On separation question for automata-theoretic hierarchies

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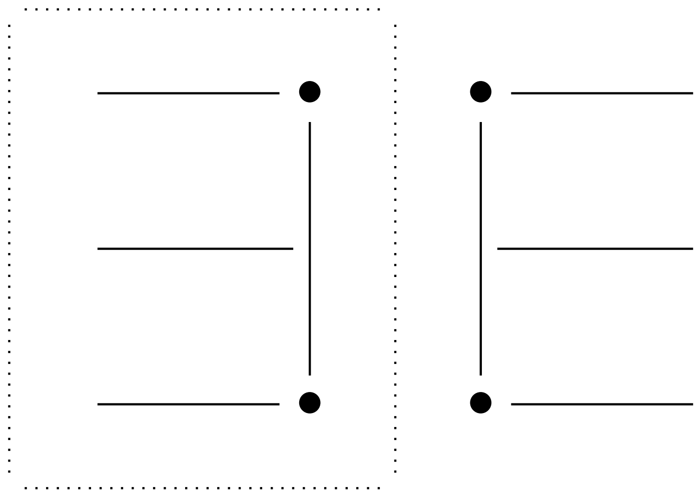
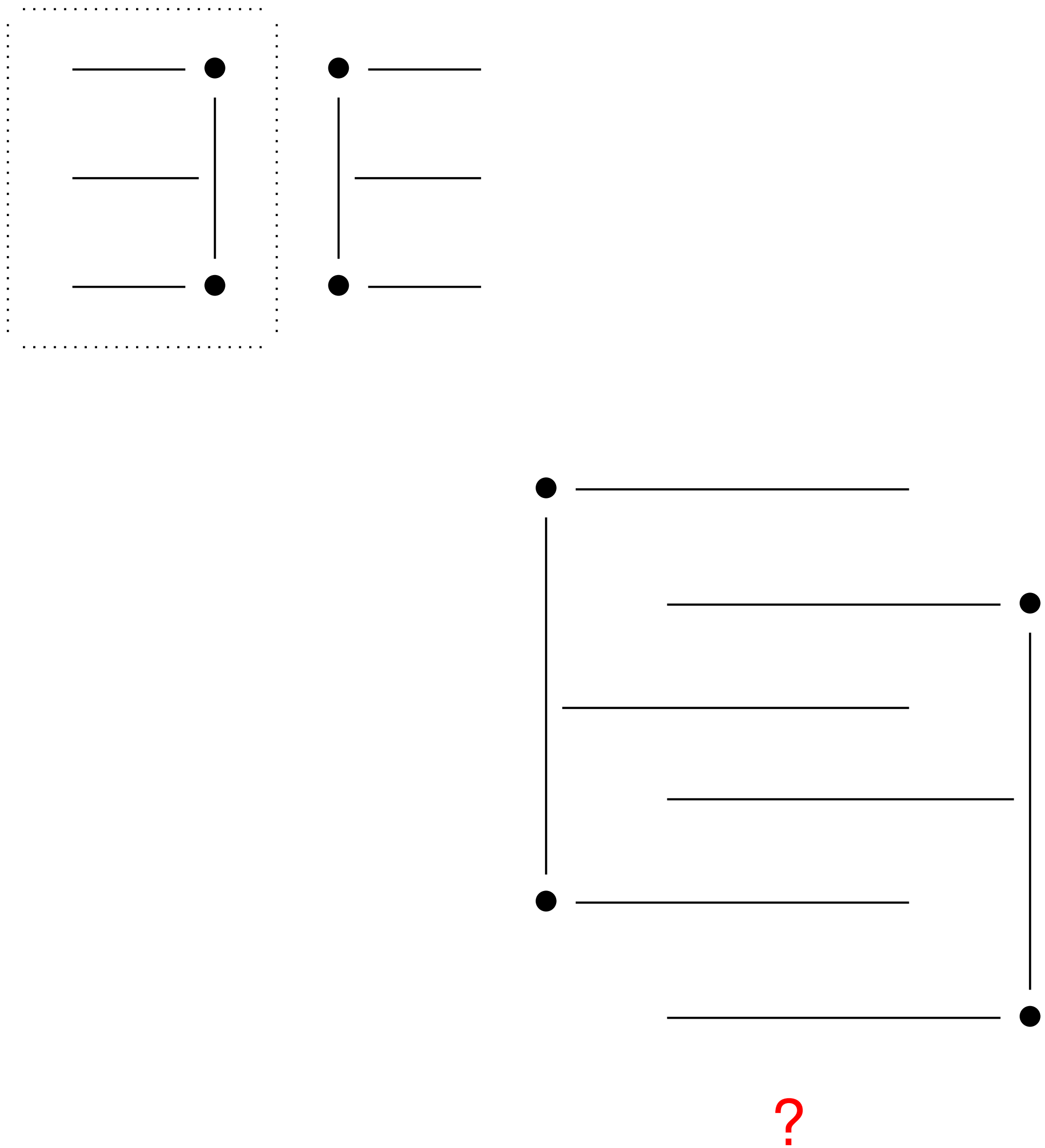
Separation problem

Given disjoint sets A and B, find a *simpler* set C,

such that

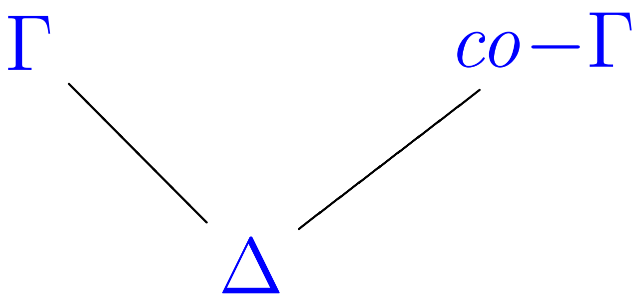
$$A \subseteq C$$
$$B \subseteq \overline{C}.$$





Separation property

A class Γ has *separation property* if any disjoint $A, B \in \Gamma$ are separable by some $C \in \Delta = \Gamma \cap co\!-\!\Gamma$ (where $co\!-\!\Gamma = \{\overline{X} : X \in \Gamma\}$).



Examples

Any two disjoint *co-recursively enumerable* sets are separable by a *recursive* set.

Not so with *r.e.*-sets, in general.

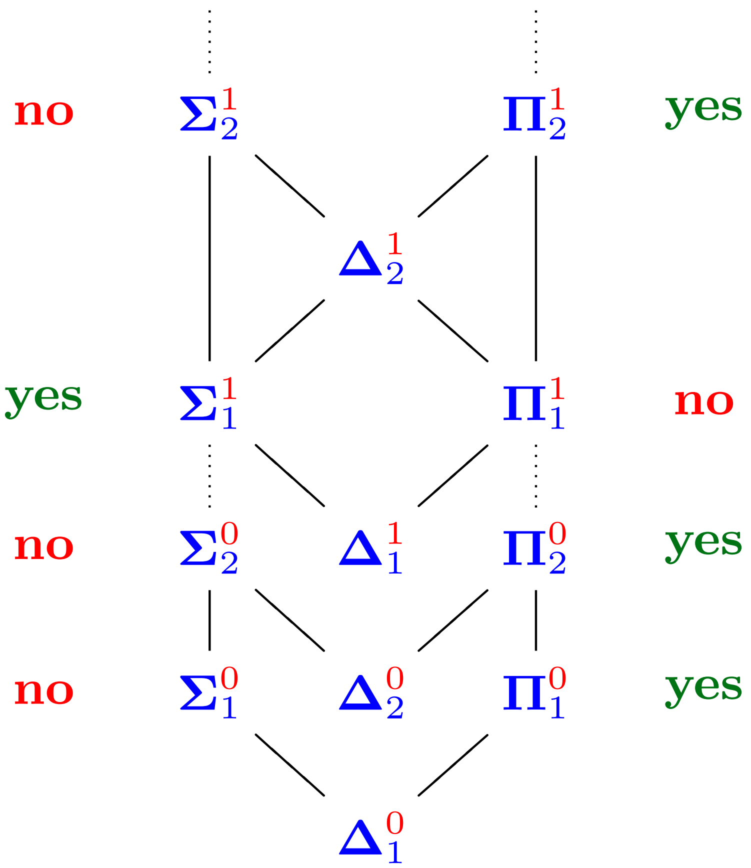
Any two *closed* subsets of the Cantor discontinuum are separable by a *clopen* set.

Not so with *open* sets, in general.

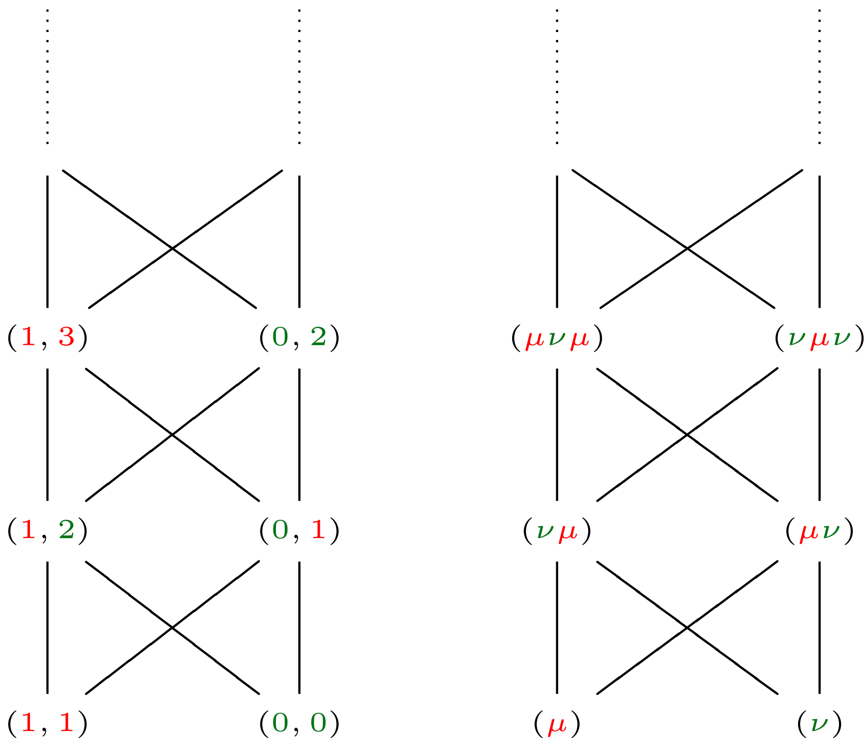
Lusin theorem. Any two disjoint *analytic* sets are separable by a *Borel* set.

Not so with *co-analytic* sets, in general.

Separation property for classical (e.g., topological) hierarchies is well understood.



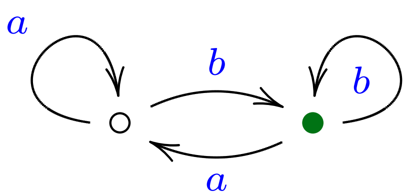
We study the problem for the **Rabin–Mostowski index hierarchy** of automata on infinite words and trees.



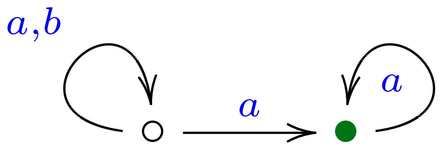
Büchi automata on infinite words

$$\mathcal{A} = \langle \Sigma, Q, q_I, Tr, F \rangle$$

where $Tr \subseteq Q \times \Sigma \times Q$, $F \subseteq Q$.



$$((a+b)^*b)^\omega$$



$$(a+b)^*a^\omega$$

The second one cannot be recognized by a deterministic automaton.

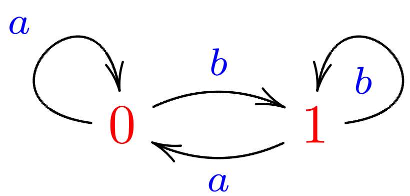
$$\xrightarrow{a}\xrightarrow{b}\xrightarrow{a}\xrightarrow{a}\xrightarrow{b}\xrightarrow{a}\xrightarrow{a}\xrightarrow{a}\xrightarrow{b}\xrightarrow{a}\xrightarrow{a}\xrightarrow{a}\xrightarrow{a}\xrightarrow{b}\xrightarrow{a} \ldots\ldots \xrightarrow{a}\xrightarrow{b}\xrightarrow{a} \ldots$$

Parity automata

$$\mathcal{A} = \langle \Sigma, Q, q_I, Tr, \textit{rank} \rangle$$

where $\textit{rank} : Q \to \omega$.

A run is accepting if $\limsup_{i\to\infty} \textit{rank}(q_i)$ is even.



$$(a+b)^* a^\omega$$

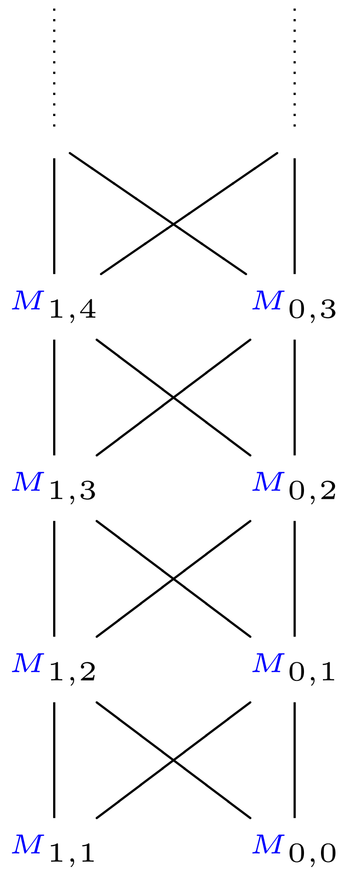
The Rabin-Mostowski **index** of a parity automaton $\mathcal{A}$ is

$$(\min \textit{rank}, \max \textit{rank})$$

We can assume $\min \textit{rank} \in \{0, 1\}$.

The McNaughton Theorem. A nondeterministic Büchi automaton can be simulated by a deterministic parity automaton of some index (i, k).

The minimal index (i, k) may be arbitrarily high (Wagner 1979, Kaminski 1985).



Proof (from *The Book* ? :-)

Let C be a set of *colours* and $\mathcal{X}$ any subset of C^ω. An $\mathcal{X}$-*automaton*

$$\mathcal{A} = \langle \Sigma, Q, q_I, Tr, \textit{rank} : Q \to C \rangle$$

accepts a word $u \in \Sigma^\omega$ iff $\textit{rank}(u) \in \mathcal{X}$.

If $\Sigma = C$ then there always exists a fixed point

$$r = \textit{rank}(r).$$

Namely,

$$\begin{aligned} r_0 &= \textit{rank}(q_I) \\ r_n &= \textit{rank}\left(\hat{Tr}(q_I, r_0 \ldots r_{n-1})\right). \end{aligned}$$

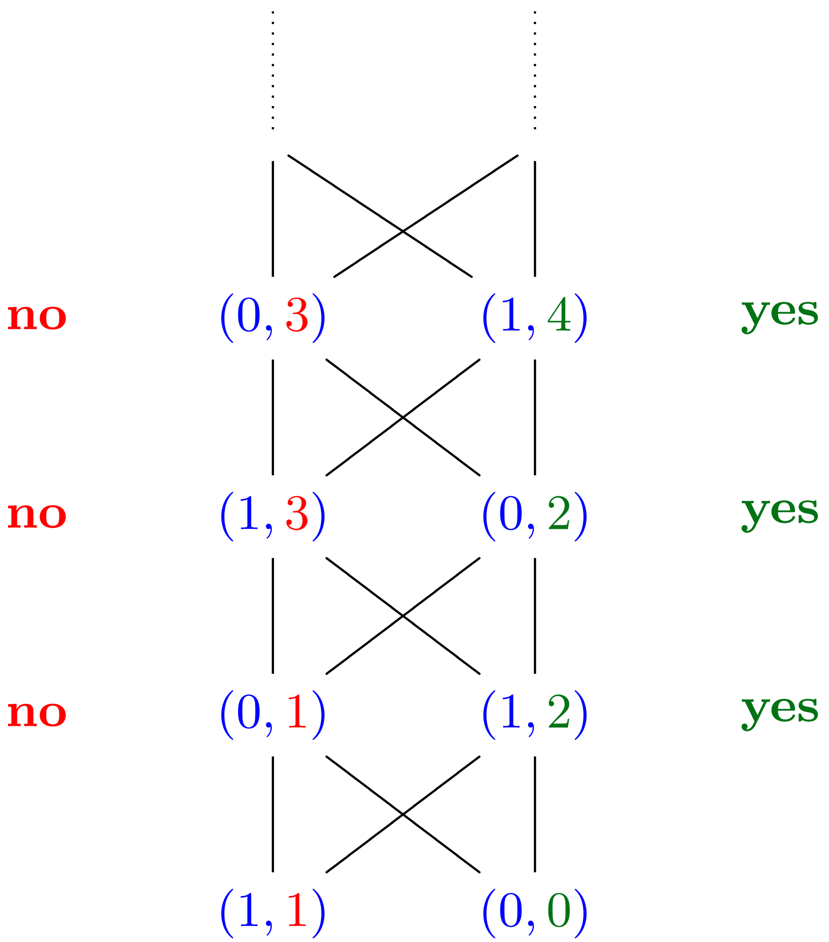
Hence no $\mathcal{X}$-automaton can recognize $\overline{\mathcal{X}}$.

In particular, no deterministic automaton of index (i, k) can recognize

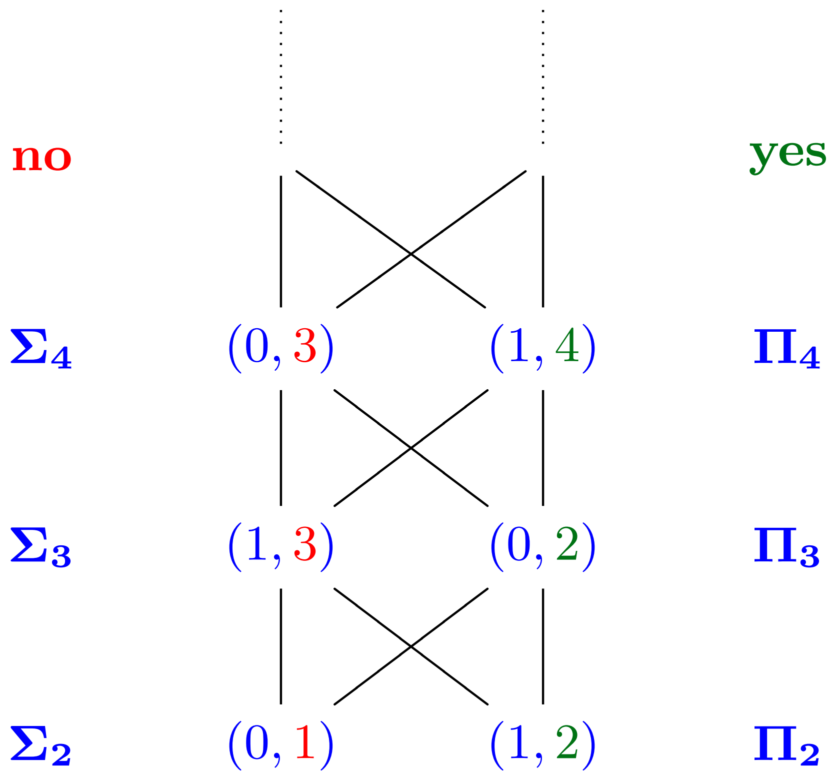
$$\{u \in \{i, \ldots, k\}^\omega : \limsup_{\ell \to \infty} u_\ell \text{ is } \textbf{odd} \}.$$

Result

Separation property for deterministic automata on infinite words:



In Σ/Π notation, separation property holds for Π classes, and fails for Σ classes.



Notation:

$$
\begin{aligned}
3 &= \{0,1,2\} \\
K_3 &= \{u \in 3^\omega : \limsup_{\ell\to\infty} u_\ell \text{ is even } \} \\
I_3 &= \{u \in 3^\omega : \limsup_{\ell\to\infty} u_\ell = 2\}
\end{aligned}
$$

Key Lemma (for $m = 3$). There exist disjoint sets $U_1, U_2 \subseteq \left(3^2\right)^\omega$ of class Σ_3, satisfying the following:

$$
\begin{aligned}
K_3 \times \overline{K_3} &\subseteq U_1 \\
\overline{K_3} \times K_3 &\subseteq U_2 \\
\overline{K_3 \times K_3} &\subseteq U_1 \cup U_2 = \overline{I_3 \times I_3}.
\end{aligned}
$$

Proof of the result.

Key Lemma:

$$\begin{array}{lcl} K_3 \times \overline{K_3} & \subseteq & U_1 \\ \overline{K_3} \times K_3 & \subseteq & U_2 \\ \overline{K_3 \times K_3} & \subseteq & U_1 \cup U_2 \quad = \quad \overline{I_3 \times I_3}. \end{array}$$

Let $\mathcal{A}$ and $\mathcal{B}$ be automata of class Π_3, s.t. $L(\mathcal{A}) \cap L(\mathcal{B}) = \emptyset$.

Then, for any $u \in \Sigma^\omega$,

$$\begin{array}{rcl} u \in L(\mathcal{A}) & \Longrightarrow & \mathit{rank}^{\mathcal{A}\times\mathcal{B}}(u) \in U_1 \\ u \in L(\mathcal{B}) & \Longrightarrow & \mathit{rank}^{\mathcal{A}\times\mathcal{B}}(u) \in U_2 \\ \mathit{rank}^{\mathcal{A}\times\mathcal{B}}(u) & \in & U_1 \cup U_2. \end{array}$$

Hence, $\mathcal{A}$ and $\mathcal{B}$ are separated by

$$\left(\mathit{rank}^{\mathcal{A}\times\mathcal{B}}\right)^{-1}(U_1).$$

$$
\begin{aligned}
u \in L(\mathcal{A}) &\implies \mathit{rank}^{\mathcal{A}\times\mathcal{B}}(u) \in U_1 \\
u \in L(\mathcal{B}) &\implies \mathit{rank}^{\mathcal{A}\times\mathcal{B}}(u) \in U_2
\end{aligned}
$$

Suppose $U_1 \subseteq X$ and $U_2 \subseteq \overline{X}$, for some $X \in \Delta_3$. Let

$$
\begin{aligned}
L(\mathcal{A}) &= \overline{X} \\
L(\mathcal{B}) &= X
\end{aligned}
$$

Then

$$
\begin{aligned}
u \in \overline{X} &\implies \mathit{rank}^{\mathcal{A}\times\mathcal{B}}(u) \in U_1 \subseteq X \\
u \in X &\implies \mathit{rank}^{\mathcal{A}\times\mathcal{B}}(u) \in U_2 \subseteq \overline{X}
\end{aligned}
$$

But the mapping

$$
u \mapsto \mathit{rank}^{\mathcal{A}\times\mathcal{B}}(u)
$$

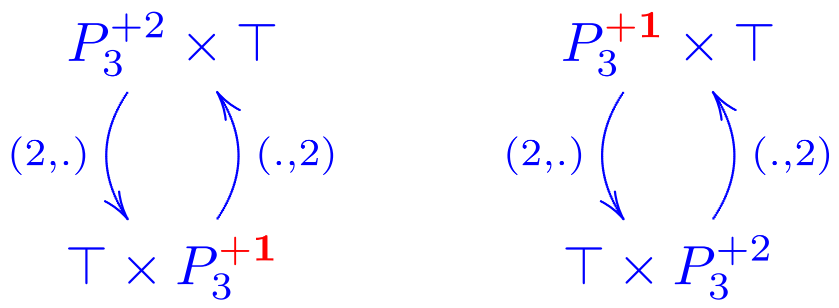
has a **fixed point**, a contradiction **!**

$$
\begin{array}{lcllcl}
K_3 \times \overline{K_3} & \subseteq & U_1 & & & \\
\overline{K_3} \times K_3 & \subseteq & U_2 & & & \\
\overline{K_3 \times K_3} & \subseteq & U_1 \cup U_2 & = & \overline{I_3 \times I_3}. &
\end{array}
$$

Proof of the Key Lemma.

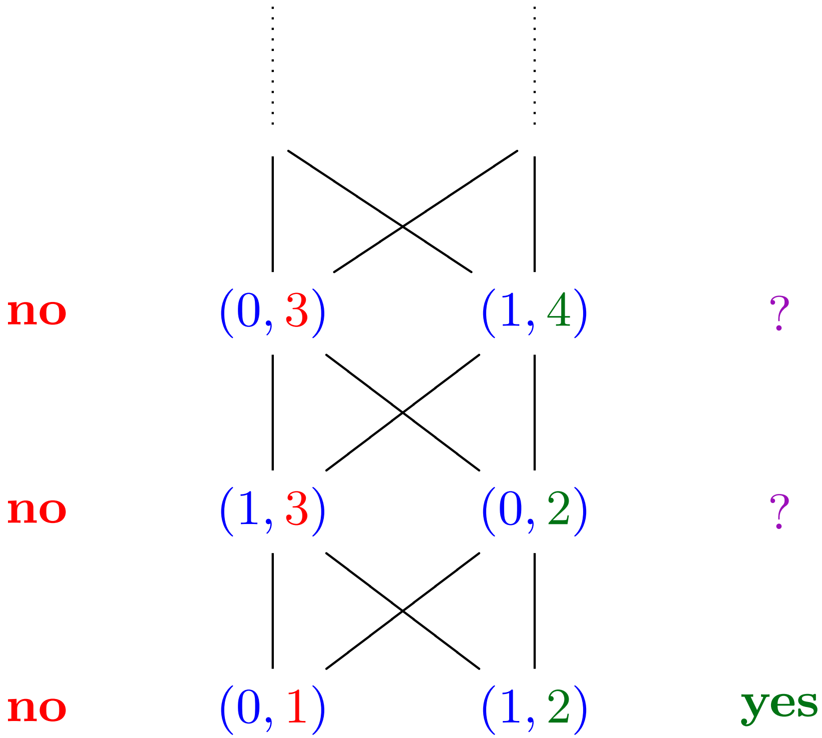
Let P_3 be the standard automaton accepting K_3.

U_1 $\qquad\qquad$ U_2



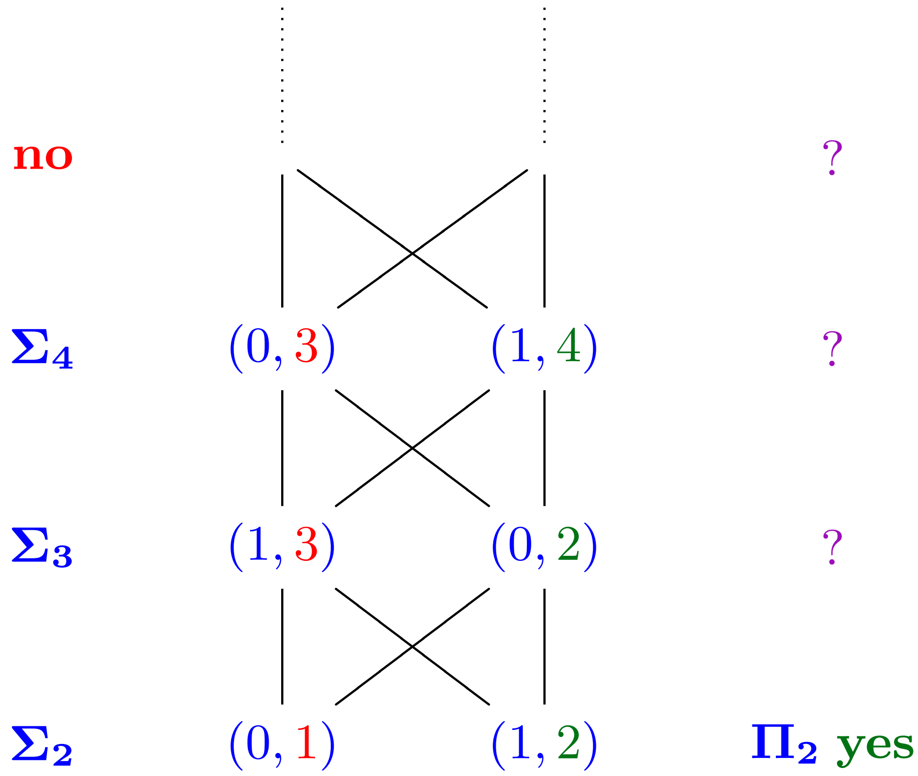
Result

Separation property for alternating automata on infinite trees:



For $(0, 1)$: Hummel, Michalewski, N., 2009. For $(1, 2)$: Rabin 1970.

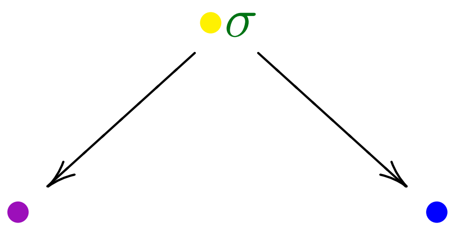
In Σ/Π notation



Parity tree automata on ℓ-ary trees

$$\mathcal{A} = \langle \Sigma, Q, q_I, Tr, \textit{rank} \rangle$$

where $Tr \subseteq Q \times \Sigma \times Q^{\ell}$,

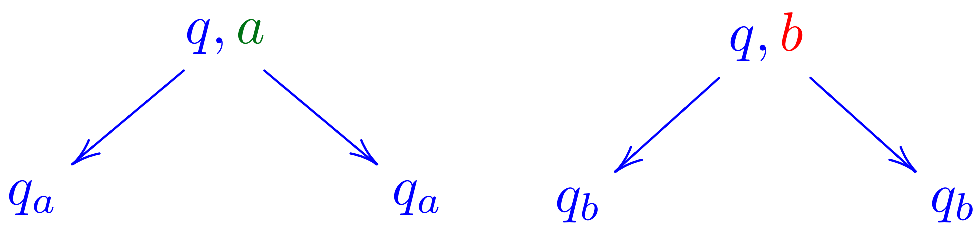


$\textit{rank} : Q \to \omega$.

The Rabin-Mostowski **index** of $\mathcal{A}$ is $(\min \textit{rank}, \max \textit{rank})$.

We can assume $\min \textit{rank} \in \{0, 1\}$.

Example



$rank(q_a) = 0$

$rank(q_b) = 1$

recognizes the set of trees where, on each branch, b appears only finitely often.

Parity games

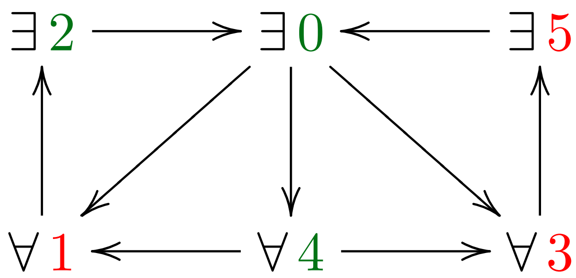
$V_\exists$	positions of Eve
$V_\forall$	positions of Adam (disjoint)
$\longrightarrow \subseteq V \times V$	possible moves (with $V = V_\exists \cup V_\forall$)
$p_1 \in V$	initial position
rank : $Q \rightarrow \omega$	the ranking function.

An infinite play $v_0 \longrightarrow v_1 \longrightarrow v_2 \longrightarrow \ldots$ is won by Eve iff $\limsup_{n\to\infty}$ *rank*(v_n) is even.

Parity games enjoy *positional determinacy*

(Emerson and Jutla 1991, Mostowski 1991).

Parity game – example



Alternating parity tree automata on ℓ-ary trees

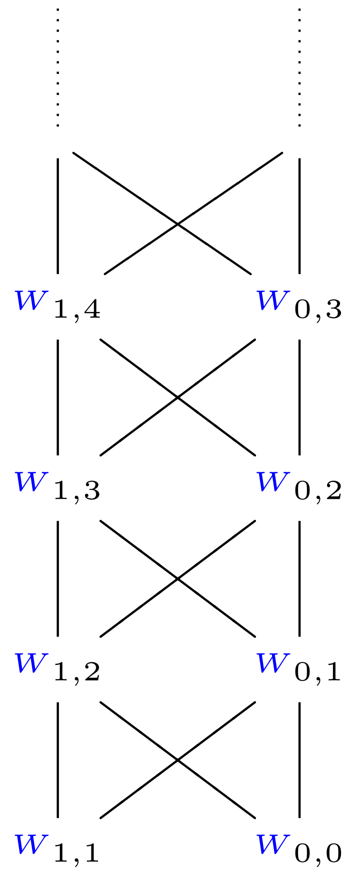
$$\mathcal{A} = \langle \Sigma, Q, Q_\exists, Q_\forall, q_I, Tr, \mathit{rank} \rangle$$

where $Q = Q_\exists \dot\cup Q_\forall$,
$Tr \subseteq Q \times \Sigma \times \{0, 1, \ldots, \ell - 1, \varepsilon\} \times Q$,
$\mathit{rank} : Q \to \omega$.

An input tree t is accepted by $\mathcal{A}$ iff Eve has a winning strategy in the parity game

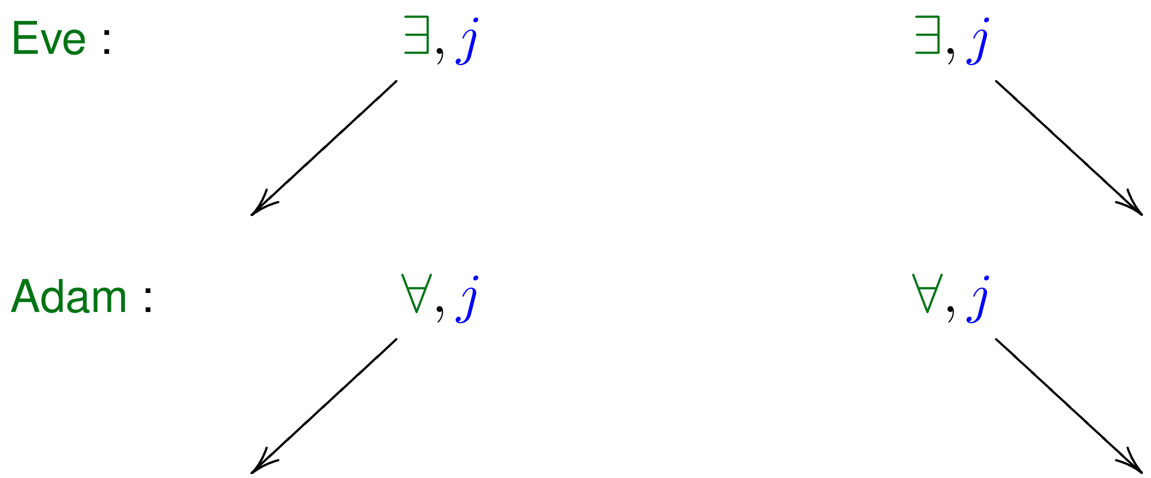
$Q_\exists \times \{0, 1, \ldots, \ell - 1\}^*$,
$Q_\forall \times \{0, 1, \ldots, \ell - 1\}^*$,
(q_I, ε),
$\mathrm{Mov} = \{((p, v), (q, vd)) \colon v \in \mathrm{dom}(t),\ (p, t(v), d, q) \in Tr\}$
$\mathit{rank}(q, v) = \mathrm{rank}(q)$.

The minimal index (i, k) may be arbitrarily high (Bradfield 1998).



Game tree languages

Alphabet : $\{\exists, \forall\} \times \{i, \ldots, k\}$, with $i \in \{0, 1\}$.

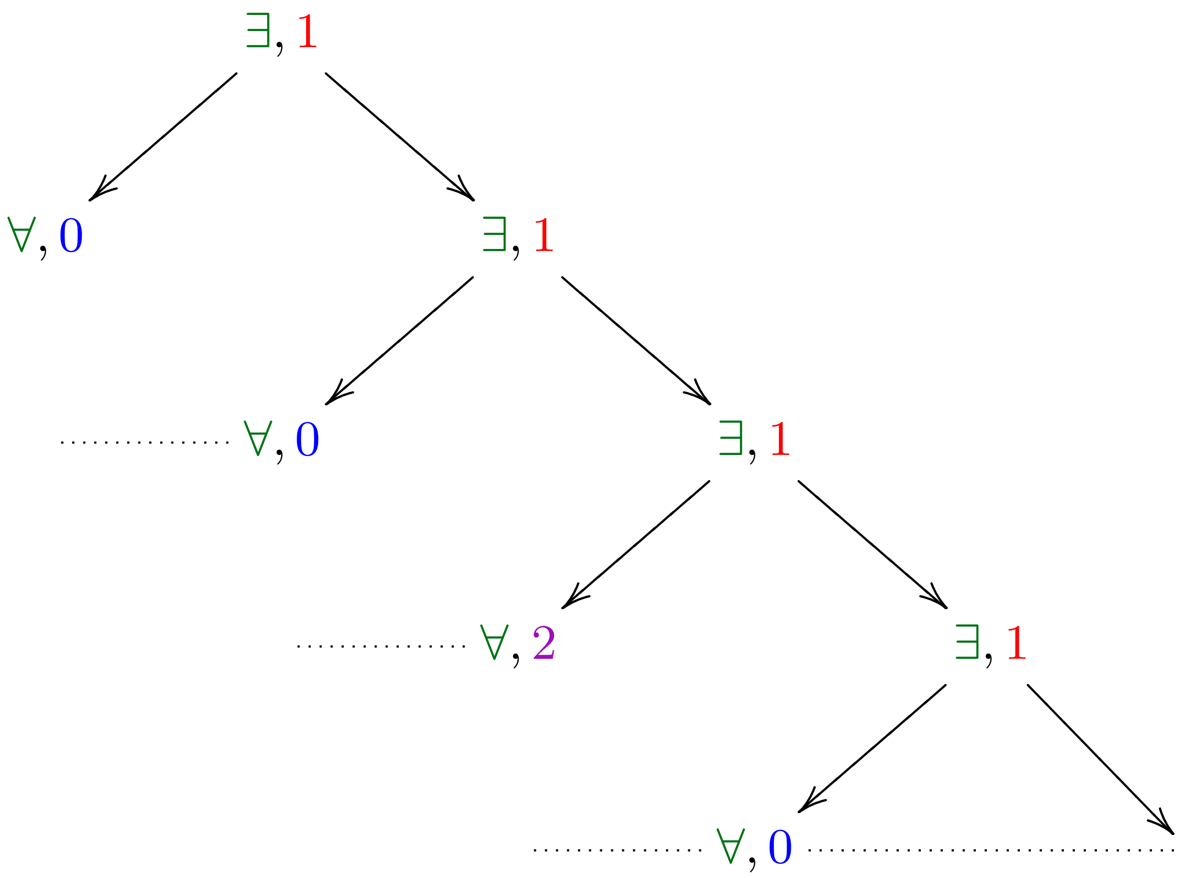


Eve wins an infinite play $(x_0, j_0),\ (x_1, j_1),\ (x_2, j_2), \ldots \quad (x_\ell \in \{\exists, \forall\})$

iff $\limsup_{\ell \to \infty} j_\ell$ is even.

The set $W_{i,k}$ consists of all trees such that Eve has a winning strategy.

Example



From words to trees

For $L \subseteq C^\omega$, let

$$Win_\ell^\exists(L) \quad = \quad \text{the set of trees } t : \ell^* \longrightarrow \{\exists, \forall\} \times C \text{, such that Eve has a strategy to force the play into } L$$

$$Win_\ell^\forall(L) \quad = \quad \text{the similar for Adam.}$$

The inseparable pair in the class Σ is provided by

$$\begin{aligned} \nabla_1 &= Win_4^\exists(U_1) \\ \nabla_2 &= Win_4^\forall(U_2), \end{aligned}$$

where U_1 and U_2 are from the Key Lemma.

Let $\mathcal{A}$ be an alternating automaton over alphabet Σ of index (i, k).

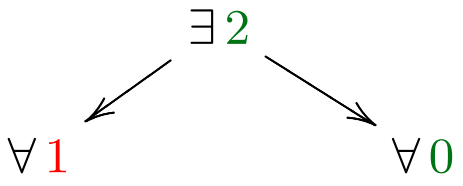
We assume that the full game tree is binary, with $\exists$ and $\forall$ alternating.

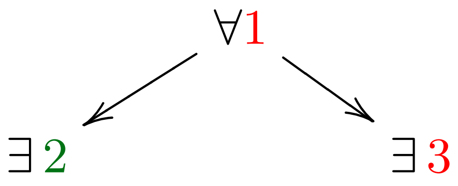
With a tree $t : \ell^* \to \Sigma$, we associate $\mathcal{T}(\mathcal{A}, t) : 2^* \to \{\exists, \forall\} \times \{\iota, \dots, \kappa\}$, such that

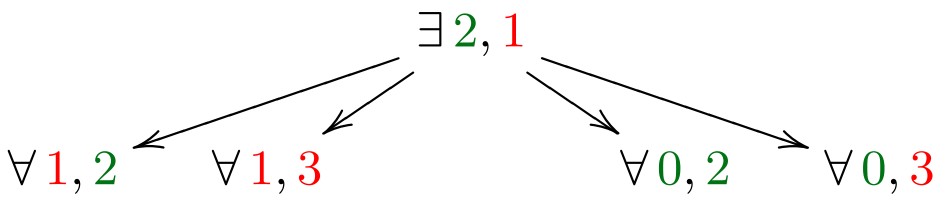
$$t \in L(\mathcal{A}) \iff \mathcal{T}(\mathcal{A}, t) \in W_{i,k}.$$

If $\mathcal{A}$ in an $\exists\forall$-automaton, and $\mathcal{B}$ an $\forall\exists$-automaton, we define

$$g^{\mathcal{A}\times\mathcal{B}}(t) \quad = \quad \mathcal{T}(\mathcal{A}, t) \star \mathcal{T}(\mathcal{B}, t).$$







Lemma. Let $\mathcal{A}$ and $\mathcal{B}$ be alternating tree automata of class Π_3, s.t. $L(\mathcal{A}) \cap L(\mathcal{B}) = \emptyset$. Then, for any tree $t : \ell^* \to \Sigma$,

$$
\begin{aligned}
t \in L(\mathcal{A}) \quad &\Longrightarrow \quad g^{\mathcal{A}\times\mathcal{B}}(t) \in \nabla_1 \\
t \in L(\mathcal{B}) \quad &\Longrightarrow \quad g^{\mathcal{A}\times\mathcal{B}}(t) \in \nabla_2 .
\end{aligned}
$$

Proof of the result.

If $\Sigma = \{\exists, \forall\} \times 3 \times 3$ then the mapping

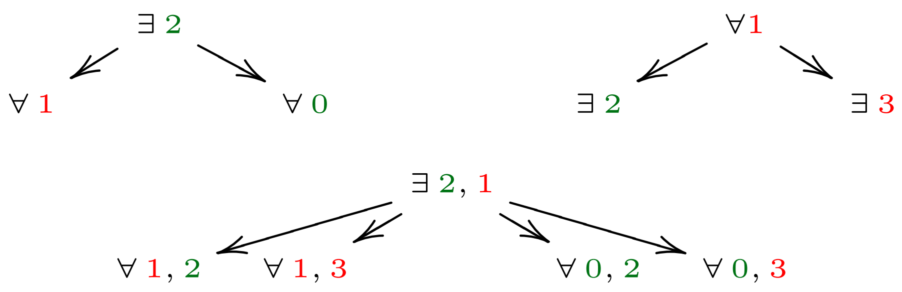
$$
t \quad \mapsto \quad g^{\mathcal{A}\times\mathcal{B}}(t)
$$

has a **fixed point**. Like before, this excludes separability of ∇_1 and ∇_2.

Lemma. $\mathcal{A}$ and $\mathcal{B}$ are Π_3, and $L(\mathcal{A}) \cap L(\mathcal{B}) = \emptyset$. Then, $\forall t$,

$$t \in L(\mathcal{A}) \implies g^{\mathcal{A}\times\mathcal{B}}(t) \in \nabla_1$$
$$t \in L(\mathcal{B}) \implies g^{\mathcal{A}\times\mathcal{B}}(t) \in \nabla_2.$$

Proof of the Lemma.



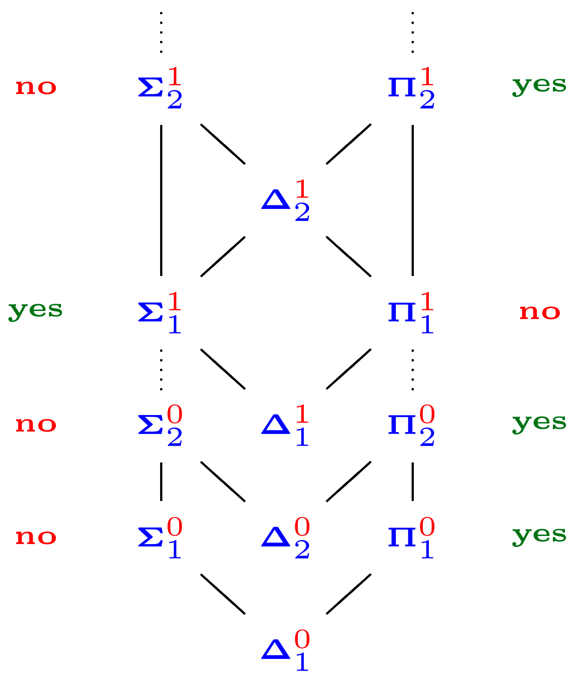
Recall $\nabla_1 = Win_4^{\exists}(U_1)$, and $K_3 \times \overline{K_3} \subseteq U_1$.

To win on $g^{\mathcal{A}\times\mathcal{B}}(t)$, Eve forces the play into $K_3 \times \overline{K_3}$.

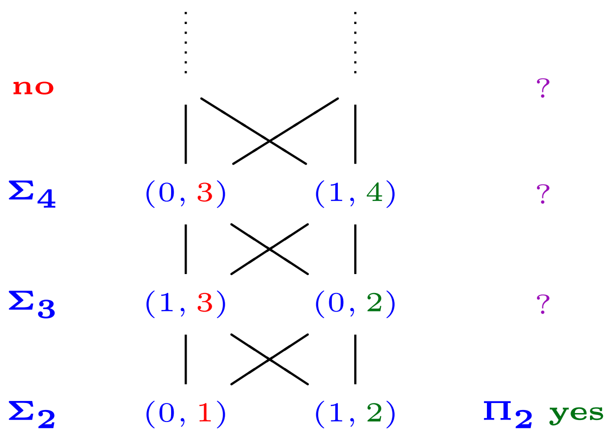
To achieve this, she combines a strategy of Eve on $\mathcal{T}(\mathcal{A}, t)$, and a strategy of Adam on $\mathcal{T}(\mathcal{B}, t)$. The argument for Adam is similar.

Conclusion

Topology



Automata



Separation property is an intriguing feature, which exhibits similar patterns in various hierarchies.