On positional strategies over finite arenas

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Disclaimer. Credits to many authors. All errors are mine own.

Perfect information games

Finite duration games (like chess) can be presented as games on graphs.

Complexity of solving such games relies on the structure of the graph ($\longrightarrow$ alternating reachability).

Infinite duration games are usually modelled as games on **colored** graphs.

Complexity relies on the structure of both: the graph and the **winning condition**.

Games on (edge colored) graphs

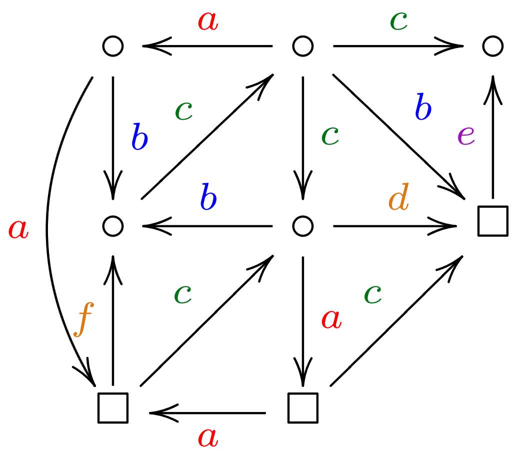
$$G = \langle Pos_\exists, Pos_\forall, Move, C, rank, W_\exists, W_\forall \rangle,$$

where $Pos = Pos_\exists \dot\cup Pos_\forall$, $Move \subseteq Pos \times Pos$,

$rank : Move \to C$, $W_\exists, W_\forall \subseteq C^\omega, W_\forall \cap W_\exists = \emptyset$.

$\circ$ *Eve*

$\square$ *Adam*



Player who cannot move, loses — the opponent wins.

An infinite play $p_0, p_1, \ldots$ is won by Q iff $rank(p_0, p_1), rank(p_1, p_2) \ldots \in W_{\mathbf{Q}}$.

Otherwise there is a draw.

Strategies

A strategy (for Eve, say) is a partial mapping $Move^* \to Move$ defined for paths ending in a position of Eve.

It is **winning** if any play π consistent with the strategy is won by Eve.

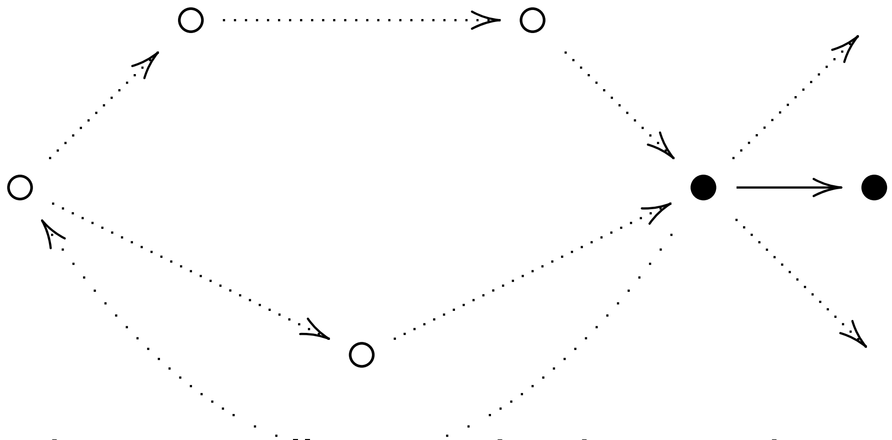
A game is **determined** if, for any position, one of the players has a winning strategy, or both players have strategies to achieve (at least) a draw.

Reachability game: No colors. Infinite play is always a draw.

Zermelo's theorem: Reachability games are determined.

Positional strategies

A **positional** strategy depends only on the actual position.



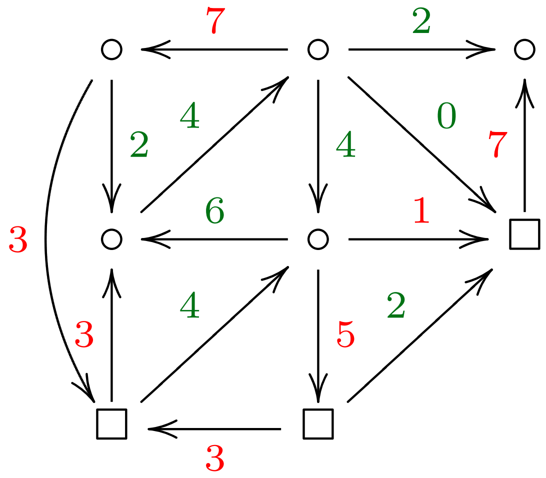
Positional determinacy — all strategies in question are positional.

Reachability games are **positionally determined** (on all graphs).

What else?

Parity games

$C = \{0, 1, \dots n\}$.



Eve wants ∞ **even**, Adam wants ∞ **odd**, **maximal** wins.

$W_\exists = \{u \in C^\omega : \limsup_{n\to\infty} u_n$ is **even** $\}$

$W_\forall = \{u \in C^\omega : \limsup_{n\to\infty} u_n$ is **odd** $\}$.

Parity games are positionally determined on all graphs (**Emerson & Jutla 1991**, **Mostowski 1991**).

Essentially, it is the **only** condition with this property.

Suppose that $W \subseteq C^\omega$ is uniform ($W = CW$), and any game

$$\langle Pos_\exists, Pos_\forall, Move, C, rank, W, \overline{W} \rangle$$

is positionally determined. Then W is a parity condition **up to renaming the letters** (not necessarily 1:1). That is, there is n and $\mathbf{h} : C \to \{0, 1, \ldots, n\}$, such that

$$u \in W \quad \text{iff} \quad \limsup_{i \to \infty} \mathbf{h}(u_i) \text{ is even}$$

(**Colcombet & N. 2006**).

Positional determinacy over finite graphs

There are more conditions that guarantee positional determinacy. For example

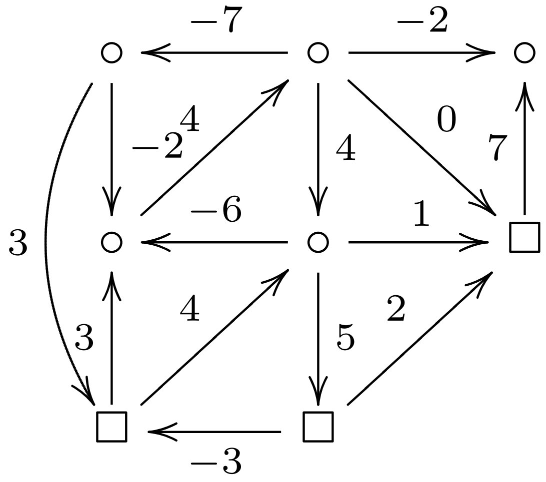
$$W = \{x \in \{0,1\}^\omega : \lim_{n \to \infty} \frac{x_1 + \ldots + x_n}{n} = 0\}$$

Clearly, W cannot be renamed to parity condition.

(Besides, it is $\mathbf{\Pi^0_3}$-complete, whereas the parity conditions are in $\mathbf{\Delta^0_3}$.)

Positional determinacy of games on finite graphs with $W_\exists = W, W_\forall = \overline{W}$ follows from a more general property.

Mean-payoff optimization games (over finite arenas)



Adam pays to Eve the amount q, while passing through an edge $\xrightarrow{q}$.

Each player wants to maximize her/his income asymptotically on average.

For each position p, there is a **compromise value** $val(p)$, which Eve and Adam can reach using **positional strategies** (**Ehrenfeucht & Mycielski 1979**).

More specifically, let, for a play $\pi = (p_0, p_1, \ldots)$ and $n \geq 1$,

$$val_n(\pi) \quad = \quad \frac{rank(p_0, p_1) + rank(p_1, p_2) \ldots + rank(p_{n-1}, p_n)}{n}.$$

Let $play(s_E, s_A, p)$ be a unique play determined by strategies s_E and s_A, and position p.

Ehrenfeucht & Mycielski 1979 show that, for any p, there are positional strategies $\overline{\mathbf{s_E}}, \overline{\mathbf{s_A}}$, such that

$$val(p) \quad =_{def} \quad \lim_{n \to \infty} val_n \left(play(\overline{\mathbf{s_E}}, \overline{\mathbf{s_A}}, p) \right),$$

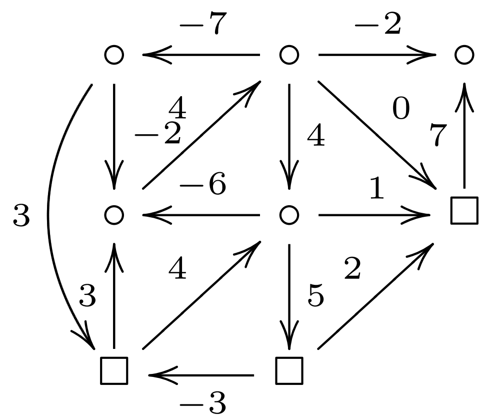
satisfies

$$\begin{aligned} val(p) \quad &= \quad \inf_{s_A} \sup_{s_E} \limsup val_n \left(play(s_E . s_A, p) \right) \\ &= \quad \sup_{s_E} \inf_{s_A} \liminf val_n \left(play(s_E . s_A), p \right) \end{aligned}$$

where s_E, s_A range over all strategies.

Mean-payoff winning conditions

$C \subseteq \mathbb{Z}$ (finite).



For a fixed threshold d,

$$W_\exists = \{x : \liminf \frac{x_1 + \ldots + x_n}{n} \geq d\}$$
$$W_\forall = \{x : \limsup \frac{x_1 + \ldots + x_n}{n} < d\}.$$

Can we characterize positional determinacy on finite arenas by a class of winning conditions?

(Such a class should somehow subsume parity games.)

Gimbert 2006 gave elegant structural conditions that characterize positional determinacy (not necessarily uniform) on all finite graphs.

Note. For finite arenas, winning conditions may admit various presentations.

Equivalence of winning conditions

For example, the aforementioned condition

$$W = \{x \in \{0,1\}^\omega : \lim_{n\to\infty} \frac{x_1 + \ldots + x_n}{n} = 0\}$$

is over finite arenas equivalent to

$$W' = \{x \in \{0,1\}^\omega : \lim_{n\to\infty} x_n = 0\}$$

More generally,

Periodicity lemma. Two winning conditions $(W_\exists, W_\forall)$ and $(W'_\exists, W'_\forall)$ are equivalent over finite arenas iff they contain the same ultimately periodic words.

Proof of the lemma.

(only if)

If an ultimately periodic word u separates the two conditions, we can take a game that essentially consists of this word.

(if)

Let s_E be a positional strategy for Eve winning from position p with the condition $(W_\exists, W_\forall)$.

Suppose Adam has a positional strategy s'_A from p to achieve at least a draw with the condition $(W'_\exists, W'_\forall)$.

Then the labeling of $play(s_E, s'_A, p)$ separates the two conditions.

Some consequences of periodicity lemma.

Parity *vs.* boundedness

If $|w|_a$ denotes the number of occurrences of a in w, let

$$W'_\exists = \{u : (\exists M\, \forall a \textbf{ odd } \in C)\ |w_a| \le M, \text{ where } w \text{ ranges over all finite factors of } u \text{ s.t. the maximal color of } w \text{ is } a\}$$

$$W'_\forall = \{u : (\exists M\, \forall b \textbf{ even } \in C)\ |w_b| \le M, \text{ where } w \text{ ranges as above up to } a \rightleftharpoons b\}$$

Then $C^\omega - (W'_\exists \cup W'_\forall) \neq \emptyset$, but any game on finite arena with the winning condition $(W'_\exists, W'_\forall)$ is equivalent to parity game, cf. **Colcombet & Loeding 2009**.

What can we gain by that?

If a winning condition $(W'_\exists, W'_\forall)$ is equivalent to $(W_\exists, W_\forall)$, for some $W'_\exists \subseteq W_\exists$ and $W'_\forall \subseteq W_\forall$ then it is the same for any **separating pair** $(\mathbf{W}''_\exists, \mathbf{W}''_\forall)$, i.e.,

$$W'_\exists \subseteq \mathbf{W}''_\exists \subseteq W_\exists$$
$$W'_\forall \subseteq \mathbf{W}''_\forall \subseteq W_\forall.$$

This may have impact on complexity if $\mathbf{W}''_\exists$ and $\mathbf{W}''_\forall$ are simpler than the original condition. Cf. **Calude *et al.* 2017**, and **Bojańczyk & Czerwiński 2018**.

Specifically, for games with $< \mathbf{M}$ positions, there is a "simple" separator of

$$W_\exists^{(\mathbf{M})} = \{u : (\forall a \text{ odd}) \ |w_a| \leq \mathbf{M}\}$$
$$W_\forall^{(\mathbf{M})} = \{u : (\forall b \text{ even}) \ |w_b| \leq \mathbf{M}\}$$

where w ranges as above.

Example: intrinsically non-regular mean-payoff condition

$$W_\exists = \{x \in \{-1,0,1\}^\omega : \liminf_{n\to\infty} \frac{x_1 + \ldots + x_n}{n} > 0\}$$
$$W_\forall = \{x \in \{-1,0,1\}^\omega : \limsup_{n\to\infty} \frac{x_1 + \ldots + x_n}{n} \leq 0\}.$$

There is no ω-regular language L, such that $W_\exists$ and L contain the same ultimately periodic words.

Note. For an ultimately periodic word $x \in \mathbb{Z}^\omega$,
$\liminf_{n\to\infty} \frac{x_1+\ldots+x_n}{n} > 0$ iff $\lim_{n\to\infty} x_1 + \ldots + x_n = +\infty$.

In the search of a characterization

Let, for $x \in \mathbb{Z}^\omega$,

$$\chi(x) \quad = \quad \begin{cases} 1 & \text{if} \qquad \lim_{n\to\infty} x_1 + \ldots + x_n = +\infty \\ -1 & \text{if} \qquad \lim_{n\to\infty} x_1 + \ldots + x_n = -\infty \\ 0 & \text{otherwise} \end{cases}$$

For $x = \left(x^{(1)}, \ldots, x^{(k)}\right) \in \left(\mathbb{Z}^k\right)^\omega$, let

$$\vec{\chi}(x) \quad = \quad \left(\chi(x^{(1)}), \ldots, \chi(x^{(k)})\right)$$

The **lexicographic energy condition**:

$$\vec{\chi}(x) \quad >_{lex} \quad \vec{0}.$$

Properties

Let $W_\exists^C$ be the set of words in $\left(\mathbb{Z}^k\right)^\omega$ satisfying the LE condition over the alphabet $C \subseteq \mathbb{Z}^k$.

Let $W_\forall^C = \overline{W_\exists^C}$.

The LE condition guarantees **positional determinacy** over finite arenas.

It subsumes mean-payoff ($k = 1$), as well as parity:

rank		
0	$\rightarrow$	$(0, \quad 0, \quad 0, \quad 0, \quad 1)$
1	$\rightarrow$	$(0, \quad 0, \quad 0, \quad -1, \quad 0)$
2	$\rightarrow$	$(0, \quad 0, \quad 1, \quad 0, \quad 0)$
3	$\rightarrow$	$(0, \quad -1, \quad 0, \quad 0, \quad 0)$
4	$\rightarrow$	$(1, \quad 0, \quad 0, \quad 0, \quad 0)$

Partial characterization

Proposition. Let $W \subseteq C^\omega$ be prefix independent ($W = CW$), and suppose that all games on finite arenas with the winning condition $(W, \overline{W})$ are positionally determined.

Assume further that W satisfies the **permutation property**

$$(v\mathbf{xy}w)^\omega \in W \quad \text{iff} \quad (v\mathbf{yx}w)^\omega \in W.$$

Then W coincides with some **LE** condition on all ultimately periodic words: consequently, the respective games are equivalent.

It is **open** if the permutation property is necessary.

Further questions

Can we have a similar characterization of **finite-memory** determinacy over finite arenas?

Is there an efficient reduction of mean-payoff games to parity games?

Can we improve upon the complexity of solving mean-payoff games, e.g., to $n^{\mathcal{O}(\log n)}$?

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