{2}$
- $v_3$: $1 + \frac{1}{2} + \frac{1}{3}$
- $v_4$: $1 + \frac{1}{2}$
- $v_5$: $1 + \frac{1}{2}$
- $v_6$: $1 + \frac{1}{2}$
- $v_7$: $1 + \frac{1}{2}$
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Assume voter $v$ approves $t$ members of a committee $W$. Then $v$ gives to $W$ the following number of points:

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$v_2$: $1 + \frac{1}{2}$  
$v_3$: $1 + \frac{1}{2} + \frac{1}{3}$  
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E.g., consider a committee

Points per voter:

$v_1$: $1 + \frac{1}{2}$
$v_2$: $1 + \frac{1}{2}$
$v_3$: $1 + \frac{1}{2} + \frac{1}{3}$
$v_4$: $1 + \frac{1}{2}$
$v_5$: $1 + \frac{1}{2}$
$v_6$: $0$
$v_7$: 
$v_8$: 
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E.g., consider a committee

Points per voter:

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\( v_5: 1 + \frac{1}{2} \)
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\( v_7: 0 \)

\( v_8: \)
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Assume voter \( v \) approves \( t \) members of a committee \( W \). Then \( v \) gives to \( W \) the following number of points:

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\]

E.g., consider a committee

Points per voter:

\( v_1: \ 1 + \frac{1}{2} \)

\( v_3: \ 1 + \frac{1}{2} + \frac{1}{3} \)

\( v_5: \ 1 + \frac{1}{2} \)

\( v_7: \ 0 \)

\( v_2: \ 1 + \frac{1}{2} \)

\( v_4: \ 1 + \frac{1}{2} \)

\( v_6: \ 0 \)

\( v_8: \ 1 \)
**Proportional Approval Voting (Thiele)**

Assume voter \( v \) approves \( t \) members of a committee \( W \). Then \( v \) gives to \( W \) the following number of points:

\[
\sum_{i=1}^{t} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{t}
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E.g., consider a committee

Points per voter:

- \( v_1: 1 + \frac{1}{2} \)
- \( v_2: 1 + \frac{1}{2} \)
- \( v_3: 1 + \frac{1}{2} + \frac{1}{3} \)
- \( v_4: 1 + \frac{1}{2} \)
- \( v_5: 1 + \frac{1}{2} \)
- \( v_6: 0 \)
- \( v_7: 0 \)
- \( v_8: 1 \)

Sum of points = \( 8 + \frac{5}{6} \)
Proportional Approval Voting (Thiele)

Assume voter $v$ approves $t$ members of a committee $W$. Then $v$ gives to $W$ the following number of points:

$$\sum_{i=1}^{t} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{t}$$

E.g., consider a committee

Committee with the highest score wins the election.

Points:

$v_1$: 1
$v_2$: 1
$v_3$: $1 + \frac{1}{2} + \frac{1}{3}$
$v_4$: $1 + \frac{1}{2}$
$v_5$: $1 + \frac{1}{2}$
$v_6$: 0
$v_7$: 0
$v_8$: 1

Sum of points = $8 + \frac{5}{6}$
Proportional Approval Voting is welfarist

The welfare vector of a committee $W$ is defined as:

$$\left( |A_1 \cap W|, |A_2 \cap W|, \ldots, |A_n \cap W| \right)$$

where:

$A_i$ is the set of candidates approved by voter $i$

$|A_i \cap W|$ is the number of representatives of $i$
Proportional Approval Voting is welfarist

The welfare vector of a committee $W$ is defined as:

$\left( |A_1 \cap W|, |A_2 \cap W|, \ldots, |A_n \cap W| \right)$

where:

$A_i$ is the set of candidates approved by voter $i$

$|A_i \cap W|$ is the number of representatives of $i$

A rule is welfarist if the decision which committee to elect can be made solely based on welfare vectors of the committees.
Phragmén’s Rule

- Voters *earn money with the constant speed* ($1 per time unit).
Phragmén’s Rule

- Voters earn money with the constant speed ($1 per time unit).
- In the first moment when there is a group of voters $S$ who all have $n$ dollars in total and who all approve a not-yet selected candidate $c$, do:
Phragmén’s Rule

- Voters **earn money with the constant speed** ($1 per time unit).
- In the first moment when there is a group of voters $S$ who all have $n$ dollars in total and who **all approve a not-yet selected candidate** $c$, do:
  1. Add $c$ to the committee.
  2. Make voters from $S$ pay for $c$ (resetting their budget to 0).
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$v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ |
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k = 12
\]
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\[ t_0 = 0 \]
Phragmén’s Rule

- Voters earn money with the constant speed ($1 per time unit).
- In the first moment when there is a group of voters $S$ who all have $n$ dollars in total and who all approve a not-yet selected candidate $c$, do:
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$t_2 = 4$

$t_1 = 2$

$t_0 = 0$
Phragmén’s Rule

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$$k = 12$$
PAV versus Phragmén’s Rule

Which of the two rules is better?
PAV versus Phragmén’s Rule

Which of the two rules is better?

• Both Thiele and Phragmén argued that their rules are proportional by how they behave on party-list profiles.
PAV versus Phragmén’s Rule

Which of the two rules is better?

• Both Thiele and Phragmén argued that their rules are proportional by how they behave on party-list profiles.

• Historically PAV was preferred since it appeared simpler.
PAV versus Phragmén’s Rule

Which of the two rules is better?

- Both Thiele and Phragmén argued that their rules are proportional by how they behave on party-list profiles.

- Historically PAV was preferred since it appeared simpler.

- Current research suggest that PAV is better in terms of proportionality.
Two Arguments in Favour of PAV

First Argument: Axioms for Cohesive Groups
How to define proportionality for more complex preferences?

$v_1$: 

$v_2$: 

$v_3$: 

$v_4$: 

$v_5$: 

$v_6$: 

$v_7$: 

$v_8$: 
How to define proportionality for more complex preferences?

ν₁:

ν₂:

ν₃:

ν₄:

ν₅:

ν₆:

ν₇:

ν₈:
How to define proportionality for more complex preferences?

For $k = 4$ these voters should approve (on average) 2 candidates in the selected committee.
How to define proportionality for more complex preferences?

$v_1$: 

$v_2$: 

$v_3$: 

$v_4$: 

$v_5$: 

$v_6$: 

$v_7$: 

$v_8$: 

How to define proportionality for more complex preferences?

For $k = 4$ these voters should approve (on average) 1 candidate in the selected committee.
How to define proportionality for more complex preferences?

\[ \frac{v_1}{v_2} : \frac{v_3}{v_4} : \frac{v_5}{v_6} : \frac{v_7}{v_8} \]

**Definition:** Each group with at least \( \ell n/k \) voters who approve at least \( \ell \) same candidates should have on average at least \( \ell \) representatives in the elected committee.
How to define proportionality for more complex preferences?

**Definition:** Each group with at least $\ell n/k$ voters who approve at least $\ell$ same candidates should have on average at least $\ell$ representatives in the elected committee.

For $k = 4$ these voters should approve (on average) 2 candidates in the selected committee.
How to define proportionality for more complex preferences?

Definition: Each group with at least $\ell n/k$ voters who approve at least $\ell$ same candidates should have on average at least $\ell$ representatives in the elected committee.

Does there exist a system which satisfies this property?
How to define proportionality for more complex preferences?

**Definition:** Each group with at least \( \ell n/k \) voters who approve at least \( \ell \) same candidates should have on average at least \( \ell \) representatives in the elected committee.

Does there exist a system which satisfies this property?

\[
\begin{align*}
\nu_1 &: \{a, d\} & \nu_7 &: \{b, c\} \\
\nu_2 &: \{a\} & \nu_8 &: \{c\} \\
\nu_3 &: \{a\} & \nu_9 &: \{c\} & n &= 12 \\
\nu_4 &: \{a, b\} & \nu_{10} &: \{c, d\} \\
\nu_5 &: \{b\} & \nu_{11} &: \{d\} & k &= 3 \\
\nu_6 &: \{b\} & \nu_{12} &: \{d\}
\end{align*}
\]
How to define proportionality for more complex preferences?

**Definition:** Each group with at least $\ell n/k$ voters who approve at least $\ell$ same candidates should have on average at least $\ell - 1$ representatives in the elected committee.

But PAV satisfies a slightly weaker property!
How to define proportionality for more complex preferences?

**Definition:** Each group with at least $\ell n/k$ voters who approve at least $\ell$ same candidates should have on average at least $\ell - 1$ representatives in the elected committee.

But PAV satisfies a slightly weaker property!

Phragmén’s Rule would satisfy it only if we replaced $\ell - 1$ with $(\ell - 1)/2$. 
Two Arguments in Favour of PAV

Second Argument: Axiomatic Extensions of Apportionment Methods
Let’s look at this instance

We have 9 voters, 9 candidates, and our goal is to select a committee of size $k = 4$. 

$v_1$: 
$v_2$: 
$v_3$: 
$v_4$: 
$v_5$: 
$v_6$: 
$v_7$: 
$v_8$: 
$v_9$: 
Let’s look at this instance

We have 9 voters, 9 candidates, and our goal is to select a committee of size $k = 4$. 

$v_1$: Party 1 (5 votes) 
$v_2$: Party 2 (2 votes) 
$v_3$: Party 3 (2 votes)
Let’s look at this instance

We have 9 voters, 9 candidates, and our goal is to select a committee of size $k = 4$.

$v_1$:

$v_2$:

$v_3$:

$v_4$:

$v_5$:

$v_6$:

$v_7$:

$v_8$:

$v_9$:

Party 1 (5 votes)

- Party 1 gets 2 seats.
- Party 2 gets 1 seat.
- Party 3 gets 1 seat.

For example
Some basic axiomatic properties: Symmetry
Some basic axiomatic properties: Symmetry

\[ \nu_1: \]
\[ \nu_2: \]
\[ \nu_3: \]
\[ \nu_4: \]
\[ \nu_5: \]
\[ \nu_6: \]
\[ \nu_7: \]
\[ \nu_8: \]
Some basic axiomatic properties: Symmetry

$v_1$:
$v_2$:
$v_3$:
$v_4$:
$v_5$:
$v_6$:
$v_7$:
$v_8$:
Some basic axiomatic properties: Symmetry

\[ v_1 : \]
\[ v_2 : \]
\[ v_3 : \]
\[ v_4 : \]
\[ v_5 : \]
\[ v_6 : \]
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\[ v_8 : \]
Some basic axiomatic properties: Consistency
Some basic axiomatic properties: Consistency

\( \nu_1 \):

\( \nu_2 \):

\( \nu_3 \):

\( \nu_4 \):

\( \nu_5 \):

\( \nu_6 \):

\( \nu_7 \):

\( \nu_8 \):
Some basic axiomatic properties: Consistency

\[ \nu_1 : \]
\[ \nu_2 : \]
\[ \nu_3 : \]
\[ \nu_4 : \]
\[ \nu_5 : \]
\[ \nu_6 : \]
\[ \nu_7 : \]
\[ \nu_8 : \]
Some basic axiomatic properties: Continuity
Some basic axiomatic properties: Continuity

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<thead>
<tr>
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<td><img src="image7.png" alt="Image" /></td>
<td><img src="image8.png" alt="Image" /></td>
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Some basic axiomatic properties: Continuity

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<tr>
<th>$E_1$:</th>
<th>$v_1$:</th>
<th>$v_2$:</th>
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<td>$E_2$:</td>
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<td>$v_8$:</td>
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Then, there exists (possibly very large) value $z$ such that:

$$z \cdot E_1 + E_2$$
Theorem: Proportional Approval Voting is the only ABC ranking rule that satisfies symmetry, consistency, continuity and D’Hondt proportionality.
Theorem: Proportional Approval Voting satisfies symmetry, consistency, continuity and D’Hondt proportionality.

Axiomatic Characterisations

**Theorem:** Proportional Approval Voting is the only ABC ranking rule that satisfies symmetry, consistency, continuity and D’Hondt proportionality.

PAV versus Phragmén’s Rule
# PAV versus Phragmén’s Rule

$$k = 12$$

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**Phragmén’s Rule**

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**Thiele’s Rule (PAV)**
PAV versus Phragmén’s Rule

\[ k = 12 \]

**Phragmén’s Rule**

**Thiele’s Rule (PAV)**

**Proportionality with respect to power**

**Proportionality with respect to welfare**
PAV versus Phragmén’s Rule

\[ k = 12 \]

Phragmén’s Rule

Thiele’s Rule (PAV)

Proportionality with respect to power

- priceability,
- laminar proportionality

Proportionality with respect to welfare

- Pigou-Dalton
- EJR
PAV versus Phragmén’s Rule

\[ k = 12 \]

Phragmén’s Rule

\[
\begin{array}{c|c|c|c|c|c|c}
| c_4 & c_5 & c_6 & c_3 & c_{13} & c_{14} & c_{15} \\
| \hline
| c_2 & c_{10} & c_{11} & c_{12} & c_1 & c_7 & c_8 & c_9 \\
| \hline
\end{array}
\]

Thiele’s Rule (PAV)

\[
\begin{array}{c|c|c|c|c|c|c}
| c_4 & c_5 & c_6 & c_3 & c_{13} & c_{14} & c_{15} \\
| \hline
| c_2 & c_{10} & c_{11} & c_{12} & c_1 & c_7 & c_8 & c_9 \\
| \hline
\end{array}
\]

Proportionality with respect to power

- priceability,
- laminar proportionality

Proportionality with respect to welfare

- Pigou-Dalton
- EJR
Two New Notions of Proportionality

Fair distribution of power

(failed by PAV)
Laminar Proportionality: Examples

It describes how the rule should behave on certain well-behaved profiles
Laminar Proportionality: Examples

\[ k = 8 \]

\[
\begin{array}{ccc}
  c_4 & c_8 & c_{12} \\
  c_3 & c_7 & c_{11} \\
  c_2 & c_6 & c_{10} \\
  c_1 & c_5 & c_9 \\
\end{array}
\]

\[ v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5 \quad v_6 \quad v_7 \quad v_8 \]

Party list profiles
Laminar Proportionality: Examples

$$k = 8$$

\[
\begin{array}{ccc}
 c_4 & c_8 & c_{12} \\
 c_3 & c_7 & c_{11} \\
 c_2 & c_6 & c_{10} \\
 c_1 & c_5 & c_9 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
 v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \\
\end{array}
\]

Party list profiles
Laminar Proportionality: Examples

\[ k = 4 \]

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Party lists with a common leader
### Laminar Proportionality: Examples

**Party lists with a common leader**

\[ k = 4 \]

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<td>( c_7 )</td>
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Laminar Proportionality: Examples

\[ k = 12 \]

| \(c_{10}\) | \(c_{17}\) |
| \(c_{16}\) | \(c_{15}\) |
| \(c_{14}\) | \(c_{20}\) |
| \(c_{13}\) | \(c_{19}\) |
| \(c_{12}\) | \(c_{18}\) |

Subdivided parties
Laminar Proportionality: Examples

\[ k = 12 \]

Subdivided parties
Laminar Proportionality: Definition
Laminar Proportionality: Definition

We say that a profile \((P, k)\) is **laminar** if:

1. \(P\) is unanimous, or
Laminar Proportionality: Definition

We say that a profile \((P, k)\) is laminar if:

1. \(P\) is unanimous, or

2. There exists a **unanimously approved** candidate \(c\), and \((P\setminus \{c\}, k - 1)\) is laminar, or

\[\frac{|P_1|}{k_1} = \frac{|P_2|}{k_2}\] such that \(P = P_1 + P_2\) and \(k = k_1 + k_2\)
Laminar Proportionality: Definition

We say that a profile $(P, k)$ is laminar if:

1. $P$ is unanimous, or

2. There exists a unanimously approved candidate $c$, and $(P \setminus \{c\}, k - 1)$ is laminar, or

$$k = 4$$
Laminar Proportionality: Definition

We say that a profile \((P, k)\) is laminar if:

1. \(P\) is unanimous, or
2. There exists a \textbf{unanimously approved} candidate \(c\), and \((P\setminus\{c\}, k - 1)\) is laminar, or
3. There are two disjoint laminar instances \((P_1, k_1)\) and \((P_2, k_2)\) with \(|P_1|/k_1 = |P_2|/k_2\) such that \(P = P_1 + P_2\) and \(k = k_1 + k_2\) \((P, k)\).

\[ k = 4 \]

\[ \begin{array}{c|c|c}
| \hline
v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\
| \hline
| \hline
\end{array} \]

\[ \begin{array}{ccc}
| \hline
c_1 & c_2 & c_3 \\
| \hline
\end{array} \]

\[ \begin{array}{ccc}
| \hline
4 & 7 & 6 \\
| \hline
\end{array} \]

\[ \begin{array}{c}
| \hline
8 \\
| \hline
\end{array} \]
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\[k = 12\]

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We say that a profile \((P, k)\) is \textit{laminar} if:

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\[k_1 = 8 \quad \text{and} \quad k_2 = 4\]
Laminar Proportionality: Definition

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We say that a rule is laminar proportional if it behaves well on laminar profiles.
Welfarist Rules

The welfare vector of a committee $W$ is defined as:

$$\left( |A_1 \cap W|, |A_2 \cap W|, \ldots, |A_n \cap W| \right)$$

where:

- $A_i$ is the set of candidates approved by voter $i$
- ($|A_i \cap W|$ is the number of representatives of $i$)

A rule is welfarist if the decision which committee to elect can be made solely based on welfare vectors of the committees.
No welfarist rule can be laminar proportional
No welfarist rule can be laminar proportional

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No welfarist rule can be laminar proportional

Welfare (6, 6, 6, 6, 6, 6, 6, 6) is preferred over welfare (7, 7, 7, 7, 5, 5, 5, 5)
No welfarist rule can be laminar proportional

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Welfare (6, 6, 6, 6, 6, 6, 6, 6) is preferred over welfare (7, 7, 7, 7, 5, 5, 5, 5)

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Welfare (7, 7, 7, 7, 5, 5, 5, 5) is preferred over welfare (6, 6, 6, 6, 6, 6, 6, 6)
Priceability
Priceability

A price system is a pair $ps = (p, \{p_i\}_{i \in [n]})$, where $p > 0$ is a price, and for each voter $i \in [n]$, there is a payment function $p_i: C \to [0,1]$ such that:

1. A voter can only pay for candidates she approves of,
2. A voter can spend at most one dollar.
Priceability

A price system is a pair $ps = (p, \{p_i\}_{i \in [n]})$, where $p > 0$ is a price, and for each voter $i \in [n]$, there is a payment function $p_i : C \to [0,1]$ such that:

1. A voter can only pay for candidates she approves of,
2. A voter can spend at most one dollar.

We say that a price system $ps = (p, \{p_i\}_{i \in [n]})$ supports a committee $W$ if the following hold:

1. For each elected candidate, the sum of the payments to this candidate equals the price $p$. 
Priceability

A price system is a pair \( ps = (p, \{ p_i \}_{i \in [n]} ) \), where \( p > 0 \) is a price, and for each voter \( i \in [n] \), there is a payment function \( p_i : C \to [0,1] \) such that:
1. A voter can only pay for candidates she approves of,
2. A voter can spend at most one dollar.

We say that a price system \( ps = (p, \{ p_i \}_{i \in [n]} ) \) supports a committee \( W \) if the following hold:
1. For each elected candidate, the sum of the payments to this candidate equals the price \( p \).
2. No candidate outside of the committee gets any payment.
Priceability

A price system is a pair $ps = (p, \{p_i\}_{i \in [n]})$, where $p > 0$ is a price, and for each voter $i \in [n]$, there is a payment function $p_i: C \to [0,1]$ such that:

1. A voter can only pay for candidates she approves of,
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We say that a price system $ps = (p, \{p_i\}_{i \in [n]})$ supports a committee $W$ if the following hold:

1. For each elected candidate, the sum of the payments to this candidate equals the price $p$.
2. No candidate outside of the committee gets any payment.
3. There exists no unelected candidate whose supporters, in total, have a remaining unspent budget of more than $p$. 
Priceability: Example

The price is \( p = 0.5 \).

1. \( v_1 \) pays \( \frac{1}{6} \) for \( c_1, c_2 \) and \( c_3 \) and \( \frac{1}{2} \) for \( c_4 \).

\[
k = 12
\]

\[
\begin{array}{cccccc}
\text{c}_4 & \text{c}_5 & \text{c}_6 \\
\text{c}_3 & \text{c}_{13} & \text{c}_{14} & \text{c}_{15} \\
\text{c}_2 & \text{c}_{10} & \text{c}_{11} & \text{c}_{12} \\
\text{c}_1 & \text{c}_7 & \text{c}_8 & \text{c}_9 \\
\end{array}
\]

Phragmén’s Rule
Priceability: Example

\[ k = 12 \]

The price is \( p = 0.5 \).

1. \( v_1 \) pays \( \frac{1}{6} \) for \( c_1, c_2 \) and \( c_3 \) and \( \frac{1}{2} \) for \( c_4 \).
2. \( v_2 \) pays \( \frac{1}{6} \) for \( c_1, c_2 \) and \( c_3 \) and \( \frac{1}{2} \) for \( c_5 \).
3. \( v_3 \) pays \( \frac{1}{6} \) for \( c_1, c_2 \) and \( c_3 \) and \( \frac{1}{2} \) for \( c_6 \).
Priceability: Example

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Priceability: Example

$k = 12$

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3. $v_3$ pays $\frac{1}{6}$ for $c_1$, $c_2$ and $c_3$ and $\frac{1}{2}$ for $c_6$.
4. $v_4$ pays $\frac{1}{2}$ for $c_7$ and $c_{10}$.
5. $v_5$ pays $\frac{1}{2}$ for $c_8$ and $c_{11}$.
6. $v_6$ pays $\frac{1}{2}$ for $c_9$ and $c_{12}$. 
No welfarist rule can be priceable
No welfarist rule can be priceable

Profile 1:

Profile 2:
Core
Core: Definition

We say that a committee $W$ is in the core if there exists no group of voters $S$ and a subset of candidates $T$ such that:

1. $\frac{|T|}{k} \leq \frac{|T|}{n}$, and 

2. Each voter in $S$ prefers $T$ to $W$. 
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$$k = 12$$

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$v_1$  $v_2$  $v_3$  $v_4$  $v_5$  $v_6$
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$k = 12$

Not in the core!
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Core contradicts the Pigou-Dalton principle!

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$\text{Core contradicts the Pigou-Dalton principle!}$

$\text{Not in the core!}$

$k = 12$

$\text{Theorem: PAV gives the best possible Approximation of the core subject to Satisfying the Pigou-Dalton principle!}$
No welfarist rule can satisfy the core
No welfarist rule can satisfy the core

Profile 1:

\[
\begin{array}{cccc}
  c_1 & c_2 & c_3 & c_4 \\
  c_5 &   &   &   \\
  c_6 &   &   &   \\
  c_7 &   &   &   \\
  c_8 &   &   &   \\
  c_9 &   &   &   \\
  c_10 & c_{11} & & \\
\end{array}
\]

Profile 2:

\[
\begin{array}{cccc}
  c_1 & & & \\
  c_2 & c_8 & & \\
  c_3 & c_9 & & \\
  c_4 & c_{10} & & \\
  c_5 & c_{11} & & \\
  c_6 & c_{12} & & \\
  c_7 & c_{13} & & \\
\end{array}
\]

Figure 4: A diagram illustrating the reasoning in Theorem 10, showing that there exists no welfarist rule that satisfies the core property. Here each rectangle denotes a single candidate who is approved by the respective voters. For example, in Profile 1 candidates $c_5$ and $c_6$ are approved by $\{v_1, v_2, v_3, v_4\}$ and $c_{11}$ is approved by $\{v_5, v_6\}$. Profile 2 depicts the case for $x = 3$ and $y = 6$. Profile 3 depicts the case for $x = 2$ and $y = 5$. 
Open questions:

• Does there always exist a committee in the core?

• Does there always exist a Pareto-optimal priceable committee?

• What is the best possible core-approximation among welfarist rules?
Beyond proportionality: diversity
(extreme form of degressive proportionality)
Proportional Approval Voting (Thiele)

Assume voter $v$ approves $t$ members of a committee $W$. Then $v$ gives to $W$ the following number of points:

$$\sum_{i=1}^{t} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{t}$$

E.g., consider a committee

Points per voter:

$v_1$: $1 + \frac{1}{2}$
$v_2$: $1 + \frac{1}{2}$
$v_3$: $1 + \frac{1}{2} + \frac{1}{3}$
$v_4$: $1 + \frac{1}{2}$
$v_5$: $1 + \frac{1}{2}$
$v_6$: $0$
$v_7$: $0$
$v_8$: $1$

Sum of points $= 8 + \frac{5}{6}$
Approval Chamberlin-Courant rule

Voter $v$ gives to $W$ one point if $v$ approves someone from $W$ and zero points otherwise.
Approval Chamberlin-Courant rule

Voter $v$ gives to $W$ one point if $v$ approves someone from $W$ and zero points otherwise.

E.g., consider a committee

Points per voter:

$\begin{align*}
  v_1 &: 1 \\
  v_3 &: 1 \\
  v_5 &: 1 \\
  v_7 &: 1 \\
  v_2 &: 1 \\
  v_4 &: 1 \\
  v_6 &: 1 \\
  v_8 &: 1 
\end{align*}$

Sum of points = 8
**PAV for Euclidean Preferences**

<table>
<thead>
<tr>
<th>uniform on a circle</th>
<th>Gaussian</th>
<th>4 Gaussians</th>
<th>uniform on a square</th>
</tr>
</thead>
</table>

PAV for Euclidean Preferences

- Uniform on a circle
- Gaussian
- 4 Gaussians
- Uniform on a square

PAV for Euclidean Preferences

uniform on a circle

Gaussian

4 Gaussians

uniform on a square

Multiwinner AV for Euclidean Preferences

uniform on a circle  Gaussian  4 Gaussians  uniform on a square

PAV for Euclidean Preferences

uniform on a circle  

Gaussian

4 Gaussians

uniform on a square

Approval Chamberlin–Courant for Euclidean Preferences

uniform on a circle  
Gaussian  
4 Gaussians  
uniform on a square

One More Interesting Application
Proportionality for rankings (designing an Internet search engine)

An output is a ranking over the candidates

\[ \nu_1 : \nu_2 : \nu_3 : \nu_4 : \nu_5 \]
Proportionality for rankings
(designing an Internet search engine)

• Type “Apple” into Google image search
  – some users search for Apple **computers** (70%)
  – some users search for the **fruit** (30%)
Proportionality for rankings (designing an Internet search engine)

• Type “Apple” into Google image search
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• Google outputs an ordered list of images
Proportionality for rankings (designing an Internet search engine)

• Type “Apple” into Google image search
  – some users search for Apple computers (70%)
  – some users search for the fruit (30%)

• Google outputs an ordered list of images

• Is it appropriate for Google to
  – only output images of computers?
  – alternate fruit and computer?
Proportionality for rankings
(designing an Internet search engine)

- Type “Apple” into Google image search
  - some users search for Apple **computers** (70%)
  - some users search for the **fruit** (30%)
- Google outputs an ordered list of images
- Is it appropriate for Google to
  - only output images of **computers**?
  - alternate **fruit** and **computer**?
- Ideally, we’ll see
  - 3 pieces of **fruit** and 7 **computers** in top 10 results,
  - 6 pieces of **fruit** and 14 **computers** in top 20 results...
Proportionality for rankings
(designed an Internet search engine)

- Example „apple”

![Google search results for "apple" showing various images of apples and Apple logos.](image-url)
Proportionality for rankings
(designing an Internet search engine)

- Example „jaguar”
Proportionality for rankings
/designing an Internet search engine/

- Example „???”
Proportionality for rankings 
(designing an Internet search engine)

- Example „Armstrong“ (Neil, Lance, Louis)
Proportionality for rankings (designing an Internet search engine)

An output is a ranking over the candidates.

How should we define proportionality for rankings?
Proportionality for rankings (designing an Internet search engine)

How should we define proportionality for rankings?

We would like each prefix of the ranking to be proportional!

An output is a ranking over the candidates.
From committees to rankings

Committee monotonicity
If a rule picks some set $W$ of winners when electing parliament of size $k$, this rule should pick $W'$ such that $W \subset W'$, when electing a parliament of size $k + 1$. 
From committees to rankings

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Committee monotonicity can be used to create a ranking:
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Committee size $k = 1$:
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Committee size $k = 2$: 

Committee monotonicity can be used to create a ranking:
From committees to rankings

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If a rule picks some set \( W \) of winners when electing parliament of size \( k \), this rule should pick \( W' \) such that \( W \subseteq W' \), when electing a parliament of size \( k + 1 \).

Committee size \( k = 2 \):
From committees to rankings

**Committee monotonicity**
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Committee size $k = 3$:
From committees to rankings

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Committee monotonicity can be used to create a ranking:
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left | right
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Thanks!

https://github.com/elektronaj/MW2D