Automata-theoretic hierarchies

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joint work with André Arnold and Henryk Michalewski

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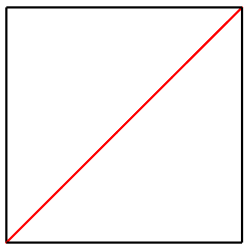
Note. This is a revised version of the slides accompanying the presentation to the Games workshop. Thanks go to Filip Murlak and Michał Skrzypczak for correcting some erroneous claims and answering some questions stated in the talk.

Why hierarchies ?

Do all intervals have rational length ?

Pythagoras: plausibly yes

Anonymous: **no !**



Are all definable subsets of a real line

Borel measurable ?

Lebesgue: plausibly yes

Suslin: **no !**

Borel sets are those generated from open intervals by countable union and complement.

$$B_0 \subseteq B_1 \subseteq \ldots B_\omega \subseteq B_{\omega+1} \ldots \ldots B_{\omega^2} \ldots$$

Behind this question...

Cantor: is every subset of a real line either countable or equinumerable with the whole line **?** (*Continuum hypothesis*)

Alexandrov & Hausdorff: true for Borel sets

Suslin: but projection of a Borel (even closed) relation may be non-Borel **!**

70 years later. Interesting sets of trees recognized by automata are usually non-Borel.

MSO definable sets of infinite words can be recognized by finite automata with the Büchi acceptance criterion.

Is this criterion sufficient for MSO definable sets of **trees** ?

Rabin: **no !**

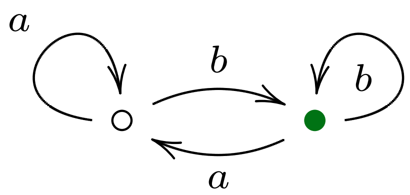
more colors needed

Plan.

- Why the index hierarchies are strict
  - deterministic automata on words
  - game languages
  - alternating automata on trees
- Separation and reduction properties
- Relation to other hierarchies and decidability issues

More colors are needed.

To understand why, we first look at deterministic automata on infinite words.



$$((a+b)^*b)^\omega$$

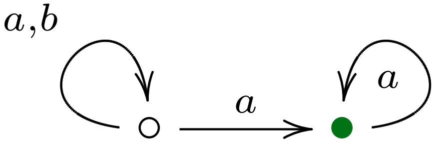
The complement $\overline{((a+b)^*b)^\omega} = (a+b)^*a^\omega$

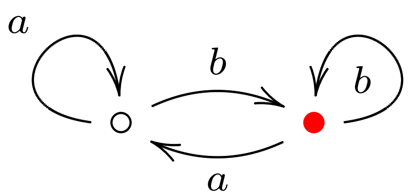
cannot be recognized by deterministic Büchi automaton.

$$\xrightarrow{a}\xrightarrow{b}\xrightarrow{a}\xrightarrow{a}\xrightarrow{b}\xrightarrow{a}\xrightarrow{a}\xrightarrow{a}\xrightarrow{b}\xrightarrow{a}\xrightarrow{a}\xrightarrow{a}\xrightarrow{a}\xrightarrow{b}\xrightarrow{a} \ldots\ldots \xrightarrow{a}\xrightarrow{b}\xrightarrow{a} \ldots$$

$$(a+b)^* a^\omega$$

Nondeterminism or dual criterion helps





Is there any pattern sufficient for all deterministic automata ?

Note: the Büchi pattern $((\circ + \bullet)^* \bullet)^\omega$ is sufficient for non-deterministic automata.

More precisely, let $R \subseteq C^\omega$.

A deterministic R-automaton is $\langle A, Q, q_I, Tr, rank \rangle$,
where $q_I \in Q$, $Tr : Q \times A \to Q$, $rank : Q \to C$.

$$
\begin{array}{llllllllll}
 & q_I & & & & & & & & \\
 & \| & & & & & & & & \\
\text{A run} & q_0 \xrightarrow{a_0} & q_1 \xrightarrow{a_1} & q_2 \xrightarrow{a_2} & q_3 \xrightarrow{a_3} & \cdots & \text{is} & \textbf{accepting} \\
\text{iff} & rank(q_0) & rank(q_1) & rank(q_2) & rank(q_3) & \cdots & \in & R.
\end{array}
$$

Is there an $R \subseteq C^\omega$, such that deterministic R-automata recognize all ω-regular languages **?**

No deterministic R-automaton (over alphabet C) may accept $\overline{R}$.

Suppose

$$L(\mathcal{A}) \quad = \quad \overline{R}$$

Create a word

$$q_0 \xrightarrow{rank(q_0)} q_1 \xrightarrow{rank(q_1)} q_2 \xrightarrow{rank(q_2)} q_3 \xrightarrow{rank(q_3)} \quad \cdots$$

$$u = \quad rank(q_0) \quad rank(q_1) \quad rank(q_2) \quad rank(q_3) \quad \cdots$$

Then

$$u \in \overline{R} \quad \Longleftrightarrow \quad u \in R,$$

a contradiction.

Remark

$$q_0 \xrightarrow{rank(q_0)} q_1 \xrightarrow{rank(q_1)} q_2 \xrightarrow{rank(q_2)} q_3 \xrightarrow{rank(q_3)} \quad \cdots$$

$$u = \quad rank(q_0) \quad rank(q_1) \quad rank(q_2) \quad rank(q_3) \quad \cdots$$

This word u is a fixed point of the mapping

$$w \quad \mapsto \quad rank \circ run(w)$$

By the Banach Fixed Point Theorem, it is a *unique* fixed point.

Other analogies — strategy stealing ?

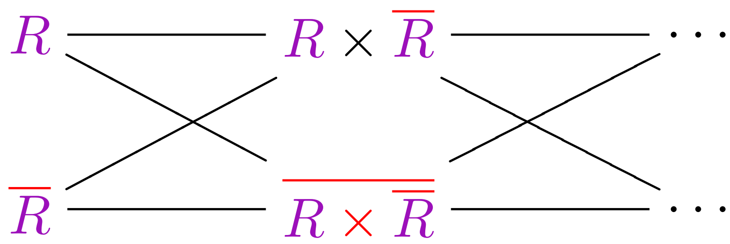
Remark

The $\overline{R}$-automata recognize the complements of languages recognized by R-automata, for any R.

Maybe, R-automata ***plus*** $\overline{R}$-automata will suffice, for some R **?**

No. Then $R \times \overline{R}$-automata would suffice, which is not the case.

Hence, each R gives rise to a strict hierarchy



Digression

There is a *non-regular* language R, such that R-automata recognize all ω-regular languages —but also some non-regular ones.

For example, a "universal" parity condition $R \subseteq \{0,1\}^\omega$ (M. Skrzypczak)

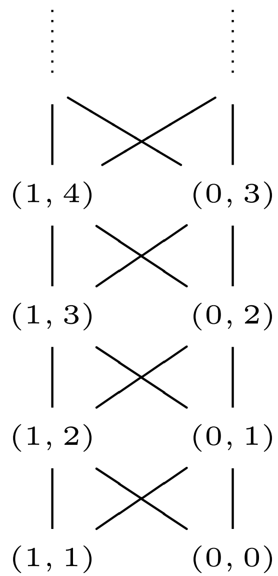
$$R = \{0^{m_0}\, 1\, 0^{m_1}\, 1\, 0^{m_2}\, 1\, \ldots : \limsup m_n \text{ is an } \textit{even} \text{ natural number } \}$$

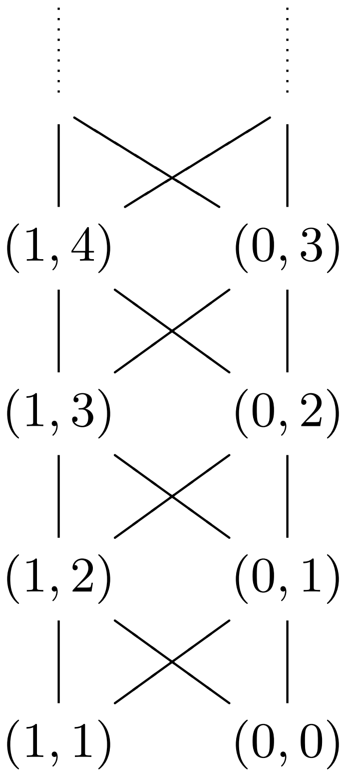
A parity automaton of **Rabin-Mostowski index** (i,k) is an R-automaton with $C = \{i, i+1, \ldots, k\}$,

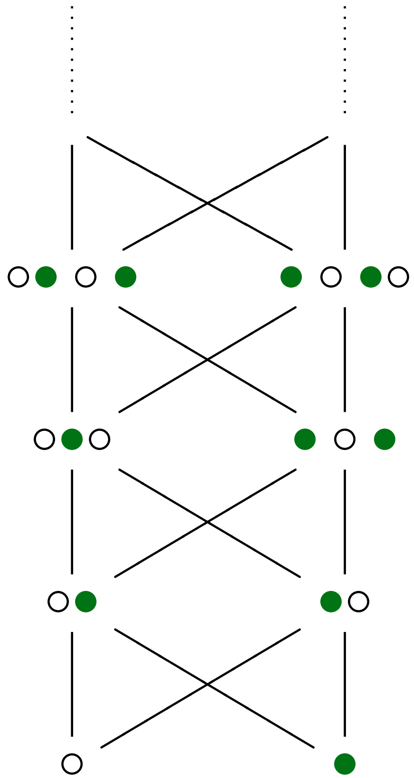
$$R = L_{i,k} = \{u : \limsup_{i \to \infty} u_i \text{is even}\}.$$

Parity automata exhaust all ω-regular languages, which is the celebrated **McNaughton Theorem** (1966).

The indices induce a hierarchy



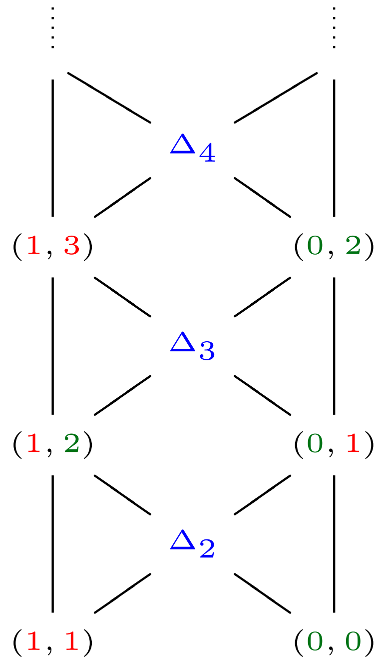




If we identify an (i, k) with a structure $\langle \{i, i+1, \ldots, k\}, \leq, Even \rangle$ then inclusions in the hierarchy correspond to embeddings of such structures.

Note that $L_{0,k} \approx \overline{L_{1,k+1}}$ **cannot** be accepted by $(1, k+1)$-automaton.

Hence the hierarchy



is **strict**, as noted by Wagner 1979, Kaminski 1985.

Note. Constructing parity condition from the Büchi condition.

Let $L \approx M$, whenever L can be recognized by an M-automaton, and M by an L-automaton. Then

$$(1,3) \quad \approx \quad (0,1) \times (1,2)$$

via transformation

$$\begin{array}{ccccccc} (1,1) & \mapsto & 3 & \quad & (1,2) & \mapsto & 1 \\ (0,2) & \mapsto & 2 & \quad & (0,1) & \mapsto & 1 \end{array}$$

Clearly

$$(0,2) \quad = \quad \overline{(1,3)}$$

We have further dependencies for $i \leq 2n$ (F. Murlak)

$$\begin{array}{rcl} (i, 2n+1) & \approx & (i, 2n) \times (0,1) \\ (i, 2n+2) & \approx & (i, 2n) \times (0,2) \end{array}$$

From words to trees.

A game on a (colored) graph.

$V_\exists$	positions of Eve
$V_\forall$	positions of Adam (disjoint)
$p_1 \in V$	initial position
$\longrightarrow \subseteq V \times V$	possible moves (with $V = V_\exists \cup V_\forall$)
$rank : V \to C$	the ranking function
$R \subseteq C^\omega$	winning condition for Eve

An infinite play $v_0 \longrightarrow v_1 \longrightarrow v_2 \longrightarrow \ldots$ is won by Eve iff $rank(v_0)\, rank(v_1)\, rank(v_2) \ldots \in R$.

A **parity game** of index (i, k) is a game with $R = L_{i,k}$.

Let $R \subseteq C^\omega$. An alternating R-automaton over binary trees $t : 2^* \longrightarrow A$ is

$$\langle A, Q, q_I, Tr, rank \rangle$$

$$Q = Q_\exists \mathbin{\dot\cup} Q_\forall \qquad Tr \subseteq Q \times A \times \{0,1,\varepsilon\} \times Q$$

$$q_I \in Q \qquad rank : Q \to C$$

An input tree t is accepted by the automaton iff Eve has a winning strategy in the game

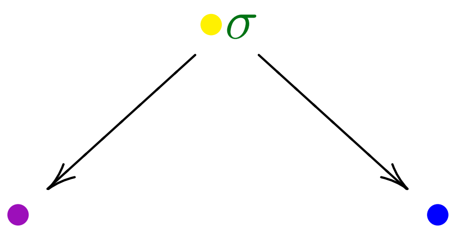
$Q_\exists \times 2^*$,	Eve's
$Q_\forall \times 2^*$,	Adam's
(q_I, ε),	initial
$\{((p,v),(q,vd)) \colon v \in \mathrm{dom}(t),\ (p,t(v),d,q) \in Tr\}$	moves
$rank(q,v) = \mathrm{rank}(q)$	ranking
R	winning Eve

In **non-deterministic** automata there are transitions

$$q_\exists \xrightarrow{\varepsilon} q_\forall$$

and, for each universal state $q_\forall$, and $a \in A$, there at most two transitions

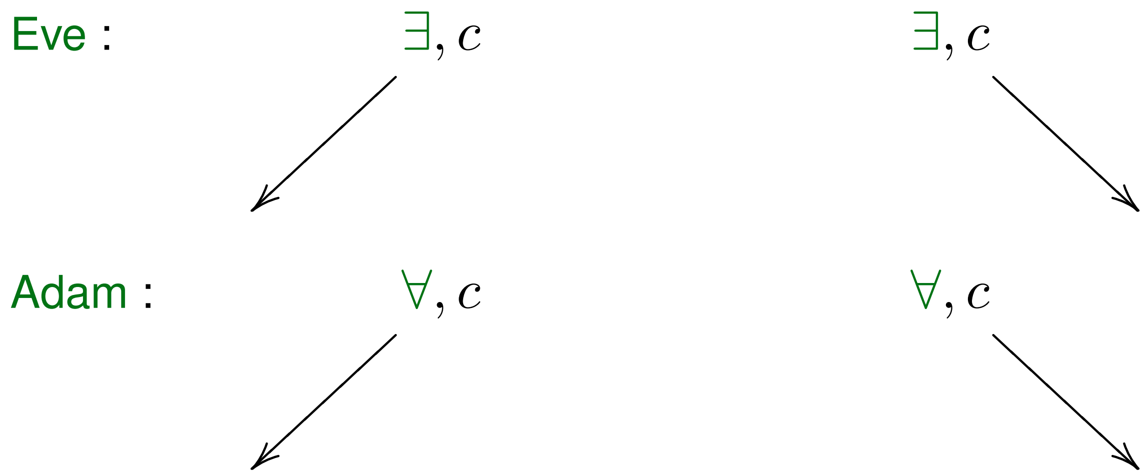
$$q_\forall \xrightarrow{a,0} p_0 \qquad q_\forall \xrightarrow{a,1} p_1$$



In **deterministic** automaton: only universal states.

Game tree languages.

A game on a tree $t : 2^* \longrightarrow \{\exists, \forall\} \times C$, with condition $L \subseteq C^\omega$.

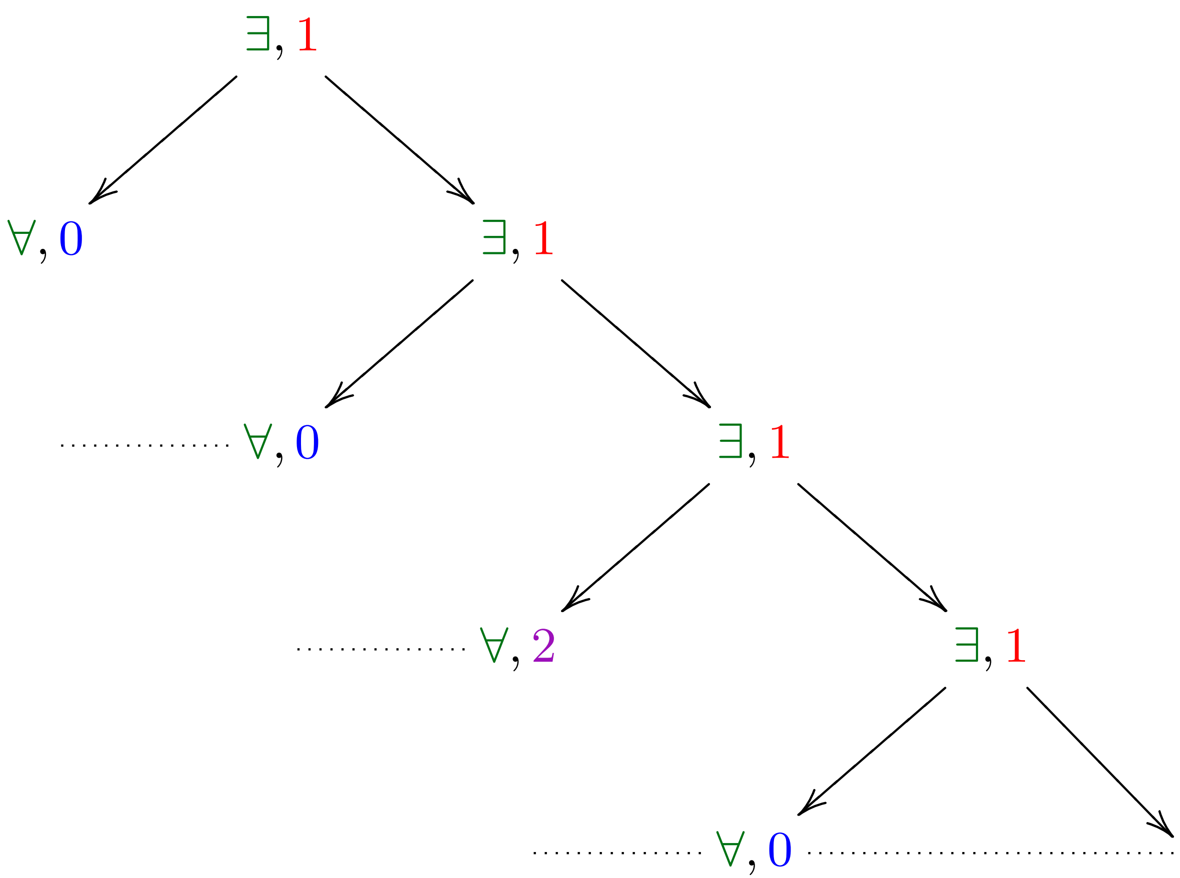


Eve wins an infinite play $(x_0, j_0),\ (x_1, j_1),\ (x_2, j_2), \ldots \quad (x_\ell \in \{\exists, \forall\})$

iff $j_0\, j_1\, j_2\ \ldots \in L$.

$$\mathrm{Win}^\exists(L) \quad = \quad \{t : \text{ Eve wins } \}$$

Example. $L = L_{0,2}$.



Easy lemma.

If $L \subseteq A^\omega$ is recognized by a deterministic R-automaton then $\mathrm{Win}^\exists(L)$ is recognized by an alternating R-automaton.

Is there an $R \subseteq C^\omega$, such that alternating R-automata recognize all recognizable tree languages **?**

A set of trees is *recognizable* if it can be recognized by an alternating (or non-deterministic) parity automaton.

No R-automaton (over alphabet $\{\exists, \forall\} \times C$) may accept $\overline{\mathrm{Win}^{\exists}(R)}$.

We use the concept of a **game tree**.

Recall that $\mathcal{A}$ accepts t iff Eve wins the game $G(\mathcal{A}, t)$ with the set of positions $2^* \times Q$ and condition R.

Unravel this game to a tree.

For a position (v, q), retain only the label $(own(q), rank(q))$, where

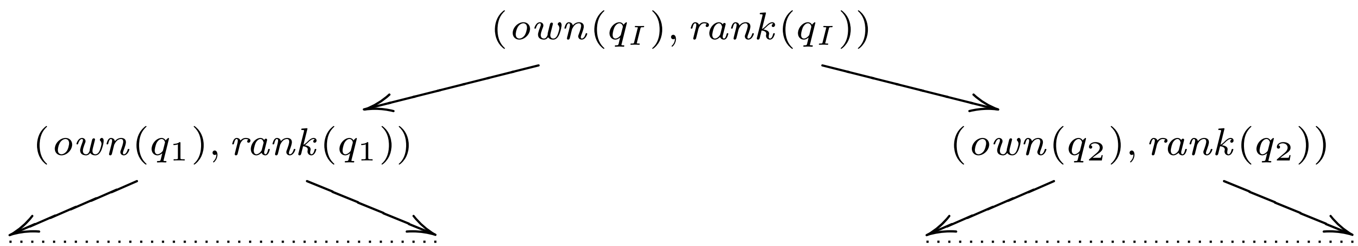
$$own(q) = \exists \quad \text{iff} \quad q \in Q_{\exists}$$
$$own(q) = \forall \quad \text{iff} \quad q \in Q_{\forall}.$$

Claim. $\mathcal{A}$ accepts t iff the game tree (*mutatis mutandis*) is in $\mathrm{Win}^{\exists}(R)$.

Suppose, for an alternating R-automaton $\mathcal{A}$,

$$L(\mathcal{A}) \quad = \quad \overline{\mathrm{Win}^{\exists}(R)}.$$

Create a tree f



where

$(q_I, (own(q_I), rank(q_I)), d_1, q_1), (q_I, (own(q_I), rank(q_I)), d_2, q_2) \in Tr.$

Then

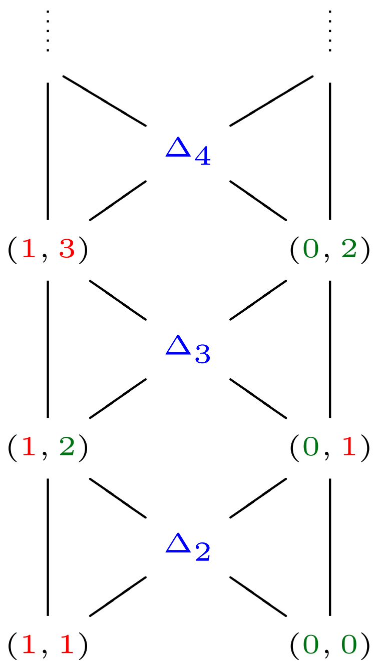
$$f \in \overline{\mathrm{Win}^{\exists}(R)} \iff f \in \mathrm{Win}^{\exists}(R),$$

a contradiction.

Abbreviate $W_{i,k} = \mathrm{Win}^{\exists}(L_{i,k})$.

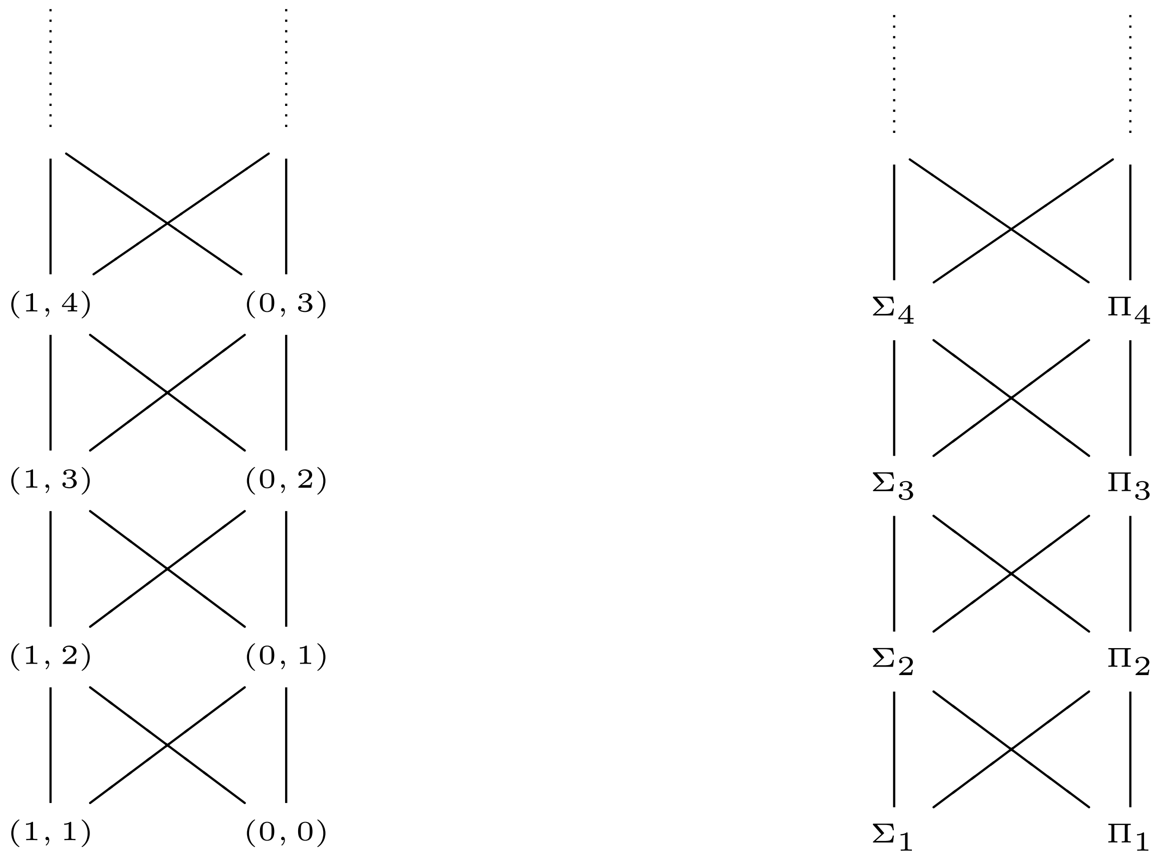
Then $W_{0,k} \approx \overline{W_{1,k+1}}$ cannot be accepted by $(1, k+1)$-automaton

Hence the hierarchy of alternating tree automata



is **strict**, as proved by Bradfield 1998.

Following mathematical logic, should we name the classes Π and Σ ?



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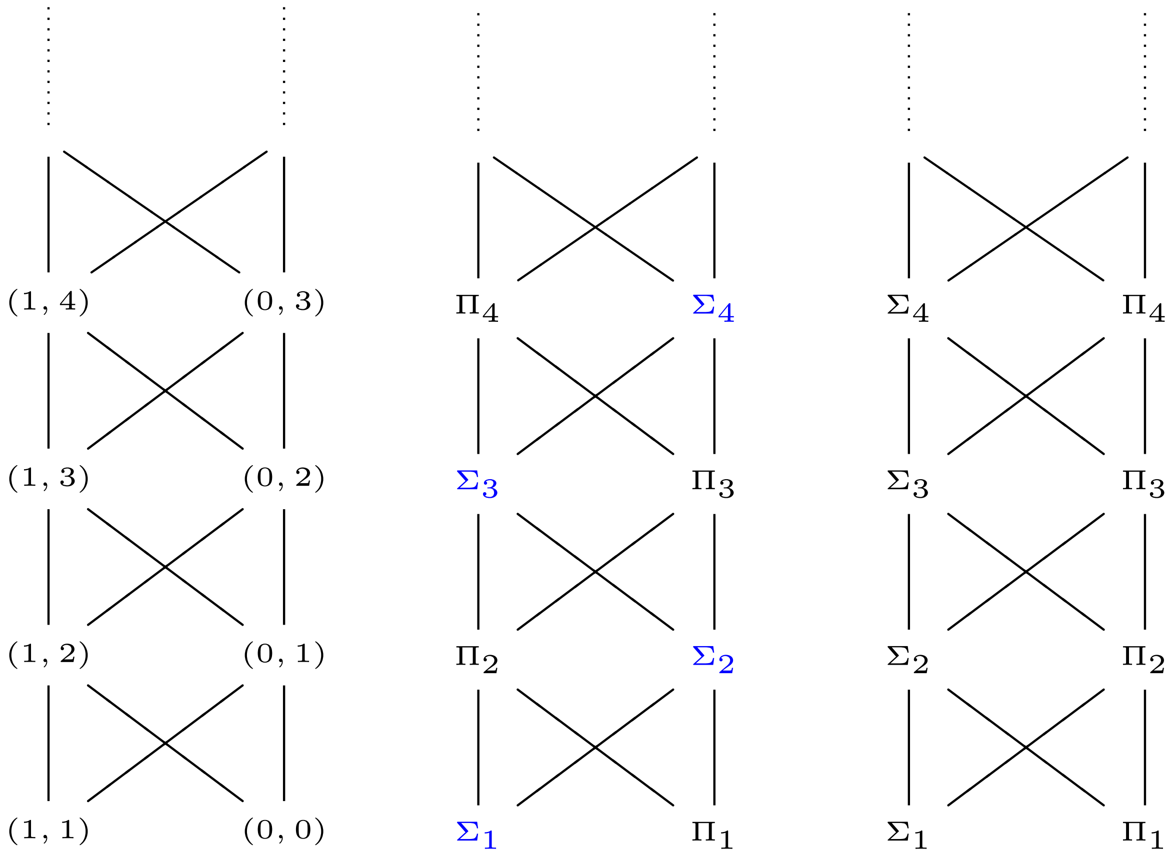


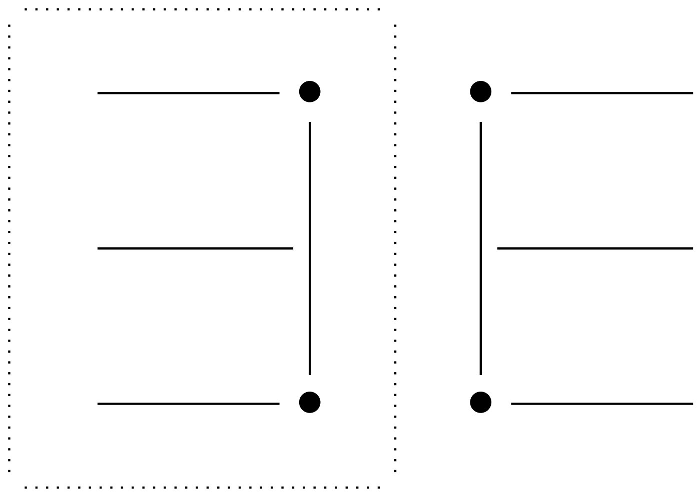
Why not like this ?

In descriptive set theory, the *orientation* of the hierarchy stems from the **separation property**.

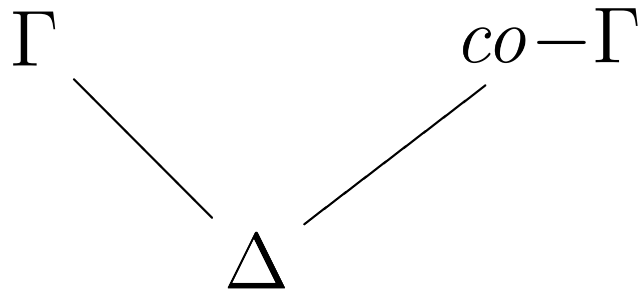
A set C separates a disjoint pair of sets A, B, if

$$\begin{aligned} A &\subseteq C \\ B \cap C &= \emptyset \end{aligned}$$

or *vice versa*.



A class of sets Γ has **separation property** if any disjoint pair of sets $A, B \in \Gamma$ is separated by some $C \in \Delta = \Gamma \cap co{-}\Gamma$ (where $co{-}\Gamma = \{\overline{X} : X \in \Gamma\}$).



Examples.

Any two disjoint *co-recursively enumerable* sets are separable by a *recursive* set.

Not so with *r.e.*-sets, in general.

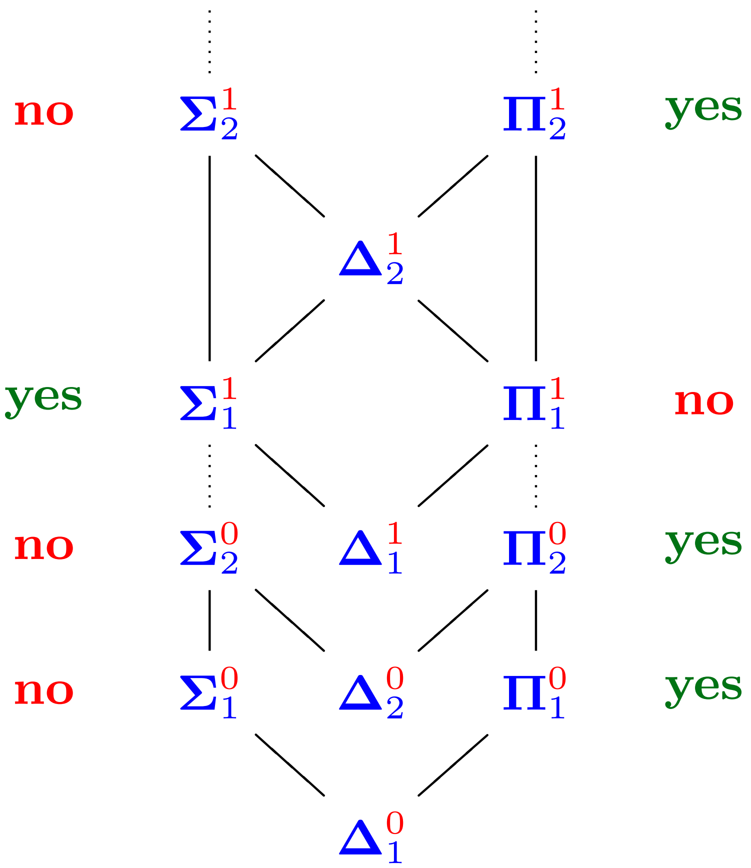
Any two *closed* subsets of the Cantor discontinuum are separable by a *clopen* set.

Not so with *open* sets, in general.

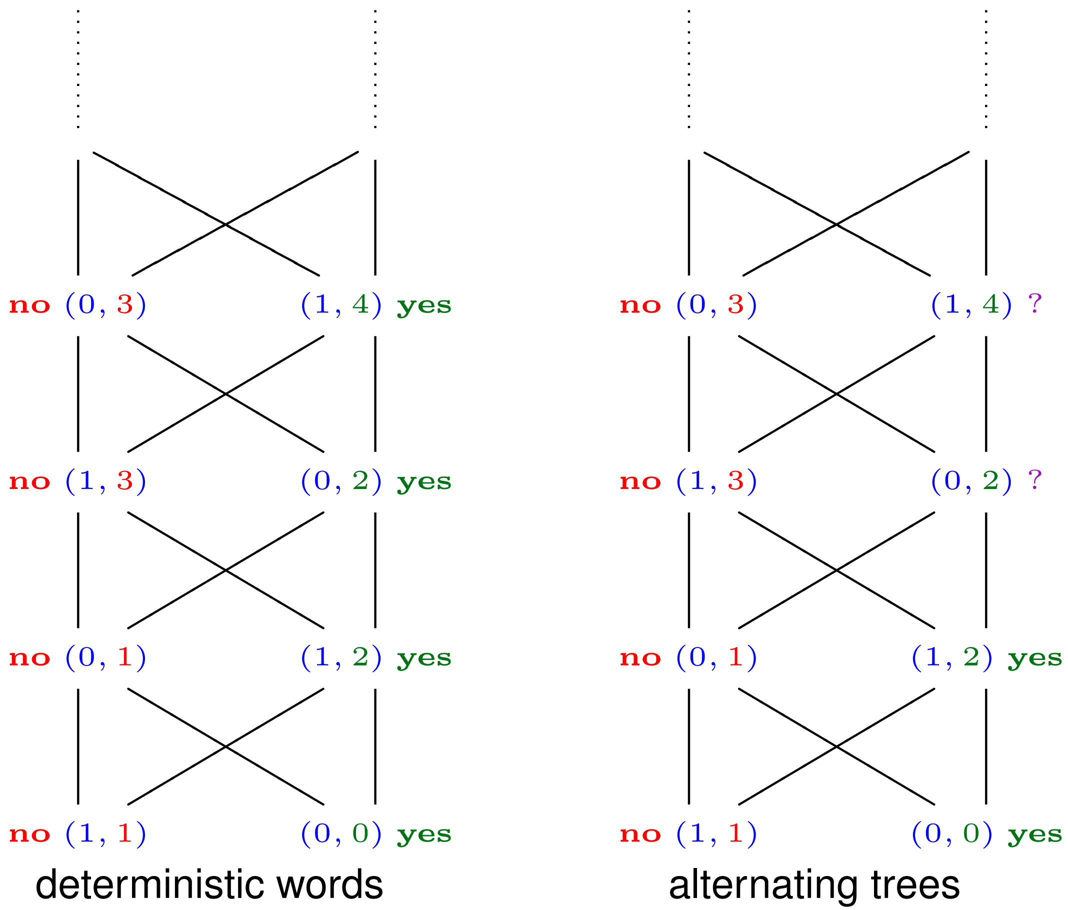
Lusin theorem. Any two disjoint *analytic* sets are separable by a *Borel* set.

Not so with *co-analytic* sets, in general.

Separation property for classical (e.g., topological) hierarchies is well understood.



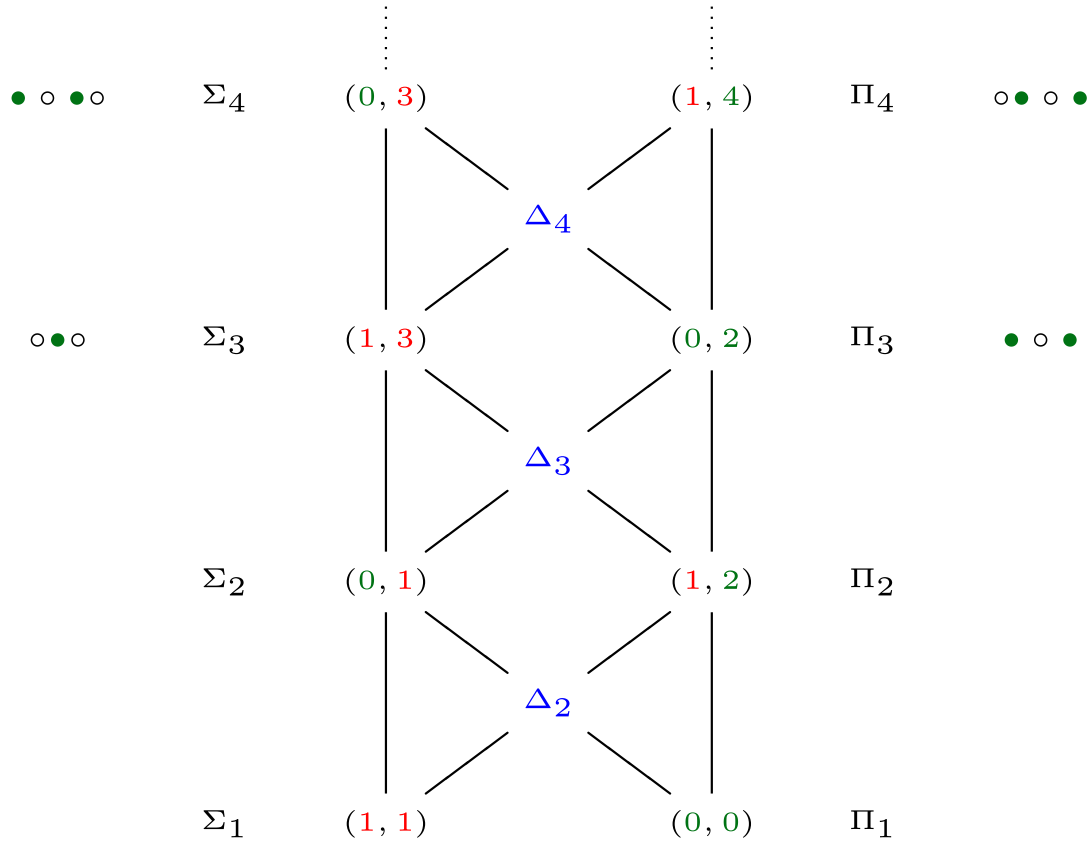
State–of–the–art for automata.



Rabin 1970, Selivanov 1998, Santocanale, Arnold 2005, Hummel, Michalewski, N. 2009,

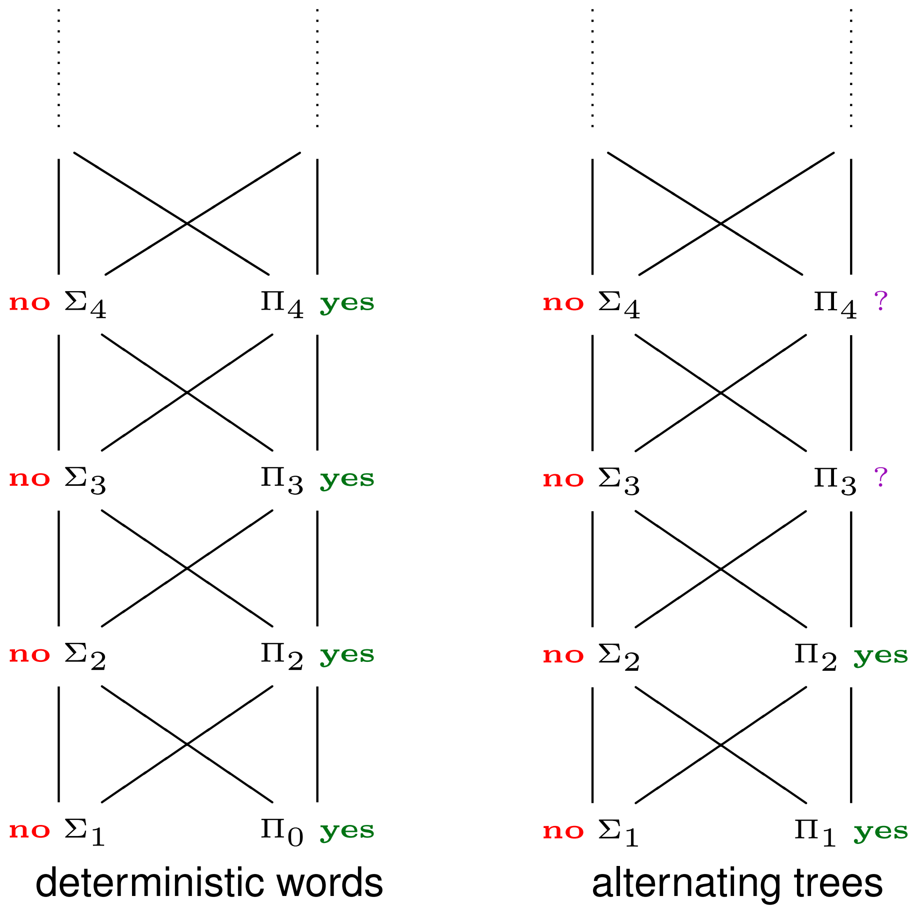
Arnold, Michalewski, N. 2012.

Convention



Σ_4
(0, 3)
(1, 4)
Π_4
Δ_4
Σ_3
(1, 3)
(0, 2)
Π_3
Δ_3
Σ_2
(0, 1)
(1, 2)
Π_2
Δ_2
Σ_1
(1, 1)
(0, 0)
Π_1

State–of–the–art for automata (in Σ/Π notation).



Rabin 1970, Selivanov 1998, Santocanale, Arnold 2005, Hummel, Michalewski, N. 2009,

Arnold, Michalewski, N. 2012.

Non-separation

For the class Σ_2, a non-separable pair $W_{0,1}$ and $W'_{0,1}$, where $W'_{0,1}$ is obtained from $W_{0,1}$ by interchanging $\exists \leftrightarrow \forall$ and $0 \leftrightarrow 1$.

$W_{0,1}$ and $W'_{0,1}$ are inseparable by any Borel set, hence *a fortiori* by any set in Δ_1. (Hummel, Michalewski, N. 2009)

Remark. The argument fails for the higher levels analogues $W'_{i,k}$ (for odd k).

For example., in $W'_{1,3}$ Adam has a strategy to force that there are infinitely many 3's, but only finitely many 1's.

But $W_{1,3}$ and $W'_{1,3}$ are separable by the Δ_3 set:

Eve has a strategy to force that 3 occurs only finitely often.

Remedy comes again *via* deterministic automata on words (Arnold, M., N. 2012).

Lemma. $L_{i,k} \times \overline{L_{i,k}}$ and $\overline{L_{i,k}} \times L_{i,k}$ are inseparable by a set in Δ.

Suppose $\mathcal{A}, \mathcal{A}'$ are deterministic automata of index (i,k) such that

$$\overline{L} \times L \subseteq L(\mathcal{A}) \qquad L(\mathcal{A}) \cap L(\mathcal{A}') = \emptyset$$
$$L \times \overline{L} \subseteq L(\mathcal{A}') \qquad L(\mathcal{A}) \cup L(\mathcal{A}') = \top.$$

Create a word u

$$q_0, q_0' \xrightarrow{rank(q_0), rank(q_0')} q_1, q_1' \xrightarrow{rank(q_1), rank(q_1')} q_2, q_2' \xrightarrow{rank(q_2), rank(q_2')} \quad \cdots$$

$$rank(q_0)rank(q_0') \qquad rank(q_1)rank(q_1') \qquad rank(q_2)rank(q_2') \qquad \cdots$$

Then

$$u \in L(\mathcal{A}) \quad \Rightarrow \quad u \in L \times \overline{L} \subseteq L(\mathcal{A}')$$
$$u \in L(\mathcal{A}') \quad \Rightarrow \quad u \in \overline{L} \times L \subseteq L(\mathcal{A})$$

a contradiction.

Remark

$$q_0, q_0' \xrightarrow{rank(q_0), rank(q_0')} q_1, q_1' \xrightarrow{rank(q_1), rank(q_1')} q_2, q_2' \xrightarrow{rank(q_2), rank(q_2')} \cdots$$

$$rank(q_0)rank(q_0') \quad rank(q_1)rank(q_1') \quad rank(q_2)rank(q_2') \quad \cdots$$

This word is a unique fixed point of the mapping

$$\left(\{i, \ldots, k\}^2\right)^\omega \ni w \quad \mapsto \quad \left(rank \circ run^{\mathcal{A}}, rank \circ run^{\mathcal{B}}\right)(w).$$

Lemma. $L_{i,k} \times \overline{L_{i,k}}$ and $\overline{L_{i,k}} \times L_{i,k}$ are inseparable by a set in Δ.

Key Lemma. For $m \geq 2$, there exist disjoint U_1^m, U_2^m in Σ_m, such that, for $L = L_{i,k}$ in Π_m,

$$\begin{aligned} L \times \overline{L} &\subseteq U_1^m \\ \overline{L} \times L &\subseteq U_2^m \end{aligned}$$

By previous Lemma, U_1^m, U_2^m form an inseparable pair.

We infer

Theorem. The tree languages $\mathrm{Win}^{\exists}(U_1^m)$ and $\mathrm{Win}^{\forall}(U_2^m)$ are inseparable by a set in Δ_k.

Thus the **separation property fails** for the classes Σ_m.

More specifically,

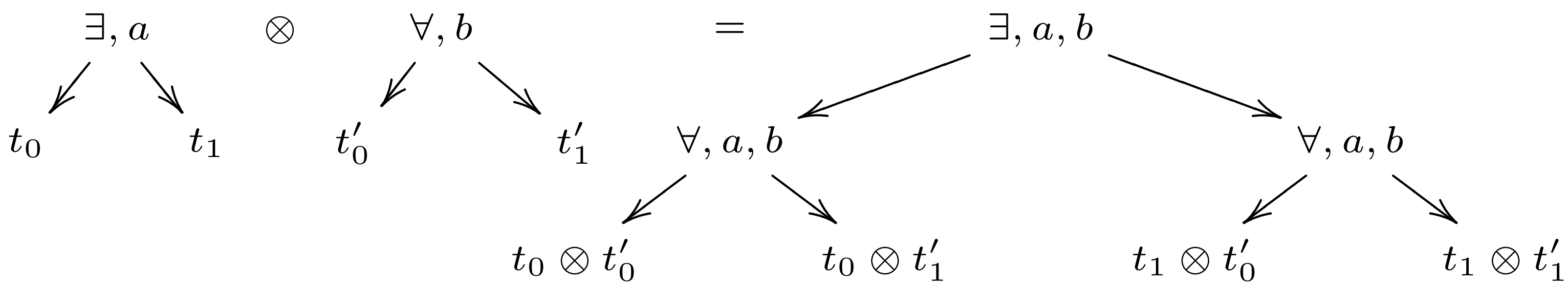
$$\begin{aligned} \mathrm{Win}^{\exists}(L \times \overline{L}) &\subseteq \mathrm{Win}^{\exists}(U_1^m) \\ \mathrm{Win}^{\forall}(\overline{L} \times L) &\subseteq \mathrm{Win}^{\forall}(U_2^m) \end{aligned}$$

and the latter cannot be separated by fixed-point argument.

Note: to simulate the $(rank \circ run^{\mathcal{A}}, rank \circ run^{\mathcal{B}})$ construction, we need a **product** of game trees.

Lemma. For any L_0, L_1,

$$\begin{aligned}
\mathrm{Win}^{\exists}(L_0) \otimes \mathrm{Win}^{\exists}(L_1) \quad &\subseteq \quad \mathrm{Win}^{\exists}(L_0 \otimes L_1) \\
\mathrm{Win}^{\forall}(L_0) \otimes \mathrm{Win}^{\forall}(L_1) \quad &\subseteq \quad \mathrm{Win}^{\forall}(L_0 \otimes L_1)
\end{aligned}$$



Further properties of U_1, U_2. (Recall they are in Σ_m.)

If L is in Π_m then

$$\begin{array}{rcl} L \times \overline{L} & \subseteq & U_1 \\ \overline{L} \times L & \subseteq & U_2 \\ \overline{L \times L} & \subseteq & U_1 \cup U_2 \end{array}$$

Hence, if $L(\mathcal{A})$ and $L(\mathcal{B})$ are disjoint languages in Π_m, they are separated by

$$\{u : (rank \circ run^{\mathcal{A}}(u), rank \circ run^{\mathcal{B}}(u)) \in U_1\}$$
$$\{u : (rank \circ run^{\mathcal{A}}(u), rank \circ run^{\mathcal{B}}(u)) \in U_2\}$$

Thus the separation property **holds** in classes Π_m (for words).

Problem. Does the separation property hold for classes Π_m, $m \geq 3$, for alternating automata on trees **?**

Rabin 1970 proved the result for $k = 2$ by combinatorial argument yielding a stronger result, which does not (provably) extend to the higher levels.

Santocanale and Arnold 2005 showed that separation of $L(\mathcal{A}), L(\mathcal{B})$ by a set in Δ is possible, whenever the automata are **non-deterministic**.
The proof uses pathfinder–automaton game.

In descriptive set theory, the separation property is often showed *via* the reduction property of the dual class.

Reduction property.

A pair of sets A', B' **reduces** pair A, B, if

$$A' \cup B' \quad = \quad A \cup B$$

$A' \subseteq A$, $B' \subseteq B$, and $A' \cap B' = \emptyset$.

A class of sets Γ has **reduction property** if any pair of sets in Γ is reduced by a pair in the same class.

The reduction property for a class Γ implies the separation property for *co*-Γ, and in descriptive set theory it is the usual way to establish the latter.

It holds as expected for the index hierarchy of deterministic automata on words (Selivanov). But...

Proposition. The reduction property fails for all alternating classes (i, k).

Lemma. Any recognizable set of trees can be presented as a finite union

$$\bigcup_d \bigcup_i d(A_i, B_i),$$

where A_i, B_i are in the same class as the original set.

Lemma. Let $\top$ be any set and $W \subseteq \top$. Assume that

$$(\top \times W) \cup (W \times \top) = X \cup Y,$$

where $X \subseteq \top \times W$, $Y \subseteq W \times \top$, and $X \cap Y = \emptyset$. Suppose further that

$$X = \bigcup_{i=1}^{m} a_i \times b_i, \qquad Y = \bigcup_{i=1}^{n} c_i \times d_i$$

for some sets $a_i, b_i, c_i, d_i \subseteq \top$. Then the set $\overline{W}$ can be generated from the sets $a_1, \ldots, a_m, d_1, \ldots, d_n$, by (finite) union and intersection.

To prove Proposition, choose $d(\top, W), d(W, \top)$, where W is hard for the class.

Relation to other hierarchies.

The **μ-calculus** expresses properties as solutions of equations.

Example — winning regions in game arenas

$$\langle V = V_\exists \cup V_\forall, \qquad \longrightarrow \subseteq V \times V \rangle$$

Players' equations:

$$\begin{array}{lcrcr} X & = & (V_\exists \cap \Diamond X) \cup (V_\forall \cap \Box X) & = & Eve(X) \\ Y & = & (V_\forall \cap \Diamond Y) \cup (V_\exists \cap \Box Y) & = & Adam(Y) \end{array}$$

where $\Diamond Z = \{p : (\exists q)\, p \longrightarrow q\}$, $\Box Z = \overline{\Diamond \overline{Z}}$.

Players' equations:

$$X \quad = \quad (V_\exists \cap \Diamond X) \cup (V_\forall \cap \Box X) \quad = \quad Eve(X)$$

$$Y \quad = \quad (V_\forall \cap \Diamond Y) \cup (V_\exists \cap \Box Y) \quad = \quad Adam(Y)$$

Then the set $W_\exists$ of Eve's **winning positions** is

in finite reachability games: $\mu X . Eve(X)$

in safety games: $\nu X . Eve(X)$

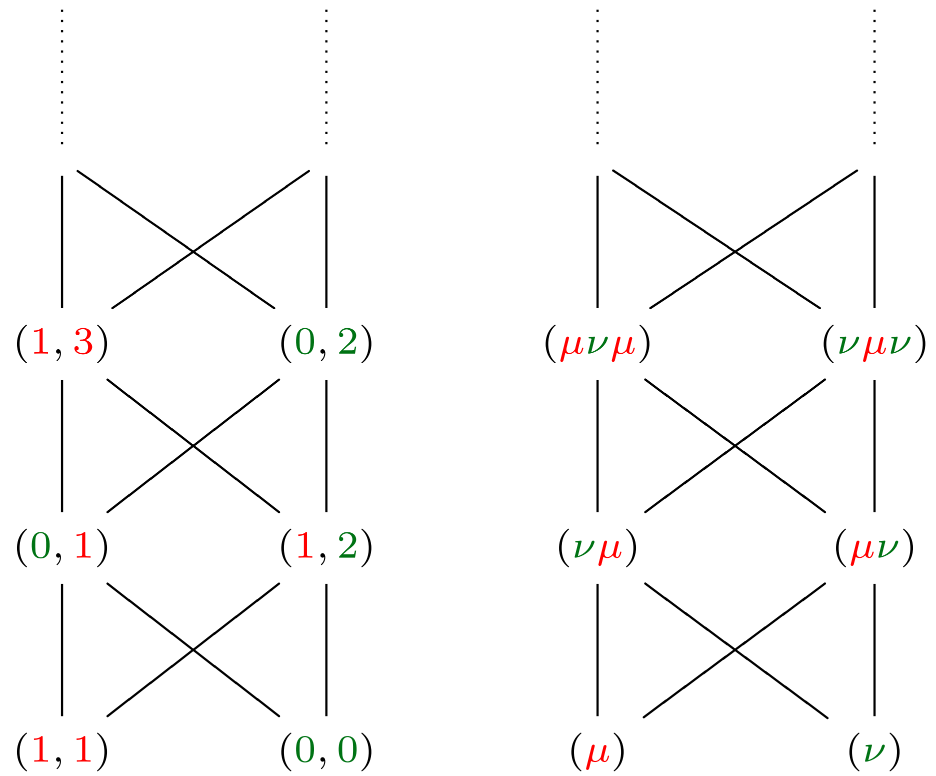
in **any** game with a prefix-independent

winning condition $C \subseteq V^\omega$: **some** fixed point of $Eve(X)$

In parity games:

$$
\begin{aligned}
\nu X_0.\mu X_1.\nu X_2.\ldots.\vartheta X_n. \quad & (V_\exists \cap rank_0 \cap \Diamond X_0) \cup \\
& (V_\exists \cap rank_1 \cap \Diamond X_1) \cup \\
& \ldots\ldots\ldots\ldots \cup \\
& (V_\exists \cap rank_n \cap \Diamond X_n) \cup \\
& \\
& (V_\forall \cap rank_0 \cap \Box X_0) \cup \\
& (V_\forall \cap rank_1 \cap \Box X_1) \cup \\
& \ldots\ldots\ldots\ldots \cup \\
& (V_\forall \cap rank_n \cap \Box X_n)
\end{aligned}
$$

There is an exact correspondence of the levels of the two hierarchies.



We link a μ-calculus formula with an automaton recognizing its tree models.

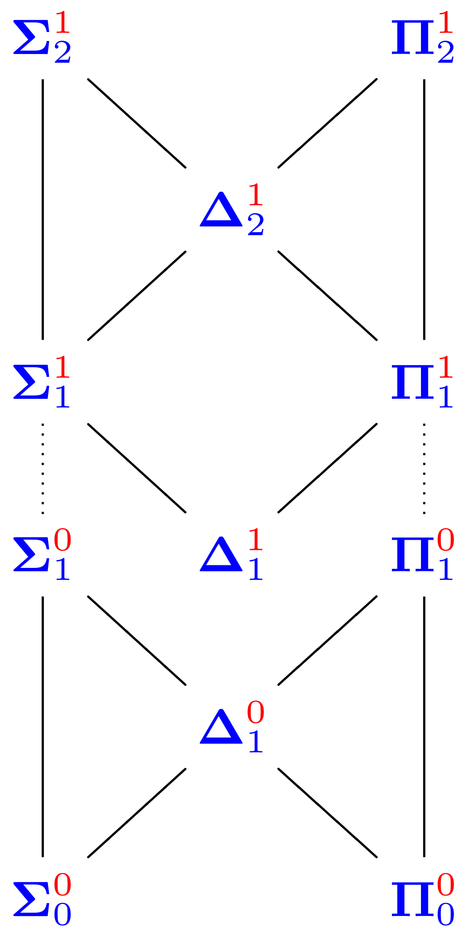
Relation to classical hierarchies – topology and logic.

1900 Borel, Baire, Lebesgue

1917 Lusin, Suslin

1929 Tarski, Kuratowski

1940 Mostowski, Kleene



Topological complexity of tree languages

recognizable	Δ^1_2	
Büchi recognizable	Σ^1_1	
deterministic	Π^1_1	
weakly recognizable	$\bigcup_{n<\omega} \Sigma^0_n$	(Borel of finite rank)
word languages	$Boole(\mathbf{\Sigma^0_2})$	

Problem. Are there recognizable tree languages which are Borel but *not* weakly recognizable **?**

Warning. There are recognizable tree languages, which are Σ^1_1, but not Büchi.

Example.

$H!$ $=$ binary trees over $\{a, b\}$ where b appears infinitely often on exactly one branch.

By Lusin Theorem $H!$ is $\mathbf{\Pi}^1_1$ (complete).

Hence $\overline{H!}$ is $\mathbf{\Sigma}^1_1$.

But it is **not** Büchi recognizable!

Decidability. Given a tree automaton $\mathcal{A}$, decide whether $L(\mathcal{A})$ is

in alternating class (i,k)	**?**
in non-deterministic class (i,k)	**?**
Büchi recognizable	**?**
non-ambiguous	**?**
deterministic	easy
weakly recognizable	**?**
in weak (alternating) class (i,k)	**?**
Borel, Σ^1_1, Π^1_1, etc.	**?**
in the Borel class Σ^0_n	**?**
$Boole(closed)$	**Bojańczyk, Place 2012**
closed	folklore

Decidability is known if $\mathcal{A}, \mathcal{B}$ are deterministic

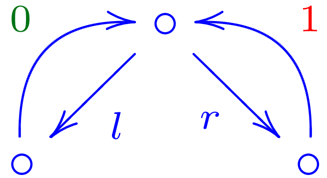
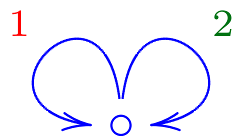
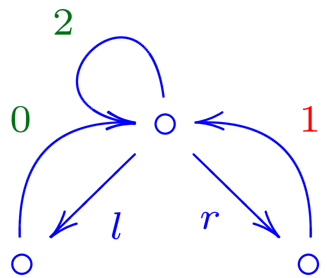
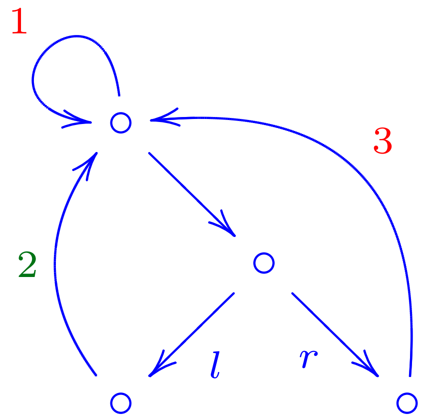
non-deterministic class (i,k)	**N., Walukiewicz 2004**
Büchi recognizable	**Urbański 2000**
weakly recognizable	**N., Walukiewicz 2003**
in weak (alternating) class (i,k)	**Murlak 2008**
Borel or Π^1_1	**N., Walukiewicz 2003**
Borel class Π^0_n/Σ^0_n $(n \leq 3)$	**Murlak 2005**
Wadge level	**Murlak 2006**
$L(\mathcal{A}) \leq_w L(\mathcal{B})$	**Murlak 2006**

Further development.

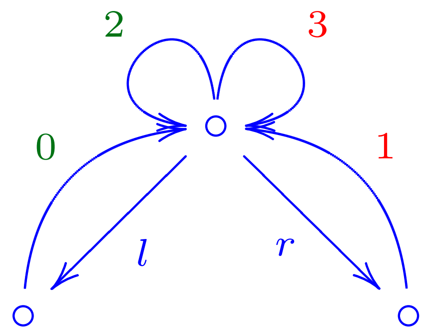
Colcombet and Loeding 2008 reduced decidability of the index in non-deterministic hierarchy to boundedness problems for distance automata.

Duparc, Facchini, and Murlak 2011 gave a decision procedure for weak alternating index, Borel index, and Wadge level of **weak game automata** (covering $W_{i,k}$).

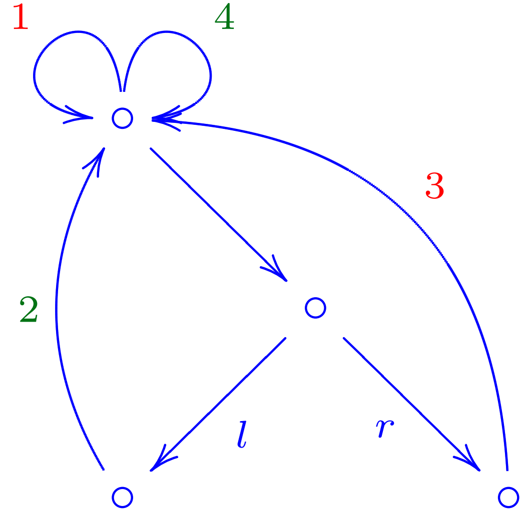
This survey is not complete !

Index class	Forbidden pattern
(1,2)	
(0,1)	
(0,2)	
(1,3)	

The (0, n) case, $n \geq 3$



The (1,n) case, $n \geq 4$



Conclusion.

The study of hierarchies helps us to understand positive aspects of *difficulty*.

I insist on this: any complicated thing, being illuminated by definitions, being laid out in them, being broken up into pieces, will be separated into pieces completely transparent even to a child...

Nicolai Lusin

Quoted from: L.Graham, J.-M.Kantor, *Naming Infinity*