On the complexity of infinite computations

interplay of automata theory and topology

Damian Niwiński, University of Warsaw

joint work with André Arnold, Igor Walukiewicz, and Filip Murlak

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Mathematical ideas

rational	irrational
fundamental theorem of algebra	no formula for solutions of order ≥ 5
continuity	Dirichlet function, Peano curve
Lebesgue measure	Banach–Tarski paradox $\bigcirc = \bigcirc + \bigcirc$
universal Turing machine	undecidability, Rice Theorem

Definability theory

rational	irrational
Borel hierarchy	Suslin counter-example (1916):
	continuous image of a Borel set
	need not be so
	● $\rightsquigarrow$ ●

There is a crack in everything. That's how the light gets in.

Leonard Cohen, *Anthem*

Finite automata appear to be on the “rational side”.

They are extremely robust — admit generalization to trees, infinite words, infinite trees. . .

Generalizations usually preserve

- elementary decidability of the emptiness problem
- closure properties (in particular, on Boolean operations),

  consequently: logical characterizations (MSO, μ-calculus)

  $\longrightarrow$ decidability of the logics. (Büchi 1960, Rabin 1969, . . .)

But. . .

Finite automata on infinite trees go beyond the Borel hierarchy.

They cannot, in general, be made deterministic, non even non-ambiguous.

Topology (art of counter-examples ?) can shed some light there.

Topics of the talk

- Automata on infinite words and trees, the index hierarchies, and they relation to classical hierarchies.
- Topological arguments in the strictness proofs.
- Where the two complexities diverge. . .
- Decidability issues – testing for forbidden patterns.

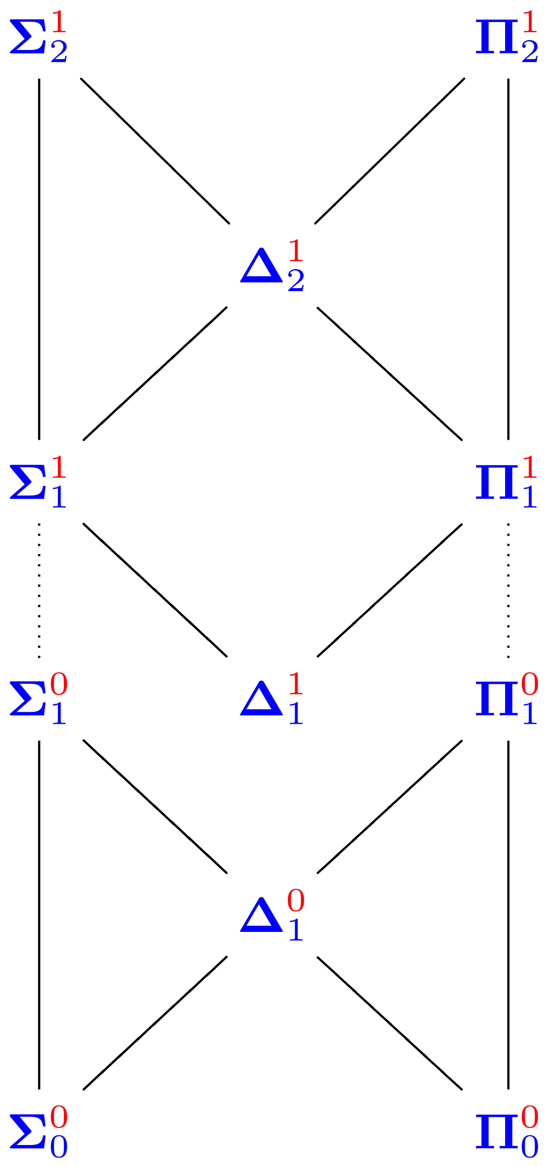
Classical definability theory

1900 Borel, Baire, Lebesgues

1917 Lusin, Suslin

1929 Tarski, Kuratowski

1940 Mostowski, Kleene



For $R \subseteq \omega^k \times (\{0,1\}^\omega)^\ell$, let

$$\exists^0 R = \{\langle \mathbf{m}, \alpha\rangle : (\exists n)\, R(\mathbf{m}, n, \alpha)\}$$

$$\exists^1 R = \{\langle \mathbf{m}, \alpha\rangle : (\exists \beta)\, R(\mathbf{m}, \alpha, \beta)\}$$

Arithmetical hierarchy			Analytical hierarchy		
Σ^0_0	$=$	recursive relations	Σ^1_0	$=$	arithmetical relations
Π^0_n	$=$	$\{\overline{R} : R \in \Sigma^0_n\}$	Π^1_n	$=$	$\{\overline{R} : R \in \Sigma^0_n\}$
Σ^0_{n+1}	$=$	$\{\exists^{\mathbf{0}} R : R \in \Pi^0_n\}$	Σ^1_{n+1}	$=$	$\{\exists^{\mathbf{1}} R : R \in \Pi^1_n\}$
Δ^0_n	$=$	$\Sigma^0_n \cap \Pi^0_n$	Δ^1_n	$=$	$\Sigma^1_n \cap \Pi^1_n$

Arithmetical hierarchy

$$\Sigma^0_0 = \text{recursive relations}$$
$$\Pi^0_n = \{\overline{R} : R \in \Sigma^0_n\}$$
$$\Sigma^0_{n+1} = \{\exists^0 R : R \in \Pi^0_n\}$$
$$\Delta^0_n = \Sigma^0_n \cap \Pi^0_n$$

Analytical hierarchy

$$\Sigma^1_0 = \text{arithmetical relations}$$
$$\Pi^1_n = \{\overline{R} : R \in \Sigma^0_n\}$$
$$\Sigma^1_{n+1} = \{\exists^1 R : R \in \Pi^1_n\}$$
$$\Delta^1_n = \Sigma^1_n \cap \Pi^1_n$$

Relativized (boldface) hierarchies

For $\beta \in \{0,1\}^\omega$, let $R[\beta] = \{\langle \mathbf{m}, \alpha\rangle : R(\mathbf{m}, \alpha, \beta)\}$.

$$\boldsymbol{\Sigma}^i_n = \{R[\beta] : R \in \Sigma^i_n, \beta \in \{0,1\}^\omega\} \qquad \boldsymbol{\Delta}^i_n = \boldsymbol{\Sigma}^i_n \cap \boldsymbol{\Pi}^i_n$$

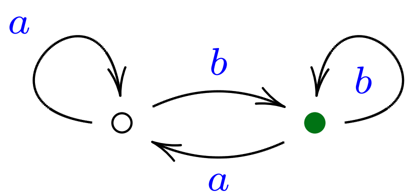
$$\boldsymbol{\Pi}^i_n = \{R[\beta] : R \in \Pi^i_n,\ \beta \in \{0,1\}^\omega\} \qquad i \in \{0,1\}$$

$$\boldsymbol{\Sigma}^0_1 = \textit{open}$$
$$\boldsymbol{\Pi}^0_1 = \textit{closed}$$
$$\boldsymbol{\Delta}^1_1 = \textit{Borel}$$

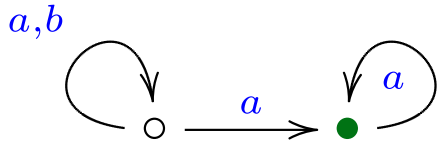
Büchi automata on infinite words

$$\mathcal{A} = \langle \Sigma, Q, q_I, Tr, F \rangle$$

where $Tr \subseteq Q \times \Sigma \times Q$, $F \subseteq Q$.



$$((a+b)^* b)^\omega$$



$$(a+b)^* a^\omega$$

The second one cannot be recognized by a deterministic automaton.

$$\xrightarrow{a}\xrightarrow{b}\xrightarrow{a}\xrightarrow{a}\xrightarrow{b}\xrightarrow{a}\xrightarrow{a}\xrightarrow{a}\xrightarrow{b}\xrightarrow{a}\xrightarrow{a}\xrightarrow{a}\xrightarrow{a}\xrightarrow{b}\xrightarrow{a} \ldots\ldots \xrightarrow{a}\xrightarrow{b}\xrightarrow{a} \ldots$$

So $(a+b)^*a^\omega$ cannot be recognized by a deterministic automaton.

But this also follows by a topological argument!

We assume the Cantor topology on X^ω, induced by the metric

$$d(u,v) = 2^{-\min\{m \,:\, u_m \neq v_m\}}$$

(or 0, if $u=v$).

If A is deterministic then the mapping

$$\Sigma^\omega \ni u \mapsto run(u) \in Q^\omega$$

continuously **reduces** $L(A)$ to $(Q^*F)^\omega$.

But

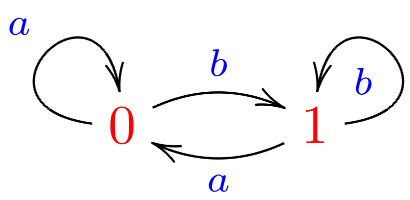
- $(Q^*F)^\omega$ is $\mathbf{\Pi}^0_2$ (G_δ) ,
- $(a+b)^*a^\omega$ is **complete** in $\mathbf{\Sigma}^0_2$ (F_σ).

Parity automata

$$\mathcal{A} = \langle \Sigma, Q, q_I, Tr, \textit{rank} \rangle$$

where *rank* $: Q \to \{0, 1, \ldots, k\}$.

$$\limsup_{i\to\infty} \textit{rank}(q_i) \text{ is even}$$



$$(a+b)^* a^\omega$$

The Rabin-Mostowski **index** of a parity automaton $\mathcal{A}$ is

(**min** *rank*(Q), **max** *rank*(Q))

We can assume **min** *rank*$(Q) \in \{0, 1\}$.

The McNaughton Theorem (1966)

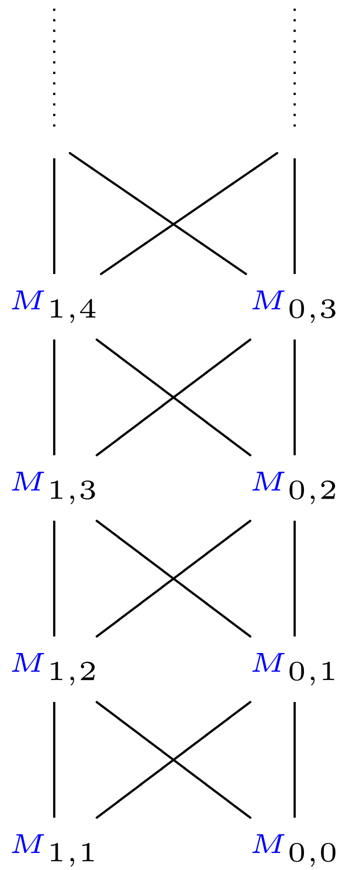
A nondeterministic Büchi automaton can be simulated by a deterministic parity automaton of some index (i, k).

The minimal index (i, k) may be arbitrarily high (Wagner 1979, Kaminski 1985).

Again, it can be inferred by a topological argument.

Let

$$M_{i,k} = \{u \in \{i, \ldots, k\}^{\omega} : \limsup_{\ell \to \infty} u_{\ell} \text{ is even}\}$$



No continuous reduction down the hierarchy.

Wadge game $G(A, B)$

Spoiler	Duplicator
$a_0 \in \Sigma$	$b_0 \in \Sigma$
a_1	b_1
a_2	b_2
$\vdots$	$\vdots$
a_{12}	b_{12}
a_{13}	wait
a_{14}	wait
a_{15}	b_{13}
$\vdots$	$\vdots$

Here $A, B \subseteq \Sigma^\omega$ (Σ finite).

Duplicator wins if $a_0 a_1 a_2 \ldots \in A \iff b_0 b_1 b_2 \in B$.

Fact

Duplicator has a winning strategy iff there is a continuous $f : \Sigma^\omega \to \Sigma^\omega$ s.t. $A = f^{-1}(B)$,

in symbols, $A \leq_w B$.

Spoiler 's strategy, e.g., in $G(M_{0,5}, M_{1,6})$

Spoiler	Duplicator
0	4
3	5
4	1
⋮	⋮
	i
$i-1$	
⋮	⋮
	wait
0	
⋮	⋮

Note

If a deterministic automaton of index $(1, 6)$, accepted $M_{0,5}$ there would be a continuous reduction of $M_{0,5}$ to $M_{1,6}$

$u \mapsto \mathit{rank} \circ run(u)$.

Contradiction!

Personal recollection

$$
\begin{array}{lcll}
A^*B & = & \mu x.Ax \cup B & A, B \subseteq \Sigma^*,\ A \neq \emptyset \\
A^\omega & = & \nu Y.AY & A \subseteq \Sigma^*,\ \varepsilon \notin A \\
(a+b)^*a^\omega & = & \mu X.\nu Y.aX \cup bY & \\
(a^*b)^\omega & = & \nu Y.\mu X.aX \cup bY & \text{Park, 1979} \\
 & \neq & \mu X.\nu Y.\ldots &
\end{array}
$$

The $\mu\nu\mu\nu\ldots$ hierarchy collapses, and any ω-regular language can be represented by a $\nu\mu$ (vectorial) expression.

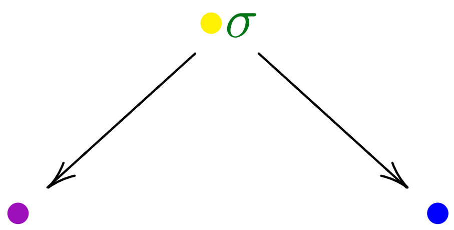
Is this hierarchy infinite in any other context ?

Yes, for infinite trees (N. 1986).

Parity tree automata

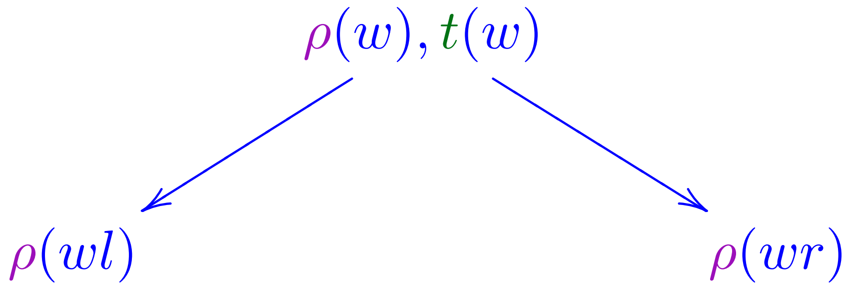
$$\mathcal{A} = \langle \Sigma, Q, q_I, Tr, \textit{rank} \rangle$$

where $Tr \subseteq Q \times \Sigma \times Q \times Q$, $\textit{rank} : Q \longrightarrow \{0, 1, \ldots, k\}$.



Parity tree automata cont'd

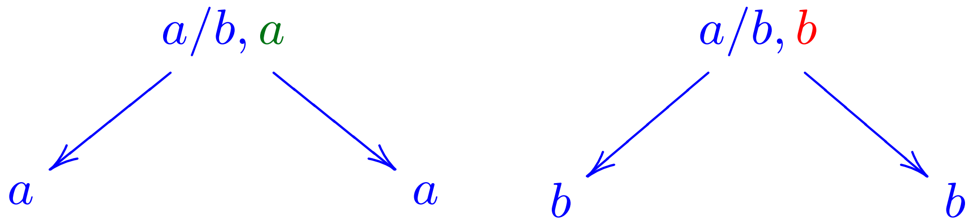
A run of $\mathcal{A}$ on a tree $t : \{l, r\}^* \rightarrow \Sigma$ is a tree $\rho : \{l, r\}^* \rightarrow Q$, such that, $\langle \rho(w), t(w), \rho(wl), \rho(wr) \rangle \in Tr$, for each $w \in$ dom (ρ)



The run is accepting if, for each path $P = p_0 p_1 \ldots \in \{l, r\}^\omega$,

$$\limsup_{k \rightarrow \infty} \textit{rank}(\rho(p_0 p_1 \ldots p_k)) \text{ is even.}$$

Example

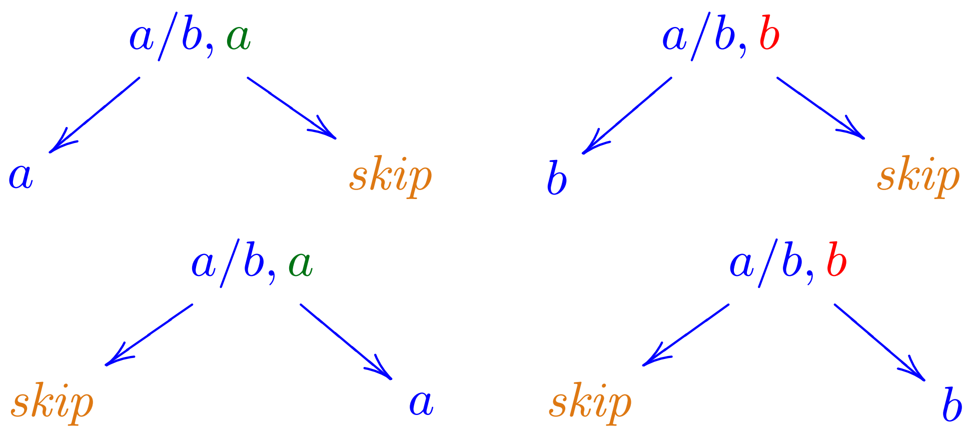


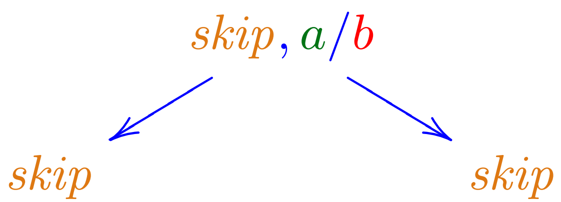
$\textit{rank}(a) = 0$

$\textit{rank}(b) = 1$

recognizes the set of trees where, on each branch, b appears only finitely often.

The complement can be recognized by





$\textit{rank}(a) = 1$

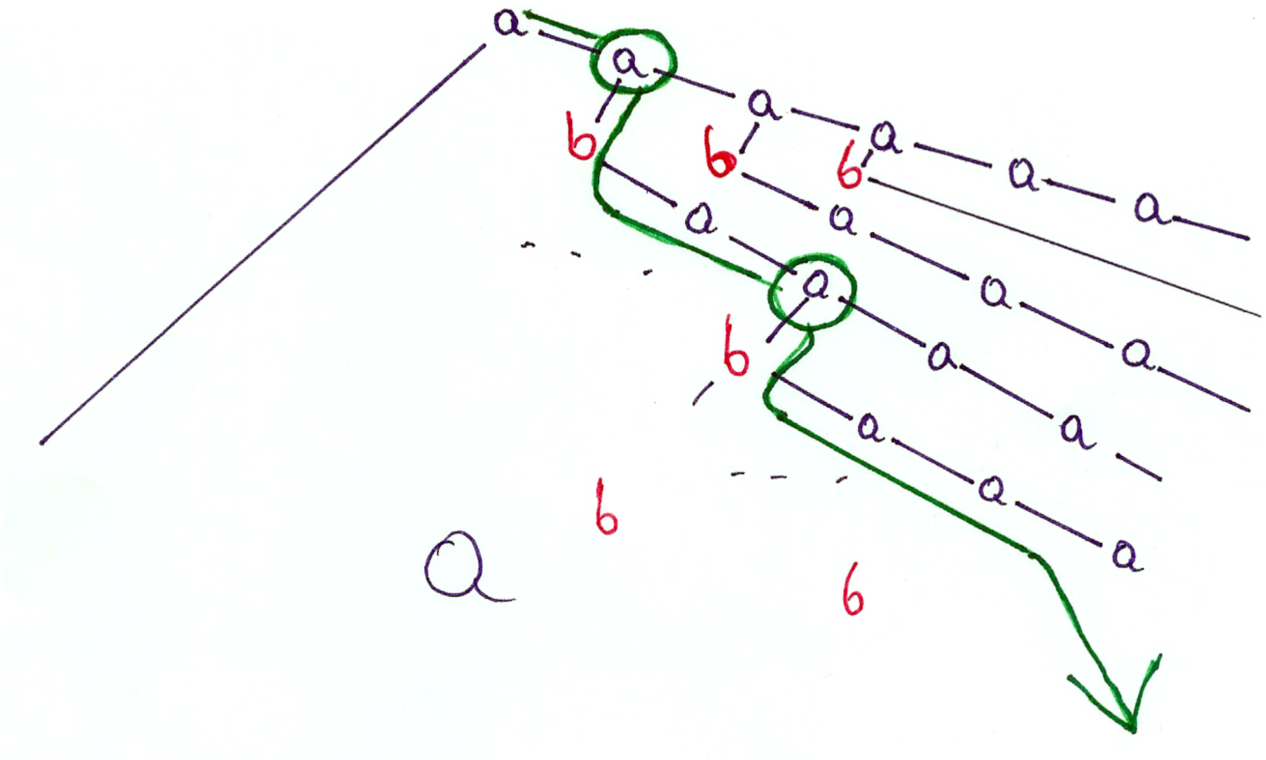
$\textit{rank}(b) = \textit{rank}(skip) = 2$

Fixed-point definitions carry over to trees.

$\nu Y.\mu X.aX \cup bY$	$=$	$(a^*b)^\omega$
$\mu X.\nu Y.aX \cup bY$	$=$	$(a+b)^*a^\omega$
$\nu y.\mu z.\, a(y,y) \cup b(z,z)$	$=$	binary trees over $\{a,b\}$ where, on each branch, b appears infinitely often.
$\mu z.\nu y.\, a(y,y) \cup b(z,z)$	$=$	binary trees over $\{a,b\}$ where, on each branch, b appears only finitely often.

Rabin 1970 proved that the last set cannot be recognized by a Büchi (i.e., index $(1,2)$) automaton.

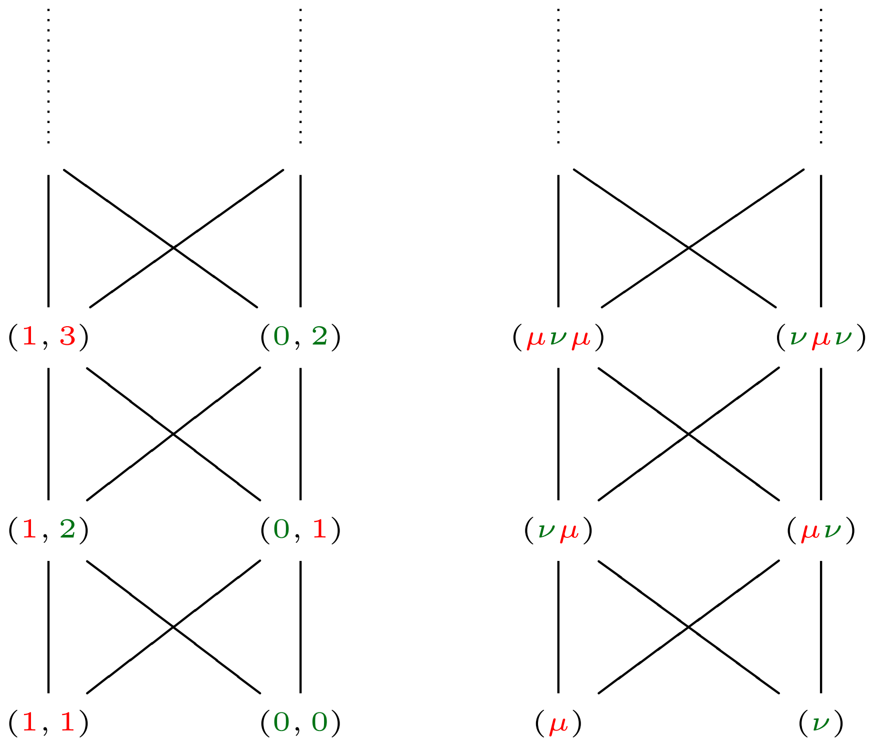
Rabin’s proof



Again, a topological argument could be used instead, as this set is Π^1_1 complete, while the Büchi automata can recognize only Σ^1_1 sets.

The following witness the strictness of the non-deterministic index hierarchy.

$$T_{i,k} = \{t \in \{i,\ldots,k\}^{\{l,r\}^*} : \text{ each branch is in } M_{i,k}\ \}$$ (N 1986)

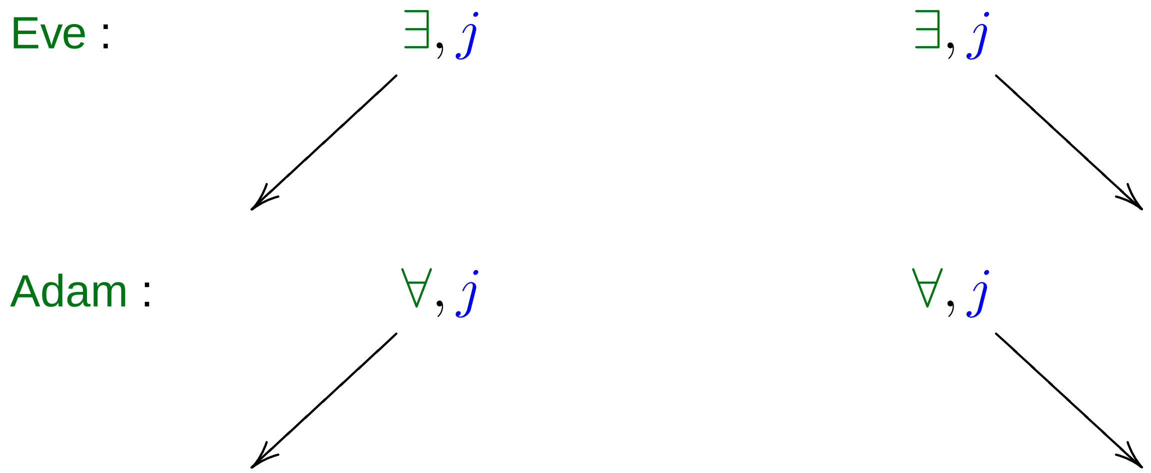


Here, a topological argument cannot be used, as all the sets $T_{i,k}$ are $\mathbf{\Pi^1_1}$ complete, hence Wadge–equivalent (except for $T_{0,0}$, $T_{1,1}$, $T_{1,2}$).

But topology comes back in the proof of the strictness of the alternating index hierarchy (Bradfield, Arnold 1998).

Game tree languages

Alphabet : $\{\exists, \forall\} \times \{i, \dots, k\}$.

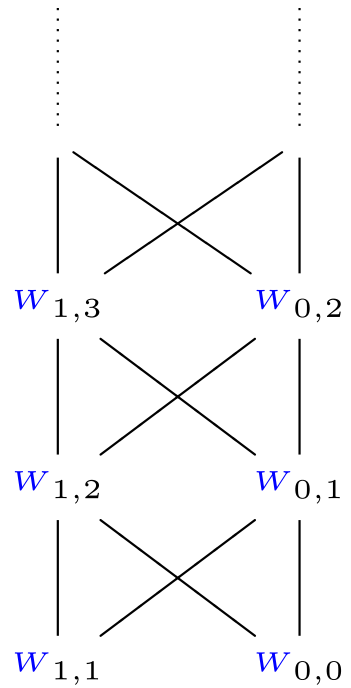


Eve wins an infinite play $(x_0, i_0), (x_1, i_1), (x_2, i_2), \dots \quad (x_\ell \in \{\exists, \forall\})$

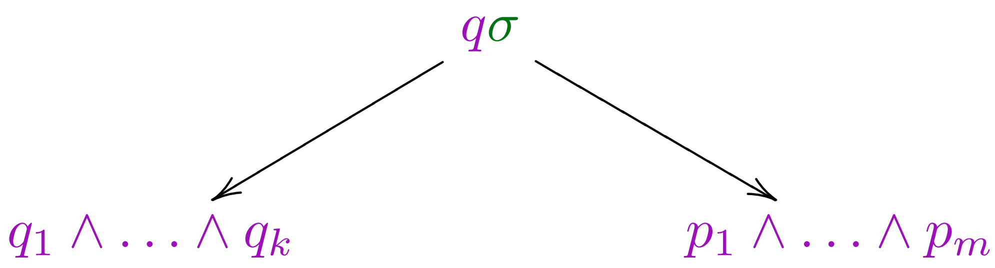
iff $\limsup_{\ell \to \infty} i_\ell$ is even.

The set $W_{i,k}$ consists of all trees such that Eve has a winning strategy.

The sets $W_{i,k}$ (like $M_{i,k}$, and unlike $T_{i,k}$) form the strict hierarchy w.r.t. the Wadge reducibility (Arnold & N, 2008).



Alternating parity tree automata



An input tree $t \in \Sigma^{\{l,r\}^*}$ induces a computation tree over states, $comp(t)$.

Composing with the function *rank* : $Q \to \{i, \ldots, k\}$, we have

$$t \in T(A) \iff \textit{rank} \circ comp(t) \in W_{i,k}$$

Hence,

$$T(A) \leq_w W_{i,k}$$

In particular, if an alternating automaton A of index (i,k) accepted $W_{\overline{i,k}}$, we would have $W_{\overline{i,k}} \leq_w W_{i,k}$, a contradiction.

Sketch of proof that $W_{\overline{i,k}} \not\leq_w W_{i,k}$

Up to renaming,

$$W_{\overline{i,k}} \approx \overline{W_{i,k}}$$

By Banach Fixed-Point Theorem, there is no contracting reduction of L to $\overline{L}$

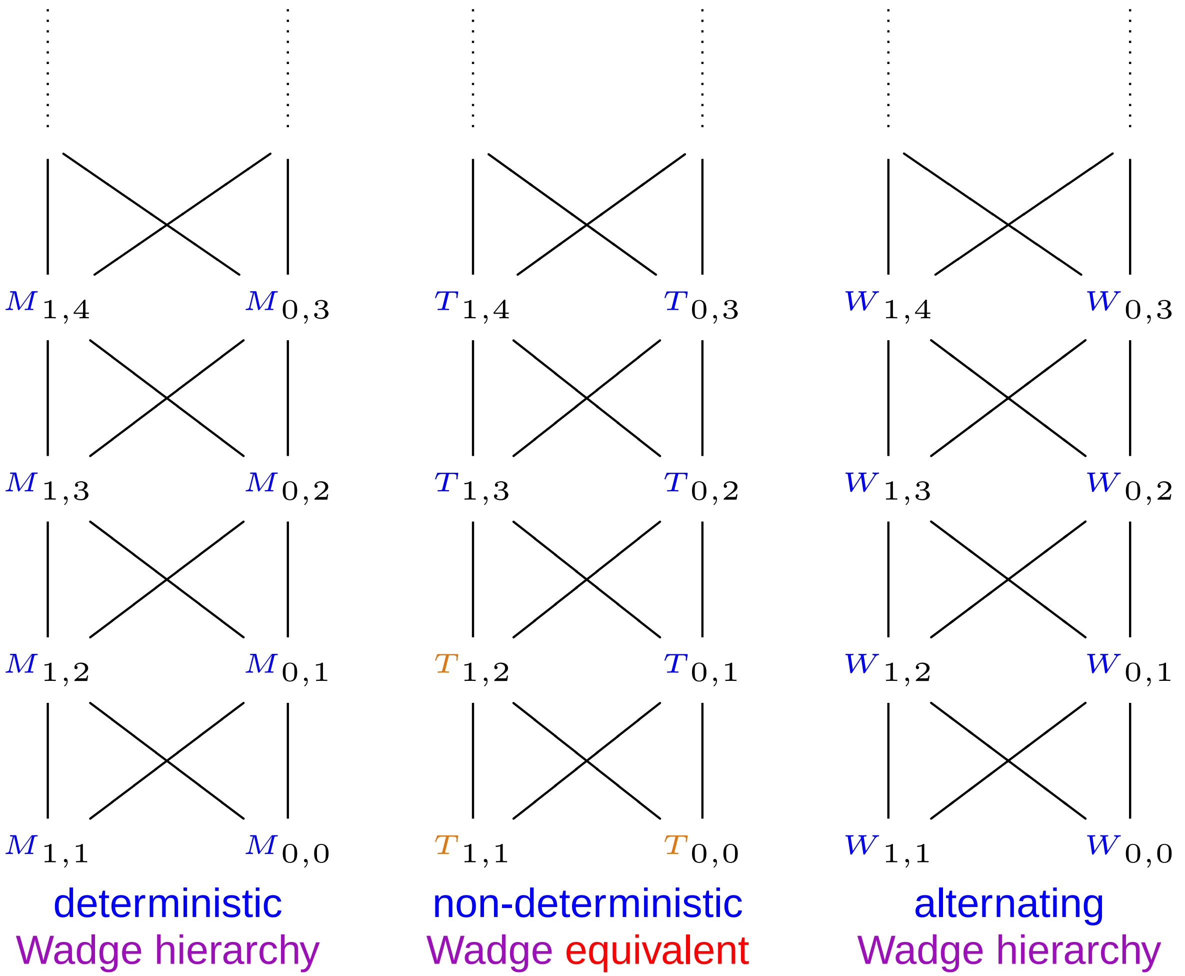
$$x_{fix} \in L \iff f(x_{fix}) \in \overline{L} \iff x_{fix} \in \overline{L}$$

Main Lemma If f reduces $W_{i,k}$ to some L then there is a mapping $h : \{i, \ldots, k\}^{\{l,r\}^*} \to \{i, \ldots, k\}^{\{l,r\}^*}$ (padding), such that

- h reduces $W_{i,k}$ to itself,
- $f \circ h$ is contracting.

About h: For $W_{0,k}$, it "stretches" the original tree completing by the nodes labeled by $(\forall, 0)$. For $W_{1,k}$, by $(\exists, 1)$.

Witnesses of the index hierarchies



When the two complexities diverge. . .

If a recognizable set of trees is Büchi recognizable (equivalently $\nu\mu$, $\exists$ S2S) then it is $\mathbf{\Sigma}^1_1$.

The converse does not hold.

Let

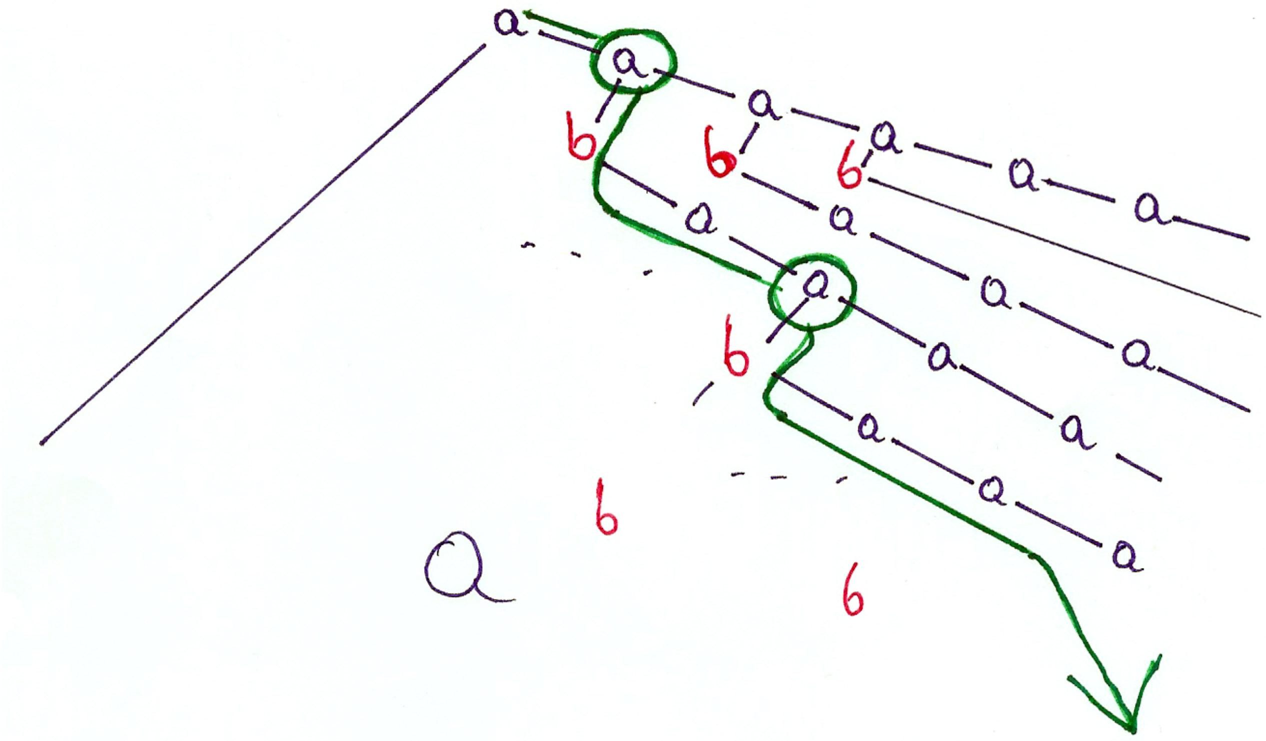
$H!$ $=$ binary trees over $\{a, b\}$ where b appears infinitely often on exactly one branch.

By Lusin Theorem ([Kechris, Thm. 18.11]), $H!$ is $\mathbf{\Pi}^1_1$ (complete).

Hence $\overline{H!}$ is $\mathbf{\Sigma}^1_1$.

But it is **not** Büchi recognizable!

Rabin’s proof works. . .



Note

$H!$ is non-ambiguous, i.e., can be recognized by a non-ambiguous parity tree automaton (exactly one accepting run).

Questions

- Are all non-ambiguous languages $\mathbf{\Pi}^1_1$?

  (It is so for deterministic languages.)

- Is it decidable, if a given tree language is non-ambiguous ?

  (It is so for determinism.)

- What is the expressive power of non-ambiguous automata ?

Fact

No non-ambiguous automaton can recognize the set of binary trees over $\{a, b\}$ such that b appears at least once (elementary proof: Carayol & Löding 2007).

$\Longleftrightarrow$ (N. & Walukiewicz 1996) The S2S formula

$$X = \emptyset \lor y \in X$$

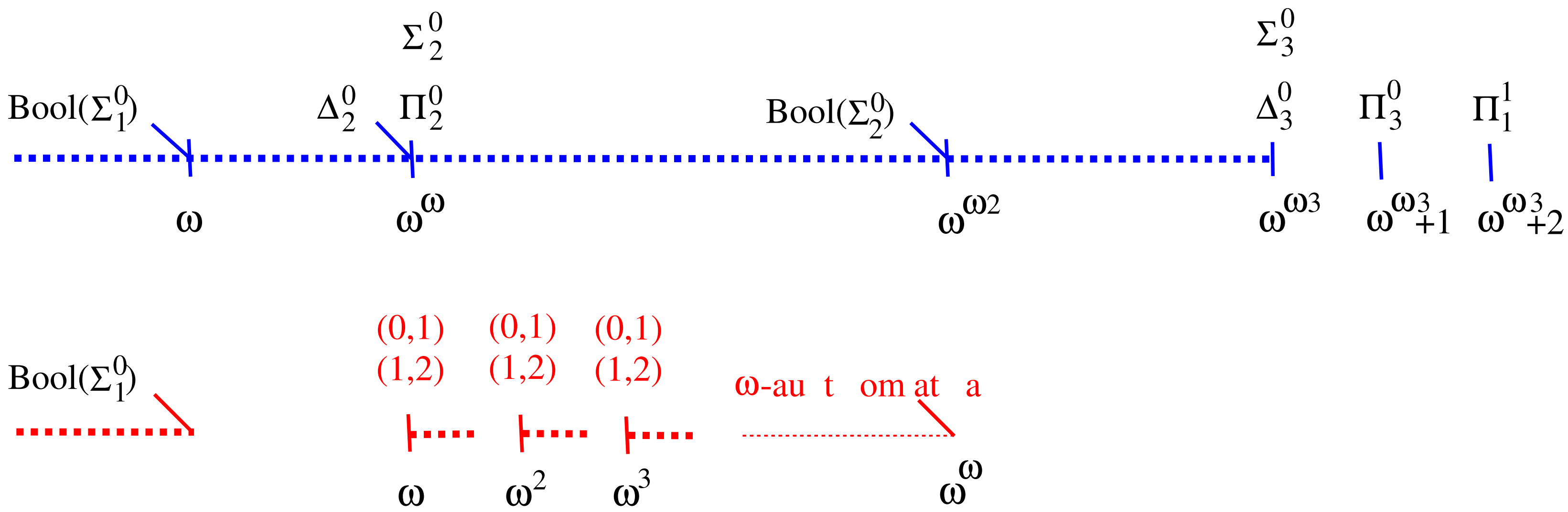
cannot be made functional $(X \mapsto y)$.

Consequently, S2S is not uniformizable (Gurevich & Shelah 1983).

In contrast to S1S...

Büchi & Landweber 1968 (?), Rabinovich 2007.

Wadge hierarchy for deterministic tree languages (Murlak 2006)



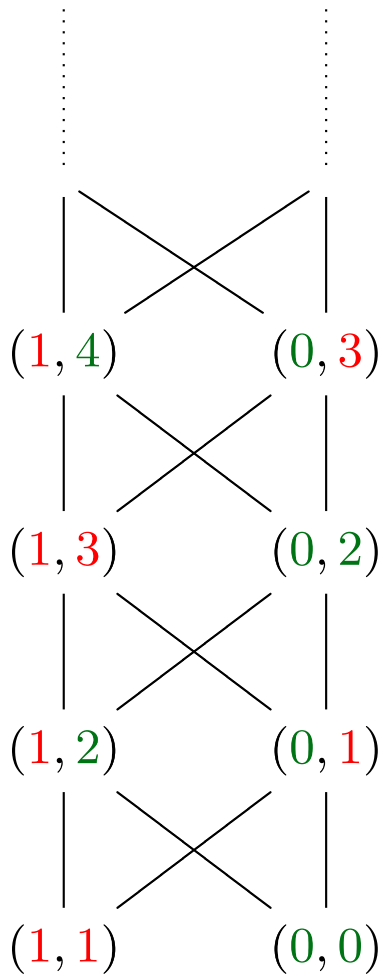
- The height is $\omega^{\omega\cdot 3} + 3$ (*vs.* ω^{ω} for word languages, Wagner 1979).
- Complete sets exist in $\mathbf{\Pi^1_1}$, $\mathbf{\Pi^0_3}$, and surprisingly, in $\mathbf{\Delta^0_3}$.

Results on Wadge hierarchies

Words	Finite automata	ω^{ω}	Wagner 1979
	Deterministic pushdown aut.	ω^{ω^2}	Duparc 2003
	Deterministic Turing machines	$\left(\omega^{CK}\right)^{\omega}$	Selivanov 2003
	Non-deterministic pushdown aut.	$> \epsilon_0$	Finkel 2001
Trees	Deterministic finite automata	$\omega^{\omega \cdot 3} + 3$	Murlak 2006
	Weak alternating automata	$\geq \epsilon_0$	Duparc & Murlak 2007

Decidability issues

Can we decide the level of a recognizable tree language in the index hierarchy ?



We know the answer only if an input automaton is deterministic.

The problem

Given : a deterministic parity tree automaton

Compute : the minimal index of a non-deterministic automaton

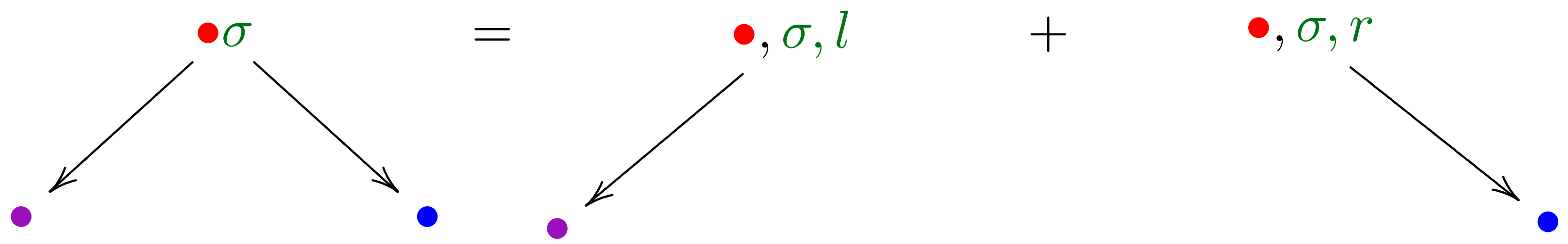
recognizing the same language.

Urbański 2000 solved the question $\equiv$ *(non-deterministic) Büchi ?*

N. & Walukiewicz 2004 settled the whole non-deterministic hierarchy.

From trees to words : path automata

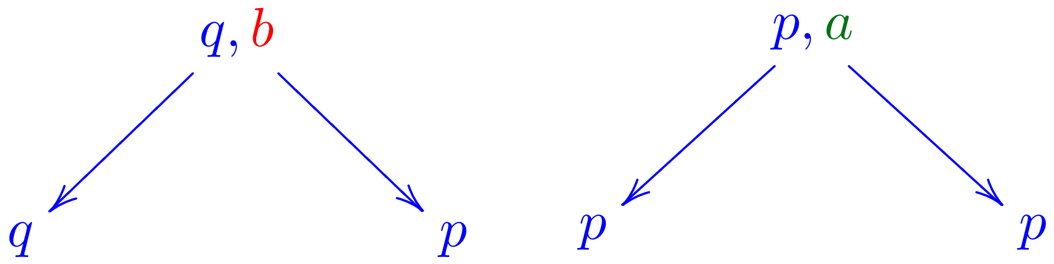
A deterministic tree automaton $\mathcal{A}$ over alphabet Σ can be identified with a deterministic word automaton $\mathcal{A}'$ over alphabet $\Sigma \times \{l, r\}$,



$\mathcal{A}$ recognizes a tree t iff $\mathcal{A}'$ recognizes all paths of t.

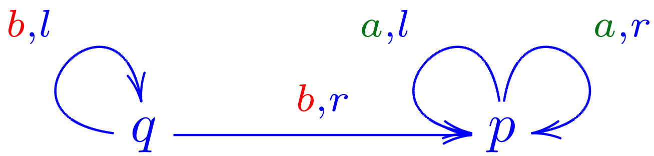
Example

Deterministic tree automaton :



rank$(q) = 0$ *rank*$(p) = 1$

Corresponding path automaton :

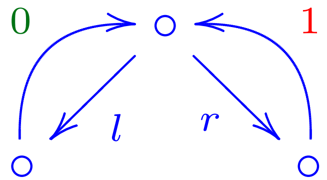
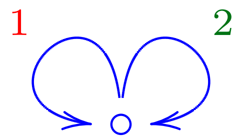
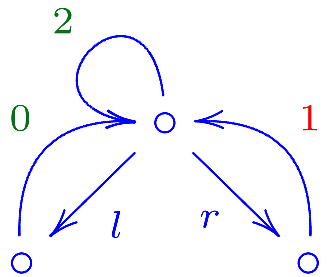
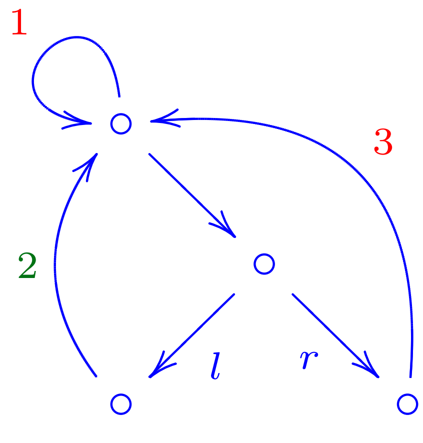


Determinization, whenever possible, is effective

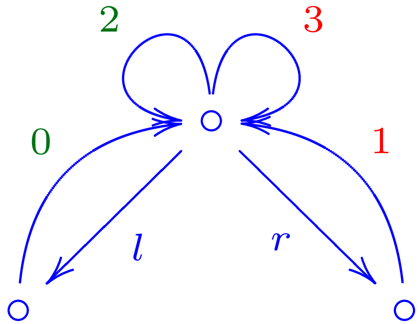
The concept of path automaton allows us to decide (in EXPTIME), if a given non-deterministic tree automaton is equivalent to a deterministic one.

It suffices to verify if

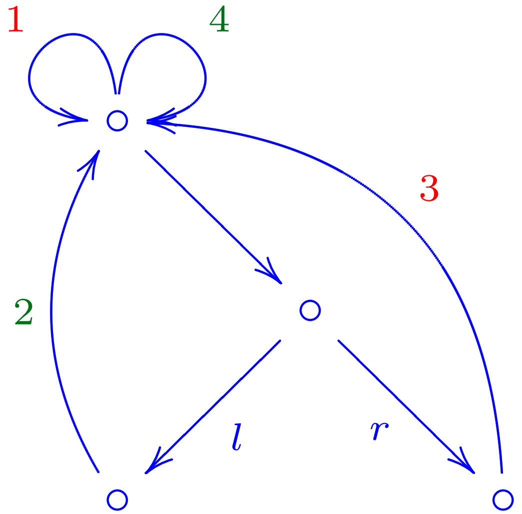
$$L(\mathcal{A}) = Trees(Paths(L(\mathcal{A})))$$

Index class	Forbidden pattern
(1,2)	
(0,1)	
(0,2)	
(1,3)	

The (0, n) case, $n \geq 3$



The (1,n) case, $n \geq 4$



Theorem. Let $\mathcal{A}$ be a deterministic tree automaton.

Then $L(\mathcal{A})$ can be recognized by a non-deterministic tree automaton of index (ι, n) if and only if the corresponding path automaton does not contain any productive $\overline{(\iota, n)}$ pattern.

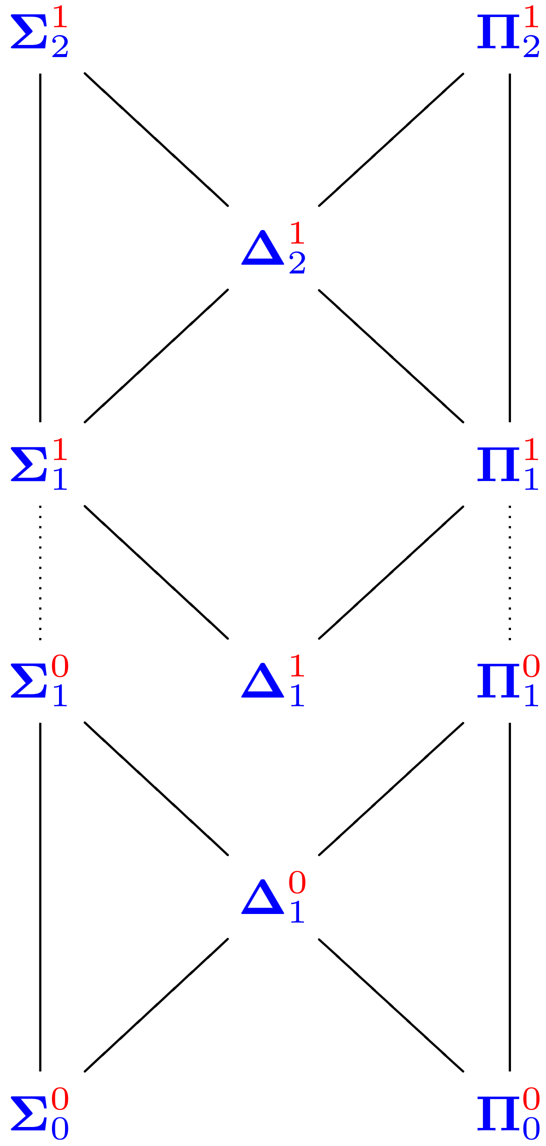
An idea of the proof.

($\Leftarrow$) Unravel a forbidden pattern into a tree and refine Rabin's argument.

($\Rightarrow$) Decompose $\mathcal{A}$ into strongly connected components, and apply inductive arguments to the sub-automata induced this way.

Corollary. Consequently, the index of a deterministic tree language can be computed within the complexity of computing productive states (i.e., NP $\cap$ co-NP), N. & Walukiewicz 2004.

Can we decide the level of a recognizable tree language in the Borel/projective hierarchies ?



For the case of infinite words, the question was settled already by Büchi & Landweber 1969.

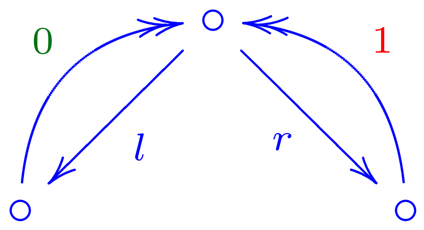
For trees, we can determine the exact level of $\mathcal{T}(\mathcal{A})$, provided that $\mathcal{A}$ is a deterministic automaton

(N. & Walukiewicz 2003, Murlak 2005).

Non-deterministic case is completely open.

Criterion : forbidden patterns

If a path automaton $\mathcal{A}'$ contains a (productive) pattern



then $\mathcal{T}(\mathcal{A})$ is $\mathbf{\Pi}^1_1$-complete, hence non-Borel.

Otherwise it is in $\mathbf{\Pi}^0_3$ (N & Walukiewicz 2003). **Dichotomy!**

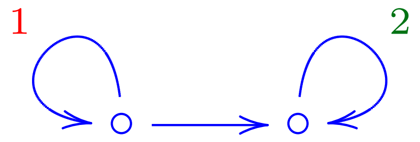
(In contrast, Skurczyński 1993 showed that there are non-deterministically recognizable tree languages on every finite level of the Borel hierarchy.)

The algorithm of detecting patterns runs in time of solving the non-emptiness problem of parity tree automata (NP $\cap$ co-NP).

Murlak 2005 settles the remaining cases :

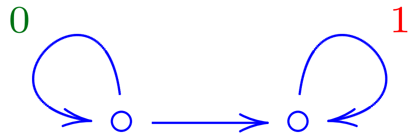
Class — **Forbidden pattern**

Π^0_1



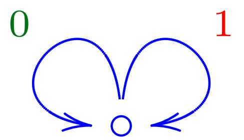
"folklore"

Σ^0_1

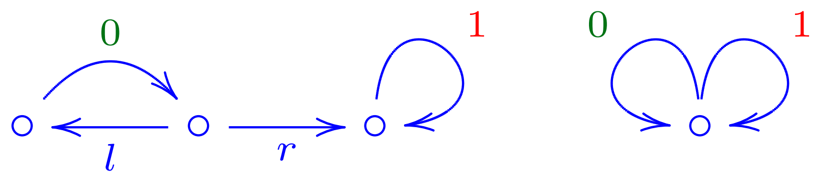


....

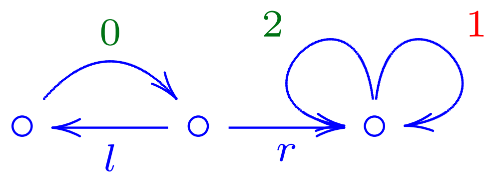
Π^0_2



Σ^0_2



Δ^0_3



Wadge reducibility—decidability issues

Fact (Büchi & Landweber 1969). For Büchi automata on infinite words:

(1) If $L(\mathcal{A}) \leq_w L(\mathcal{B})$ then there exists a finite–state transducer reducing $L(\mathcal{A})$ to $L(\mathcal{B})$.

(2) It is decidable if $L(\mathcal{A}) \leq_w L(\mathcal{B})$.

For trees, (1) does not hold. Nevertheless, Murlak 2006 shows

Fact. It is decidable if $T(\mathcal{A}) \leq_w T(\mathcal{B})$, for deterministic tree automata $\mathcal{A}$, $\mathcal{B}$.

Rather than comparing two automata "from scratch", one computes, for each deterministic automaton $\mathcal{A}$, its place in the hierarchy, i.e., an ordinal and a canonical automaton equivalent to $\mathcal{A}$.

Construction of canonical automata is the core of the proof.

Again, the non-deterministic case remains open.

Instead of conclusion

rational	irrational
closure properties,	non-uniformization of S2S
S2S characterization	ambiguity
parity condition	complexity of the non-emptiness problem ?
topological characterizations	discrepancies
effective hierarchies	non-deterministic case ?
. . . . . .	. . . . . .