Symmetry and duality in fixed-point calculus

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Eugene Onegin vs. Vladimir Lensky

Duel

1. A shuts.
2. B shuts, provided he has survived.

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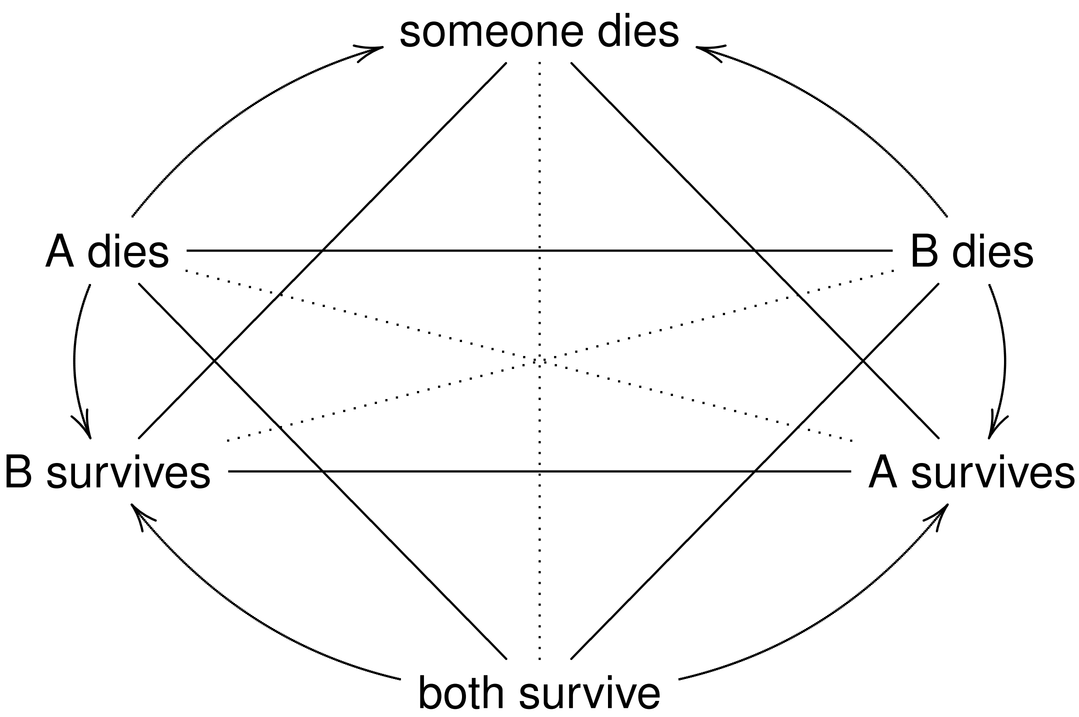
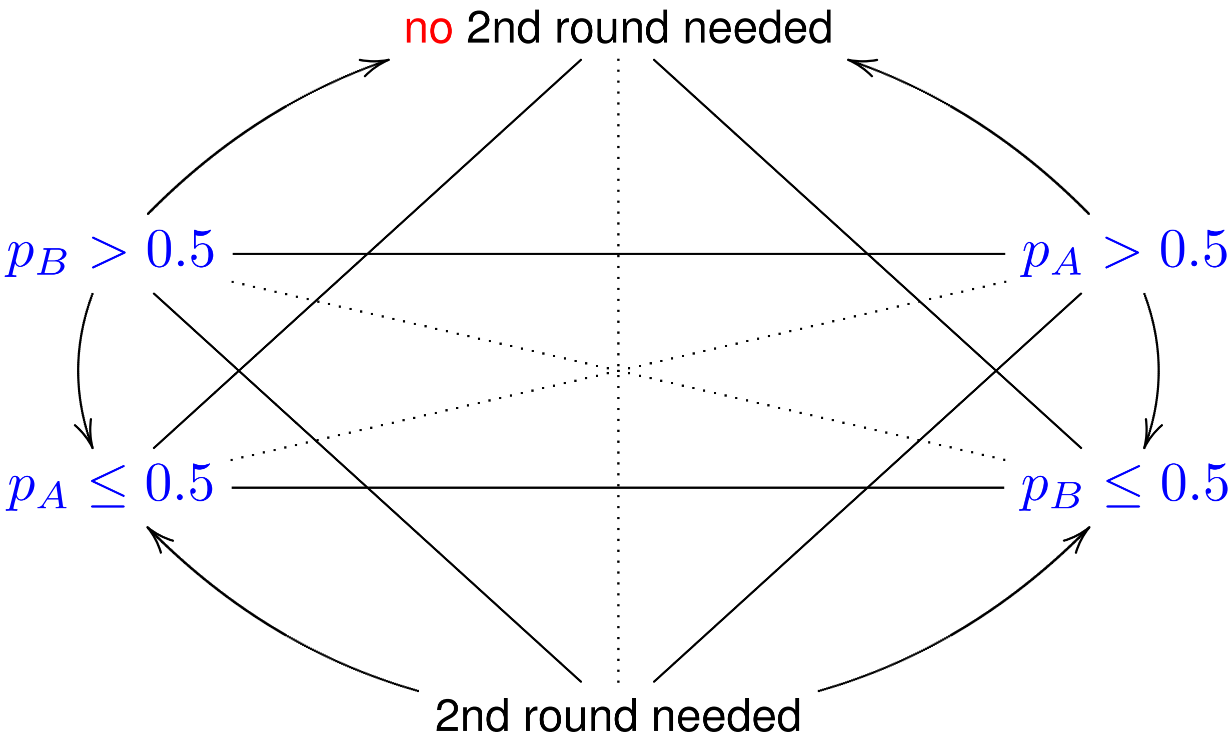
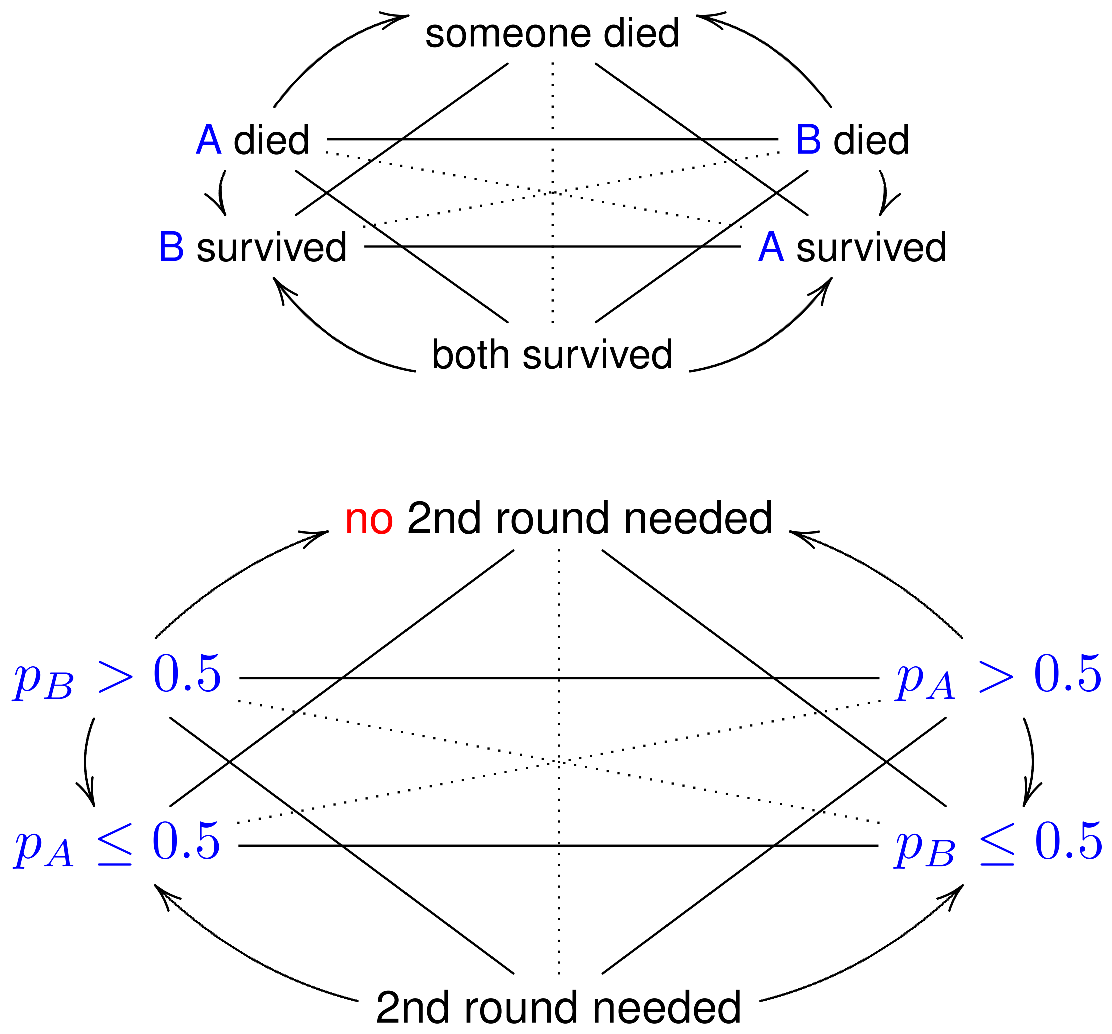


Election in two-round system

$p_X =$ the proportion of votes given for X.

If $p_X > 0.5$ then X wins the first round.



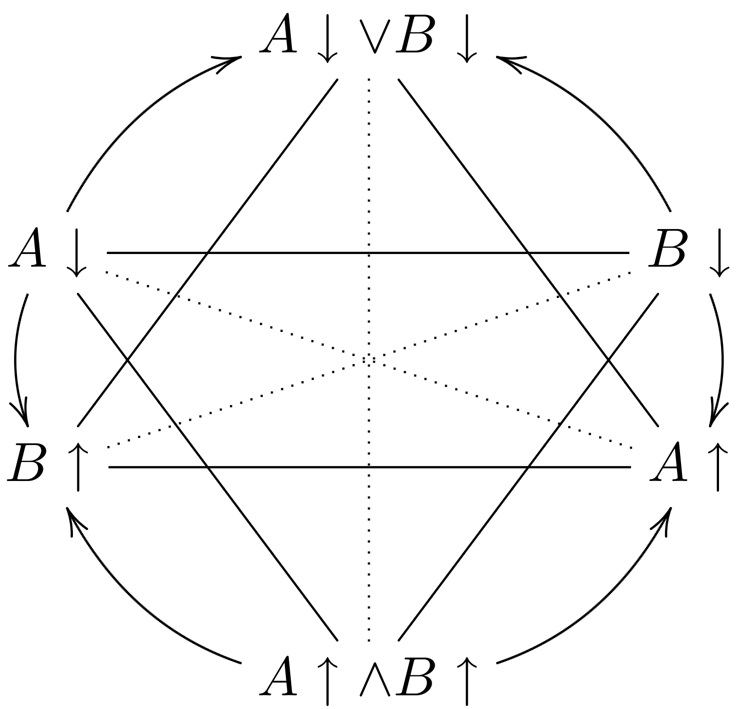

someone died
A died
B died
B survived
A survived
both survived
no 2nd round needed
$p_B > 0.5$
$p_A > 0.5$
$p_A \leq 0.5$
$p_B \leq 0.5$
2nd round needed

General scheme

A player can **win**, **loose** or **survive**.

$A \downarrow \quad \equiv \quad A$ looses $\quad \equiv \quad B$ wins

$A \uparrow \quad \equiv \quad A$ survives



But there are different concepts of winning. . .

Win the play or the game ?

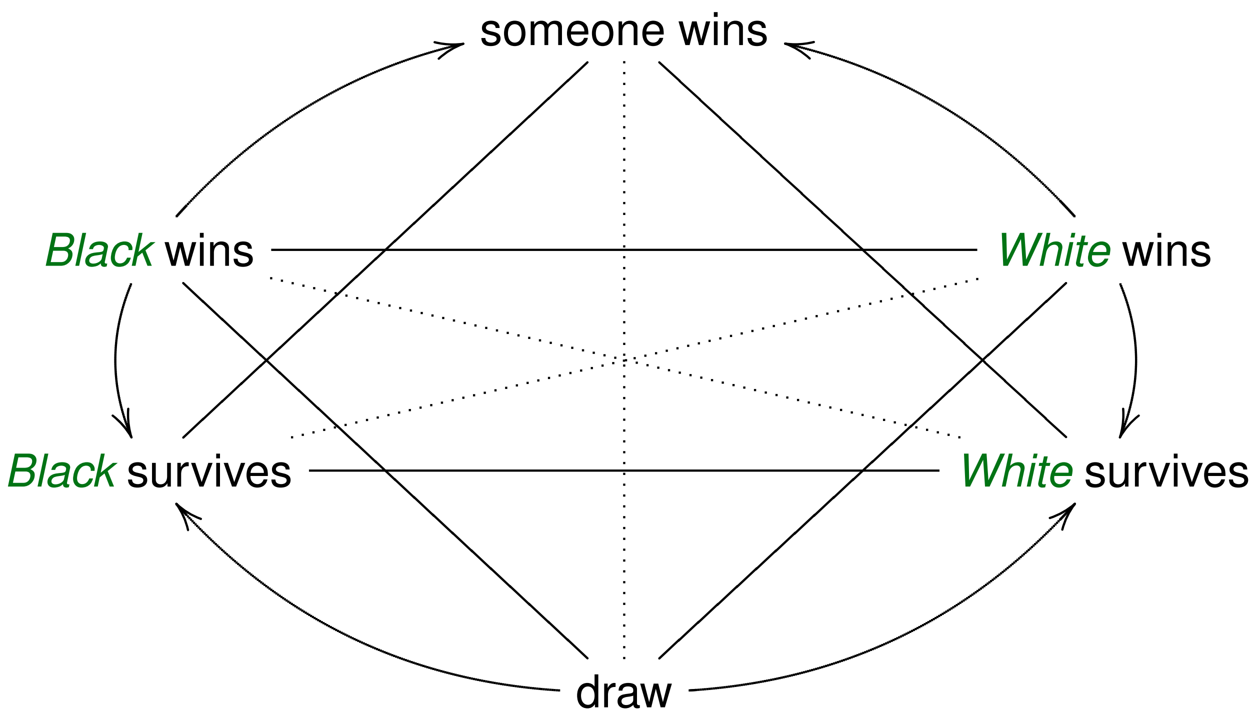
Win the play or the game ?

Win the battle or the war ?

Chess

In 1913, Ernst Zermelo proved that one of the following holds:

1. *White* has a strategy to win,
2. *Black* has a strategy to win,
3. Both parties have the strategies to **survive**, i.e., to achieve at least a draw.



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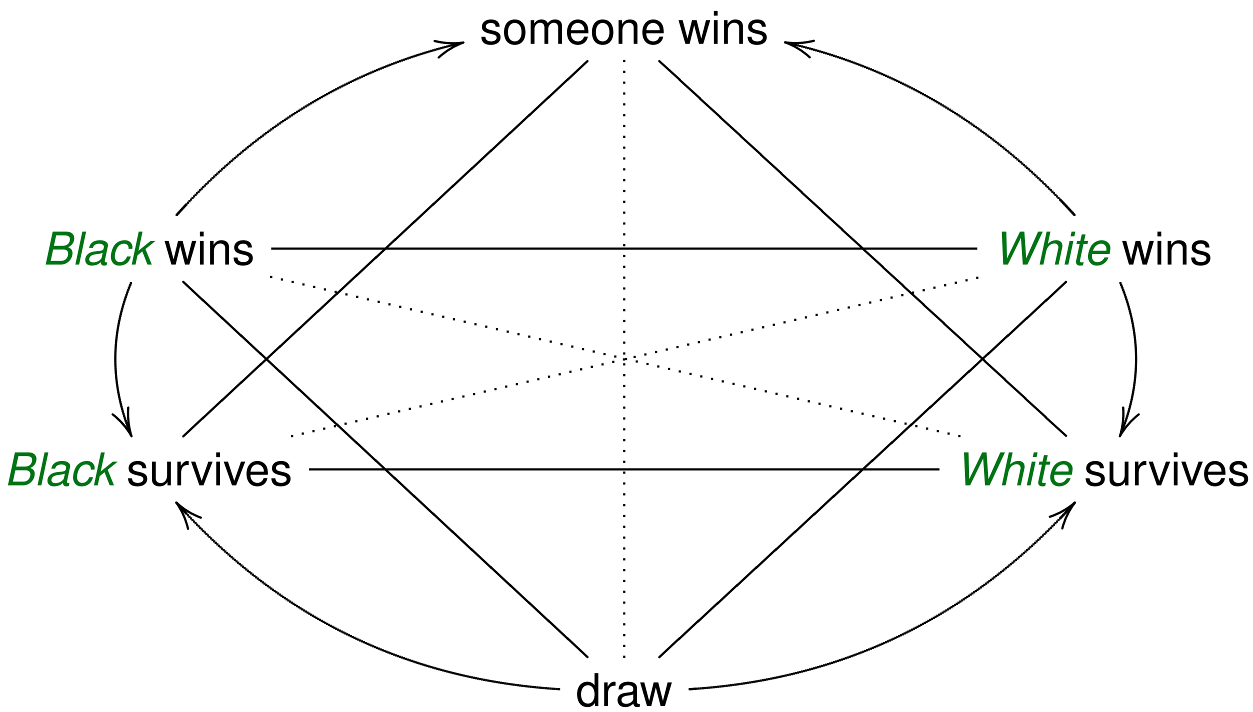

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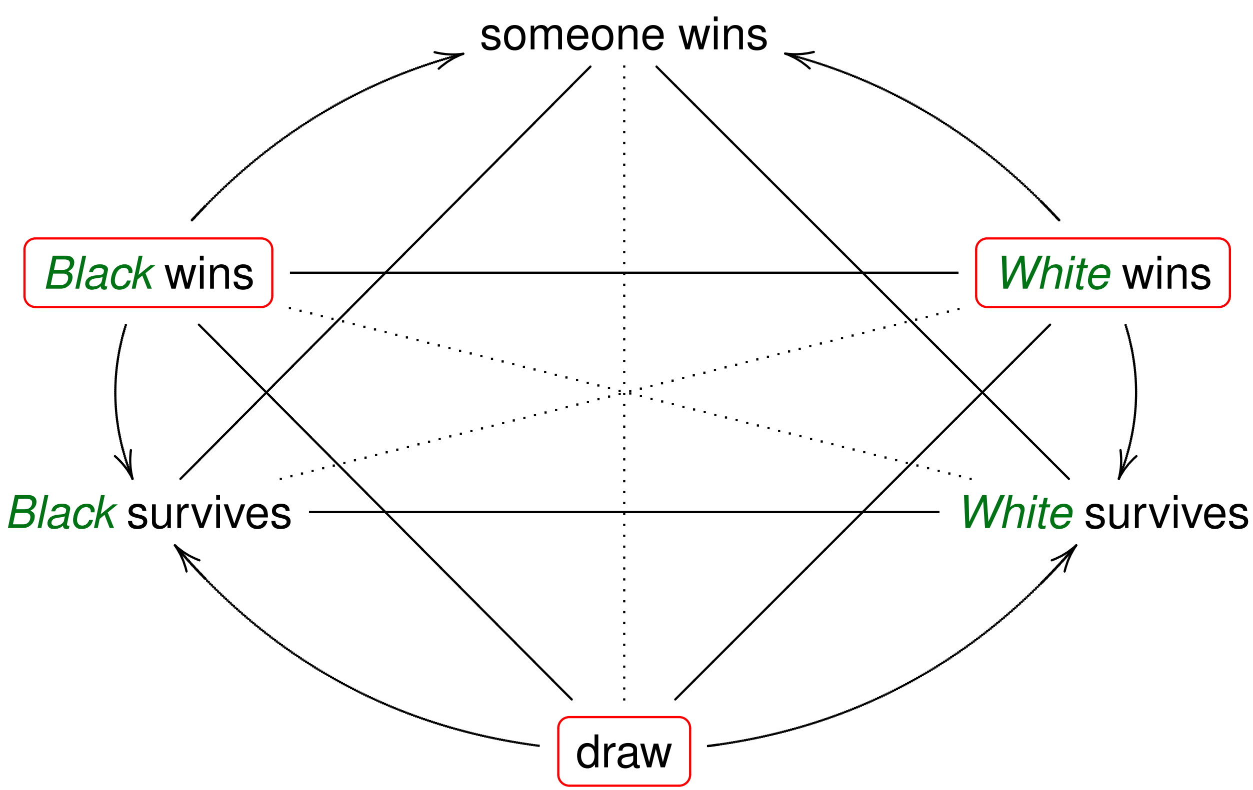


On the logical structure of Zermelo's Theorem.

We have easily

$$(\forall play)\ White(play) \vee Black(play) \vee Draw(play)$$

For a stategy S_w of White, and a strategy S_b od Black, let $S_w * S_b$ be the resulting play.

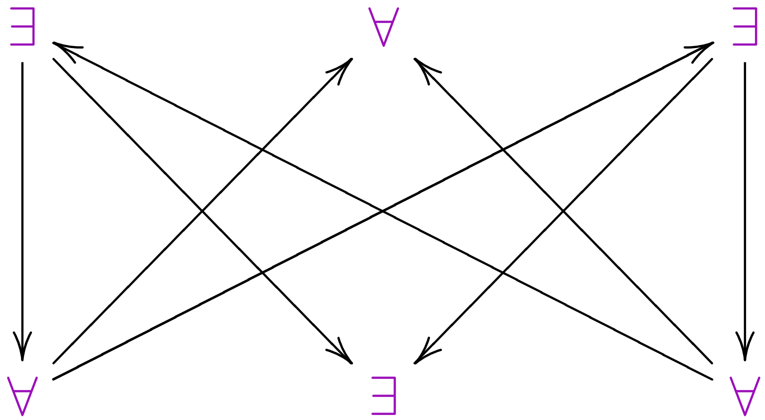
Zermelo says:

$$(\exists S_w \forall S_b)\ White(S_w * S_b) \vee (\exists S_b \forall S_w)\ Black(S_w * S_b) \vee$$
$$(\exists S_w \exists S_b\, \forall S'_w \forall S'_b)\, (White(S_w * S'_b) \vee Draw(S_w * S'_b)) \wedge$$
$$\wedge (Black(S'_w * S_b) \vee Draw(S'_w * S_b))$$

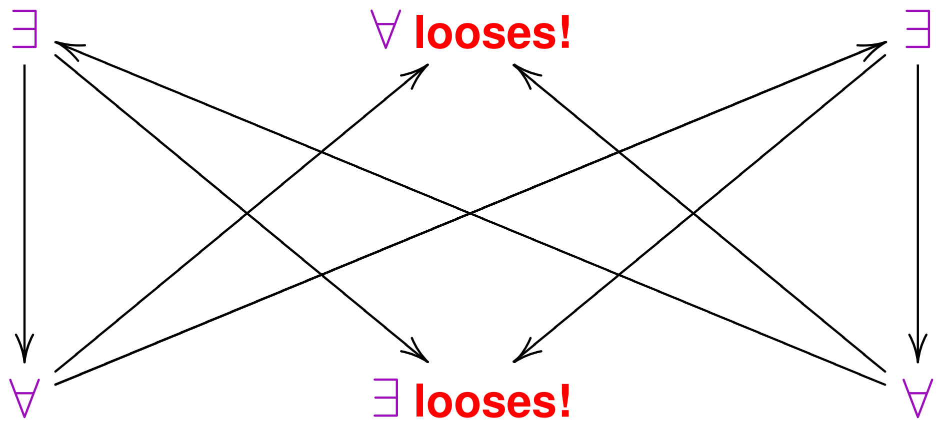
An abstract view of two-player game (like chess).

Players: *Eve* (∃) and *Adam* (∀)

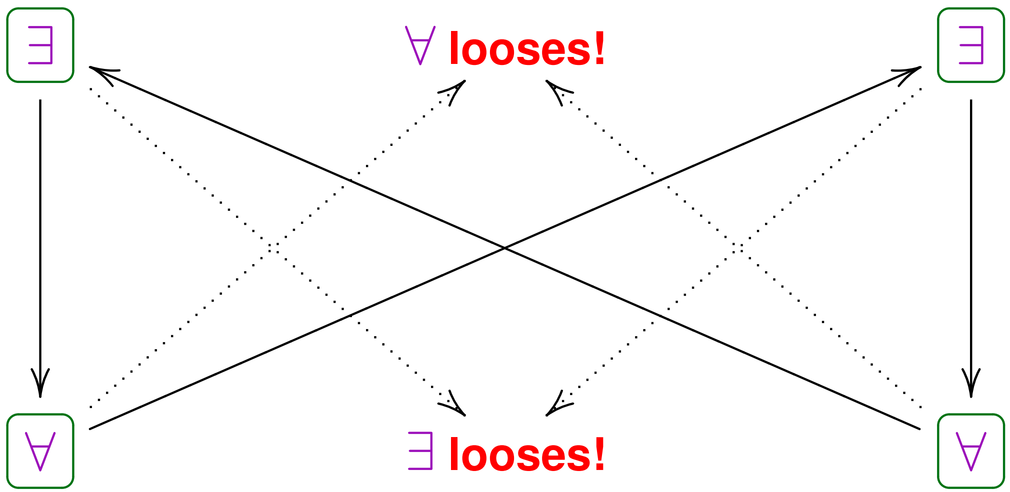
Arena:



Example cont'd



Example cont'd



Analysis

Arena:

$$\langle V = V_E \cup V_A, Mov \subseteq V \times V \rangle$$

Players' equations:

$$\begin{aligned} X &= (V_E \cap \Diamond X) \cup (V_A \cap \Box X) &= \quad Eve(X) \\ Y &= (V_A \cap \Diamond Y) \cup (V_E \cap \Box Y) &= \quad Adam(Y) \end{aligned}$$

where

$$\begin{aligned} \Diamond Z &= \{p : (\exists q \in Z)\, Mov(p, q)\} \\ \Box Z &= \overline{\Diamond \overline{Z}} \end{aligned}$$

Knaster-Tarski Theorem

$f : L \to L$

L complete lattice

f monotone

Then the fixed points of f form a complete lattice, including

the least fixed point:

$$\mu x.f(x) \quad = \quad \bigwedge\{d : f(d) \leq d\}$$

the greatest fixed point:

$$\nu x.f(x) \quad = \quad \bigvee\{d : d \leq f(d)\}$$

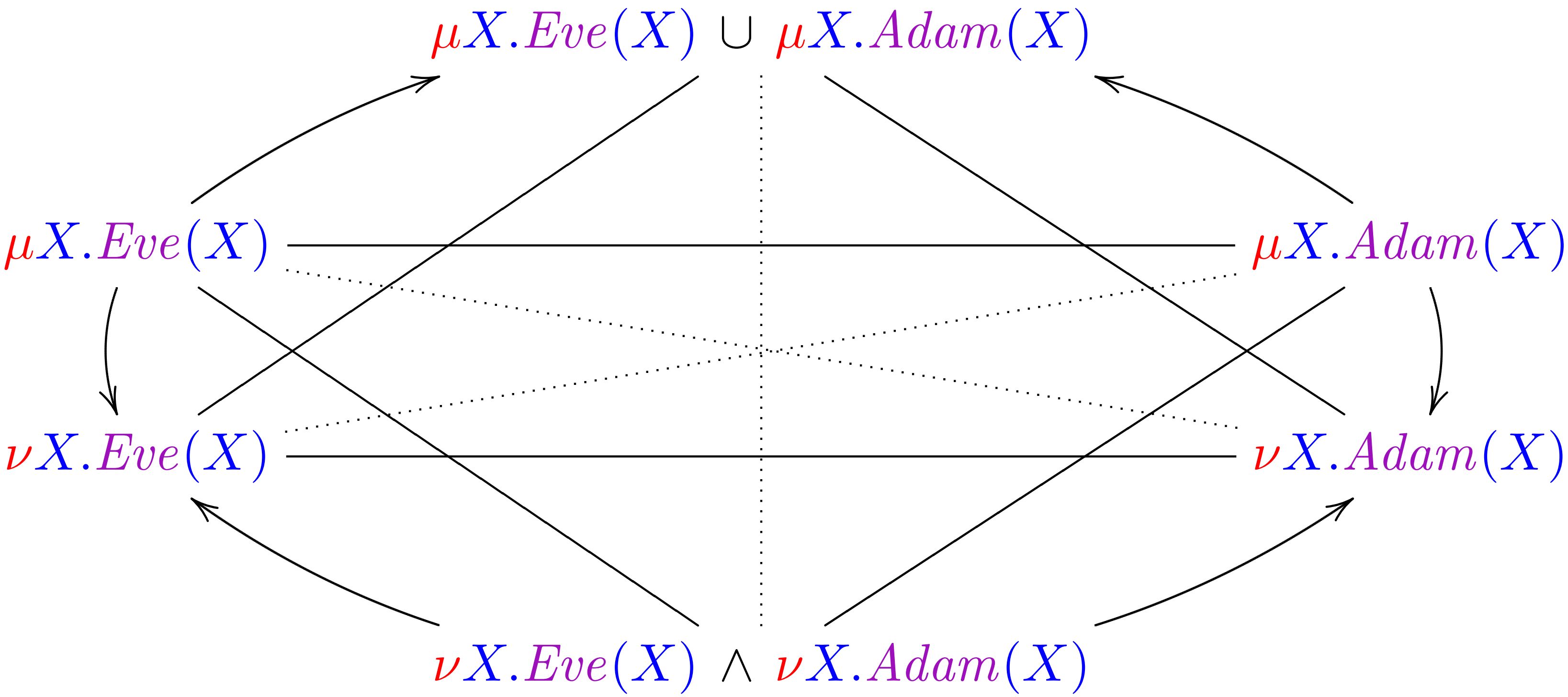
De Morgan dualities

$$
\begin{aligned}
\overline{Eve\,(X)} &= \overline{(E \cap \Diamond X) \cup (A \cap \Box X)} \\
&= \left(\overline{E \cap \Diamond X}\right) \cap \left(\overline{A \cap \Box X}\right) \\
&= \left(\overline{E} \cup \overline{\Diamond X}\right) \cap \left(\overline{A} \cup \overline{\Box X}\right) \\
&= \left(A \cup \Box \overline{X}\right) \cap \left(E \cup \Diamond \overline{X}\right) \\
&= \left(A \cap \Diamond \overline{X}\right) \cup \left(E \cap \Box \overline{X}\right) \cup \overbrace{(A \cap E)}^{\emptyset} \cup \left(\Diamond \overline{X} \cap \Box \overline{X}\right) \\
&= Adam\,(\overline{X})
\end{aligned}
$$

Hence $X = Eve\,(X) \iff \overline{X} = Adam\,(\overline{X})$,

and consequently

$$
\begin{aligned}
\overline{\mu X. Eve\,(X)} &= \nu Y. Adam\,(Y) \\
\overline{\nu X. Eve\,(X)} &= \mu Y. Adam\,(Y)
\end{aligned}
$$



How does this diagram relate to games ?

Let

$$Win_E = \{p : \textit{Eve} \text{ has a strategy to win}\}$$
$$Safe_E = \{p : \textit{Eve} \text{ has a strategy to survive }\}$$

If	X	$\subseteq$	$\overbrace{(V_E \cap \Diamond X) \cup (V_A \cap \Box X)}^{Eve(X)}$
then	X	$\subseteq$	$Safe_E$
Moreover	$Safe_E$	$\subseteq$	$Eve\,(Safe_E)$
Hence			

$$\boxed{\nu x.Eve(x) = Safe_E}$$

On the other hand,

$$\overbrace{(V_E \cap \Diamond\, Win_E) \cup (V_A \cap \Box\, Win_E)}^{Eve\,(Win_E)} \subseteq Win_E$$

Hence

$$\mu X.Eve(X) \quad \subseteq \quad Win_E$$

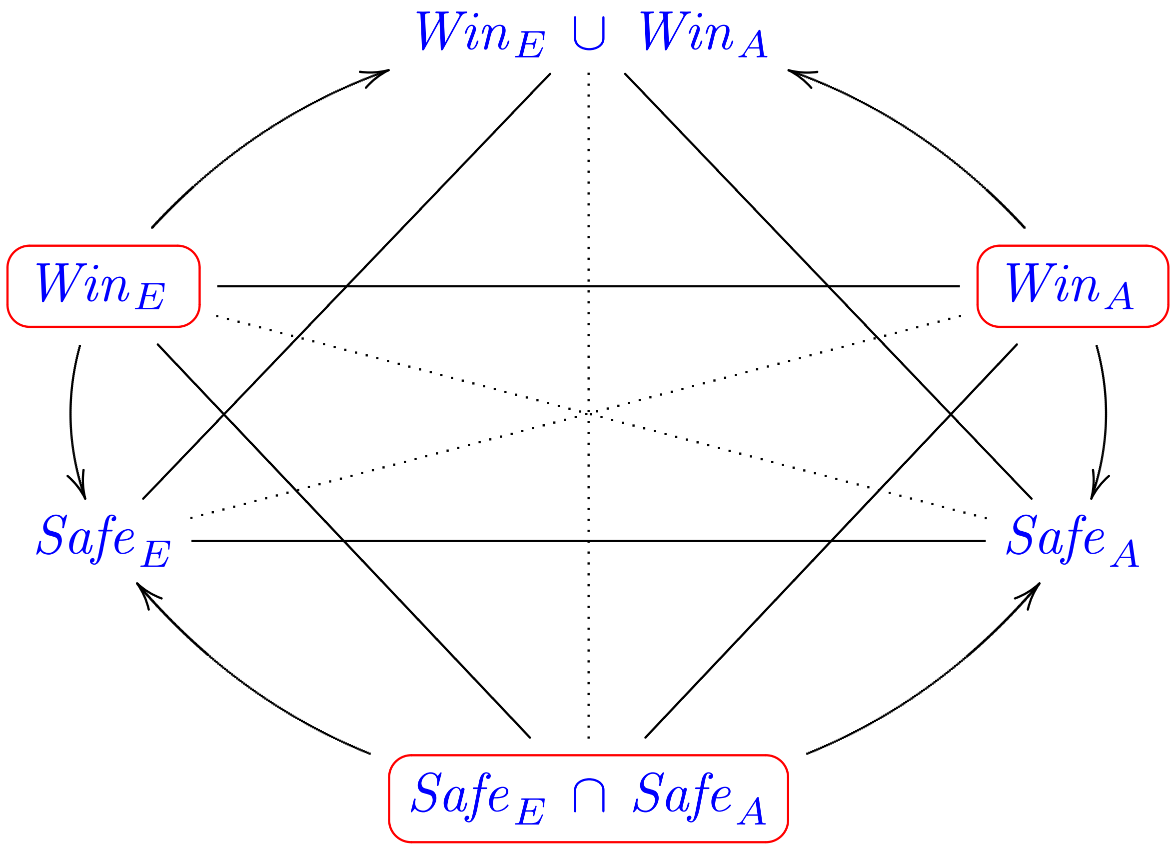
We have

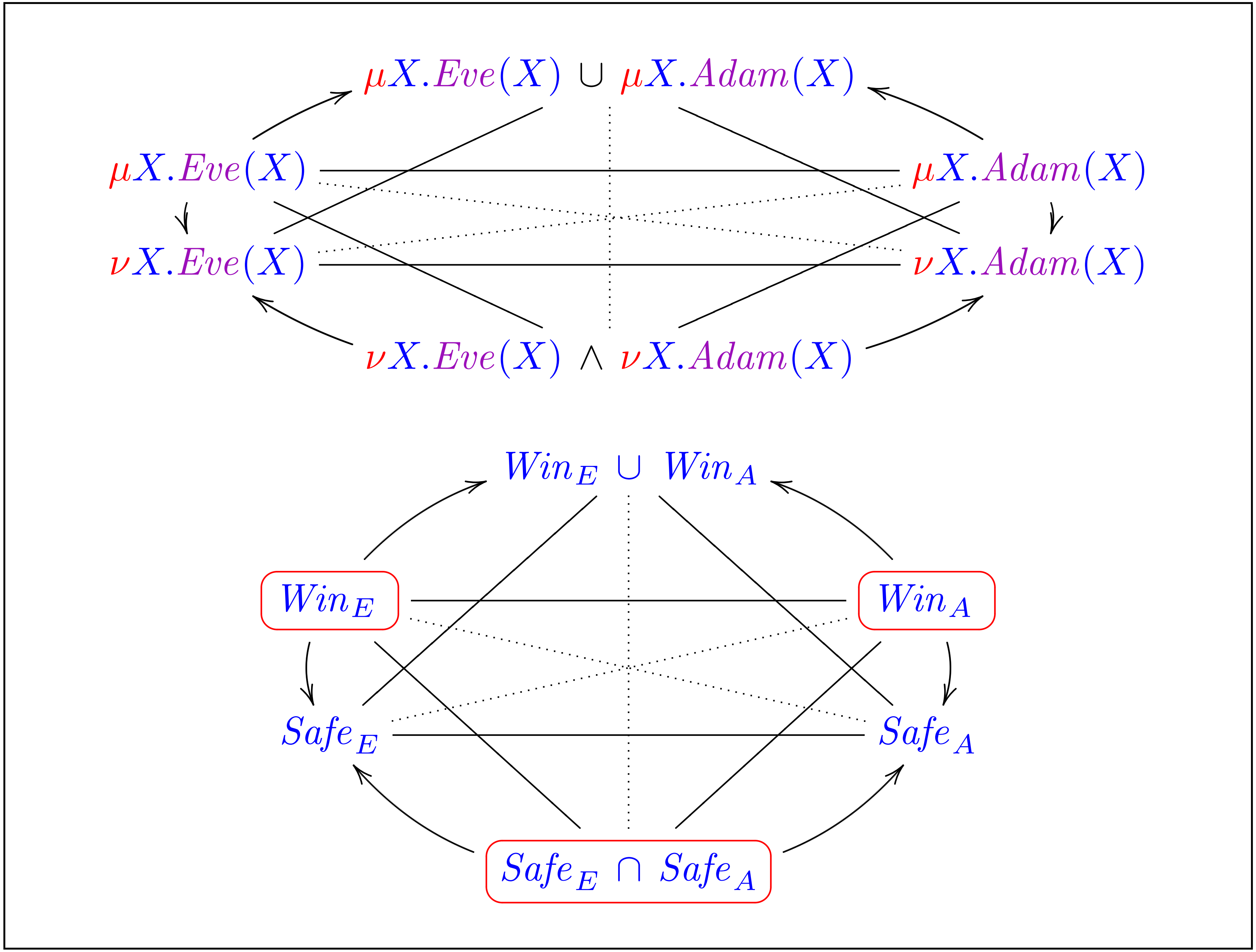
$$
\begin{aligned}
\mu X.Eve(X) &\subseteq Win_E \\
\overline{\mu X.Eve\,(X)} &= \nu Y.Adam\,(Y) \\
\nu Y.Adam\,(Y) &= Safe_A \\
Safe_A \cap Win_E &= \emptyset.
\end{aligned}
$$

Hence

$$
\boxed{\mu x.Eve(x) = Win_E}
$$

Zermelo's Theorem




$\mu X.Eve(X) \cup \mu X.Adam(X)$
$\mu X.Eve(X)$
$\mu X.Adam(X)$
$\nu X.Eve(X)$
$\nu X.Adam(X)$
$\nu X.Eve(X) \wedge \nu X.Adam(X)$
$Win_E \cup Win_A$
Win_E
Win_A
$Safe_E$
$Safe_A$
$Safe_E \cap Safe_A$

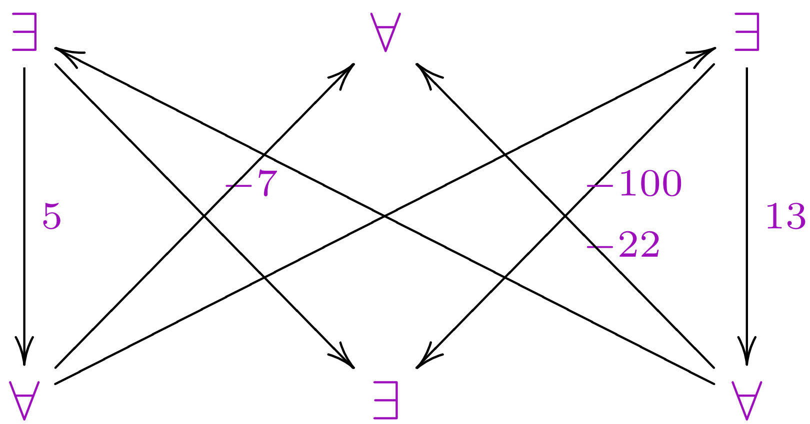
Everlasting games

Everlasting games

An infinite play need not be considered as a draw; it can be meaningful.

For example, we may require that $Adam$ pays to Eve the amount x, while passing through an edge $\xrightarrow{x}$.

Each player wants to maximize her income (e.g., asymptotically on average).



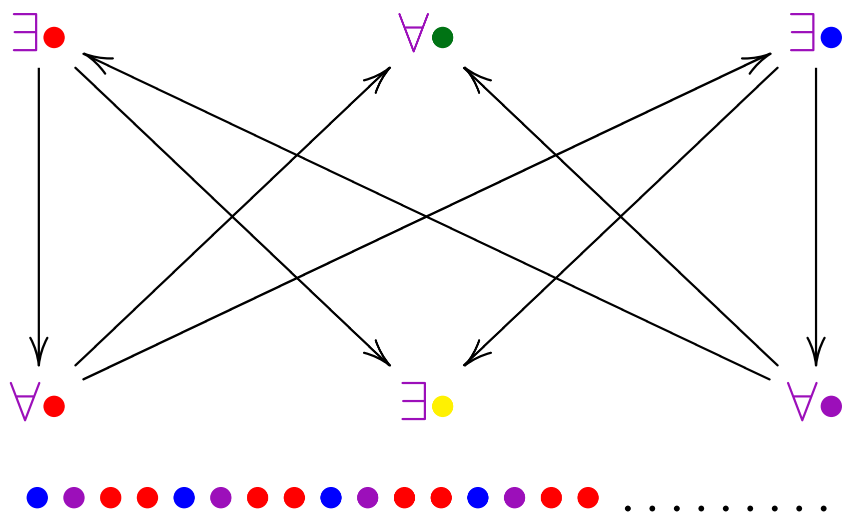
Everlasting games cont'd

In general setting, the nodes (or edges) are coloured in a set of colours Σ.

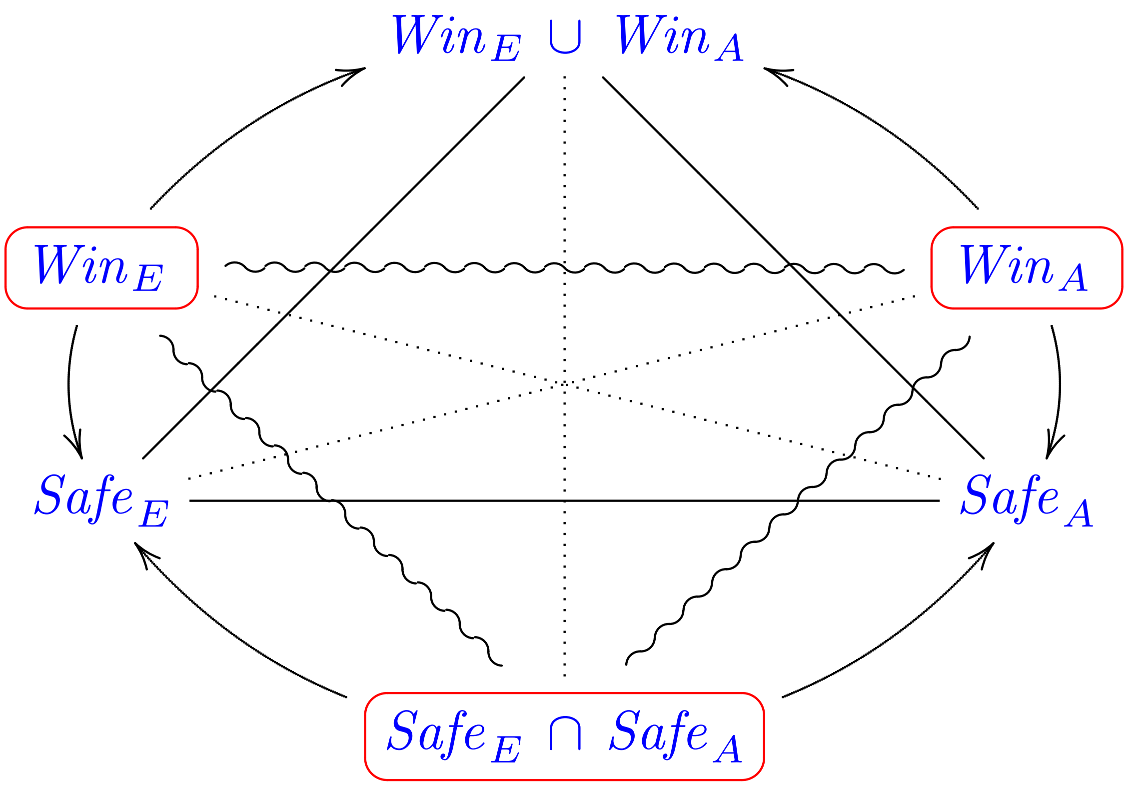
The **winning criteria** for Eve and $Adam$, respectively, are given by disjoint sets $C_E, C_A \subseteq \Sigma^\omega$.

Player X wins a play $p_0, p_1, p_2, \ldots$ iff

$$color(p_0), color(p_1), color(p_2), \ldots \in C_X.$$



In general, **Zermelo's Theorem fails** for such games.



Idea — strategy stealing

White	Mr. Kasparov	•
Black	Mr. Niwiński	

White	Mr. Niwiński
Black	Mr. Karpow

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Example of undetermined game

Let $C_E \cup C_A = \{0,1\}^\omega$ have the property that two sequences that differ in exactly one bit are winning for different players.

$$0011101101101001\,\overset{\mathbf{1}}{\not 0}0101100001011\ldots\ldots$$

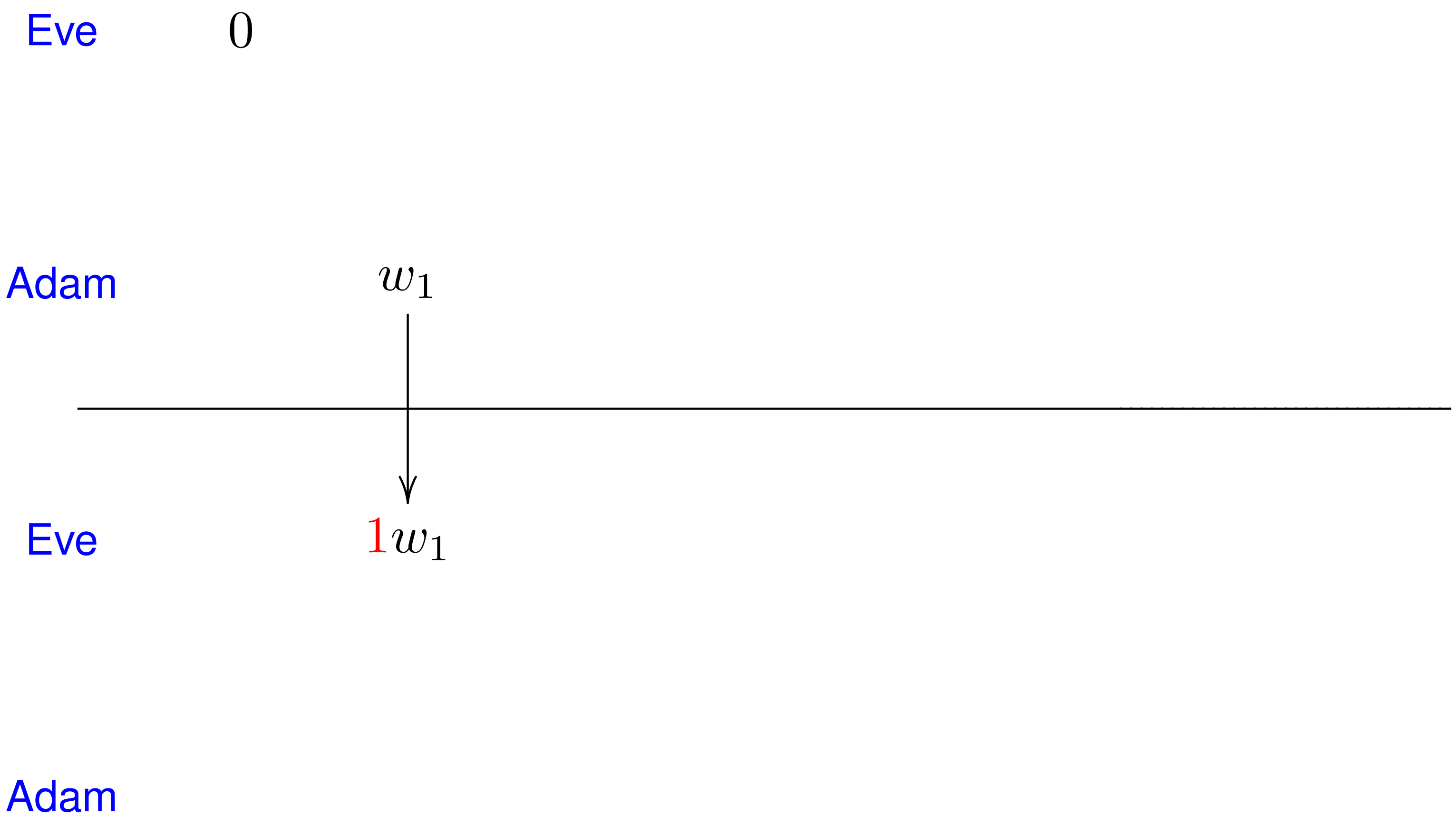
By Axiom of Choice, there exist ($2^{\aleph_0}$ many) such pairs.

Eve	w_0		w_2		w_4	
Adam		w_1		w_3		w_5

The results of the play is: $W = w_0 w_1 w_2 w_3 w_4 w_5 \ldots$

Eve wins if $W \in C_E$, otherwise Adam wins.

Suppose Adam wins



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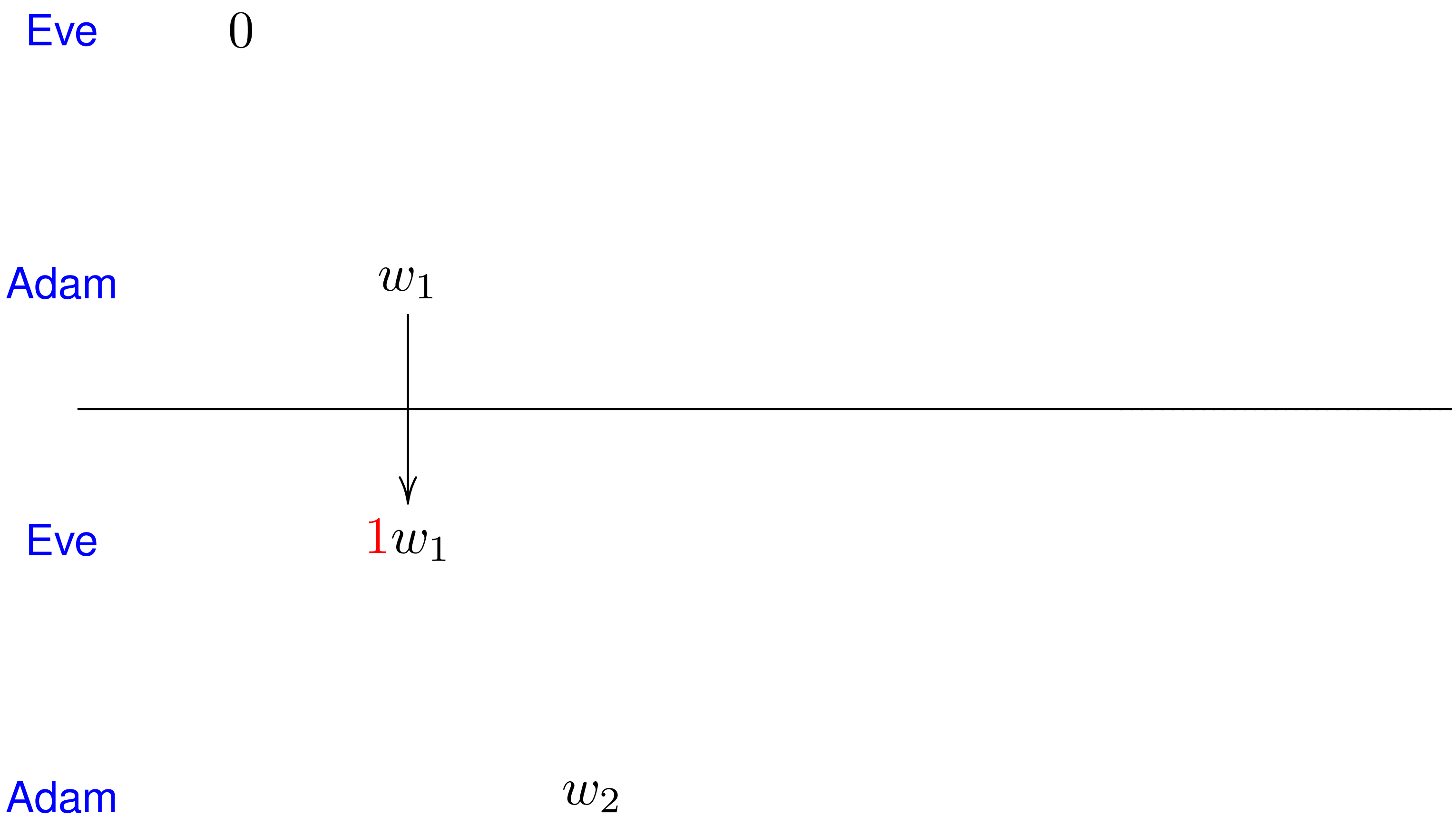

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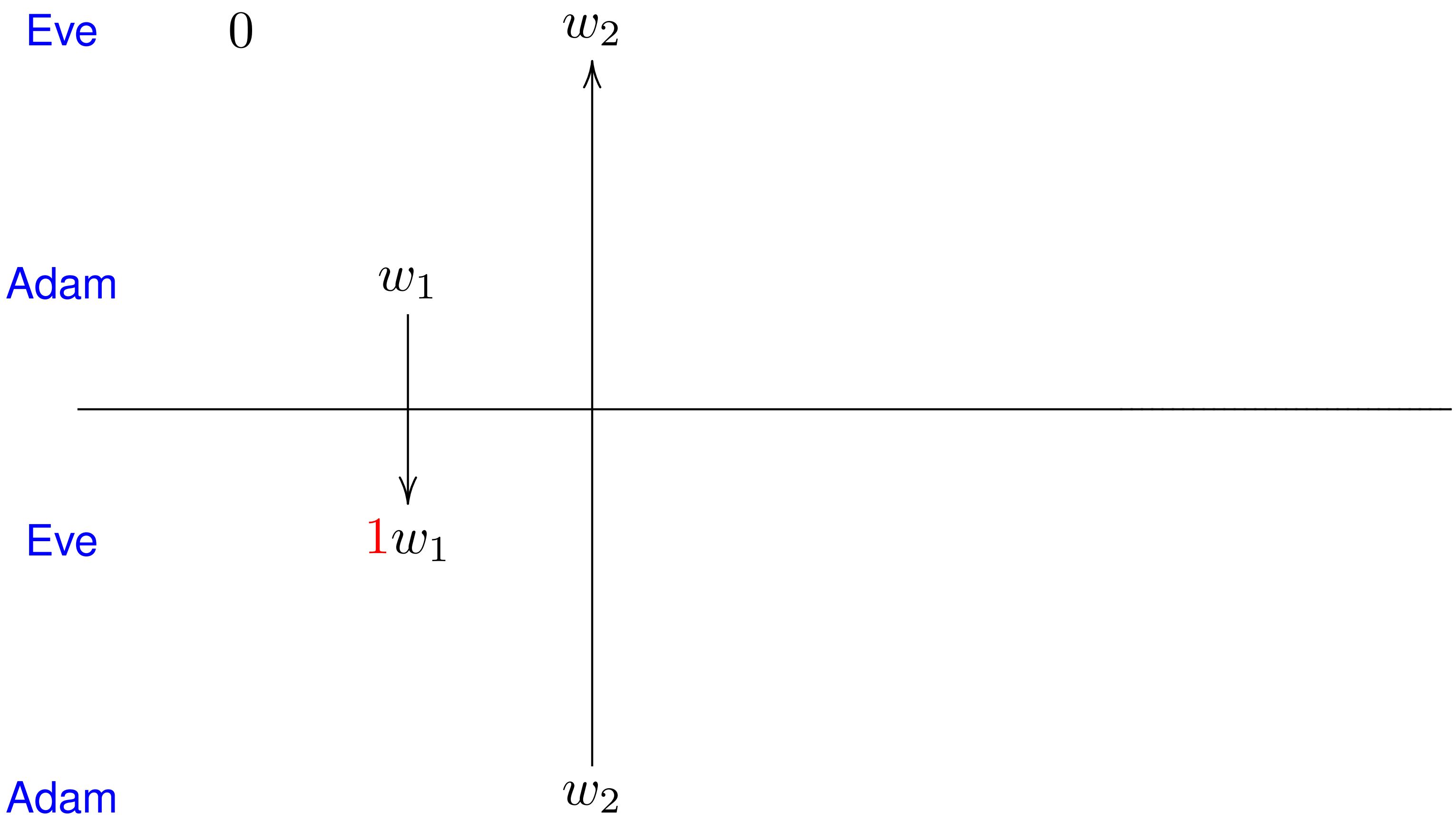

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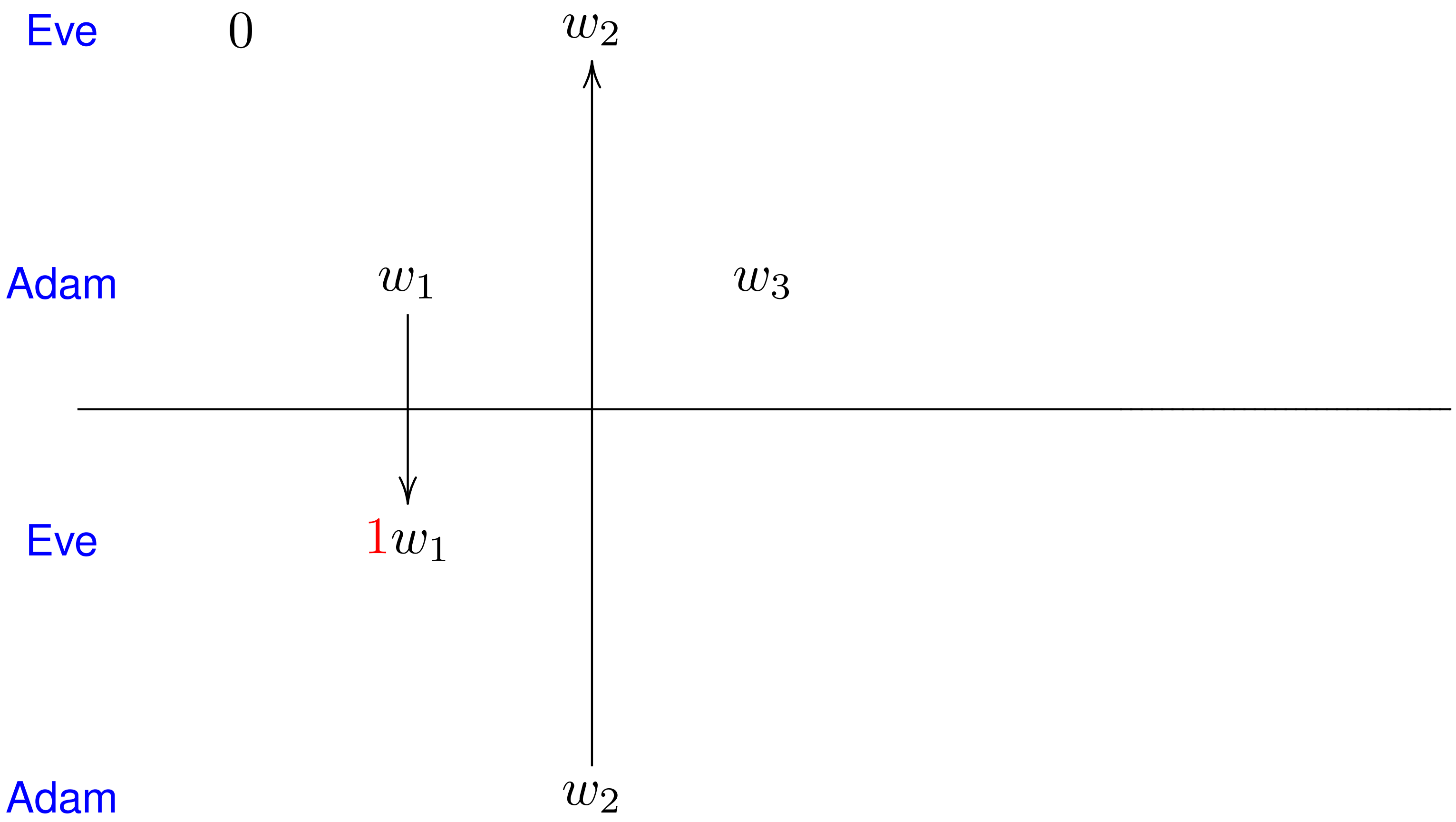

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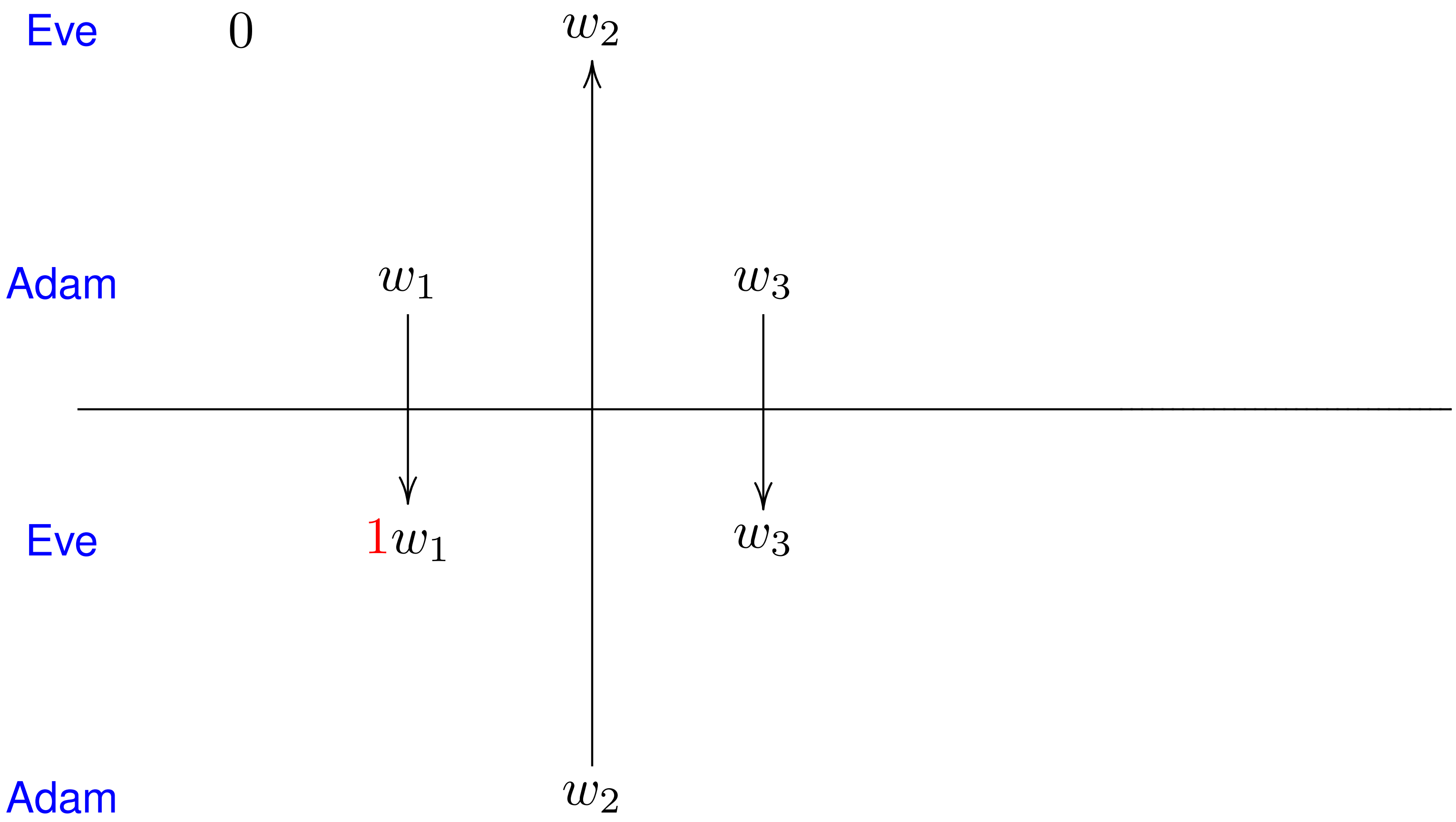

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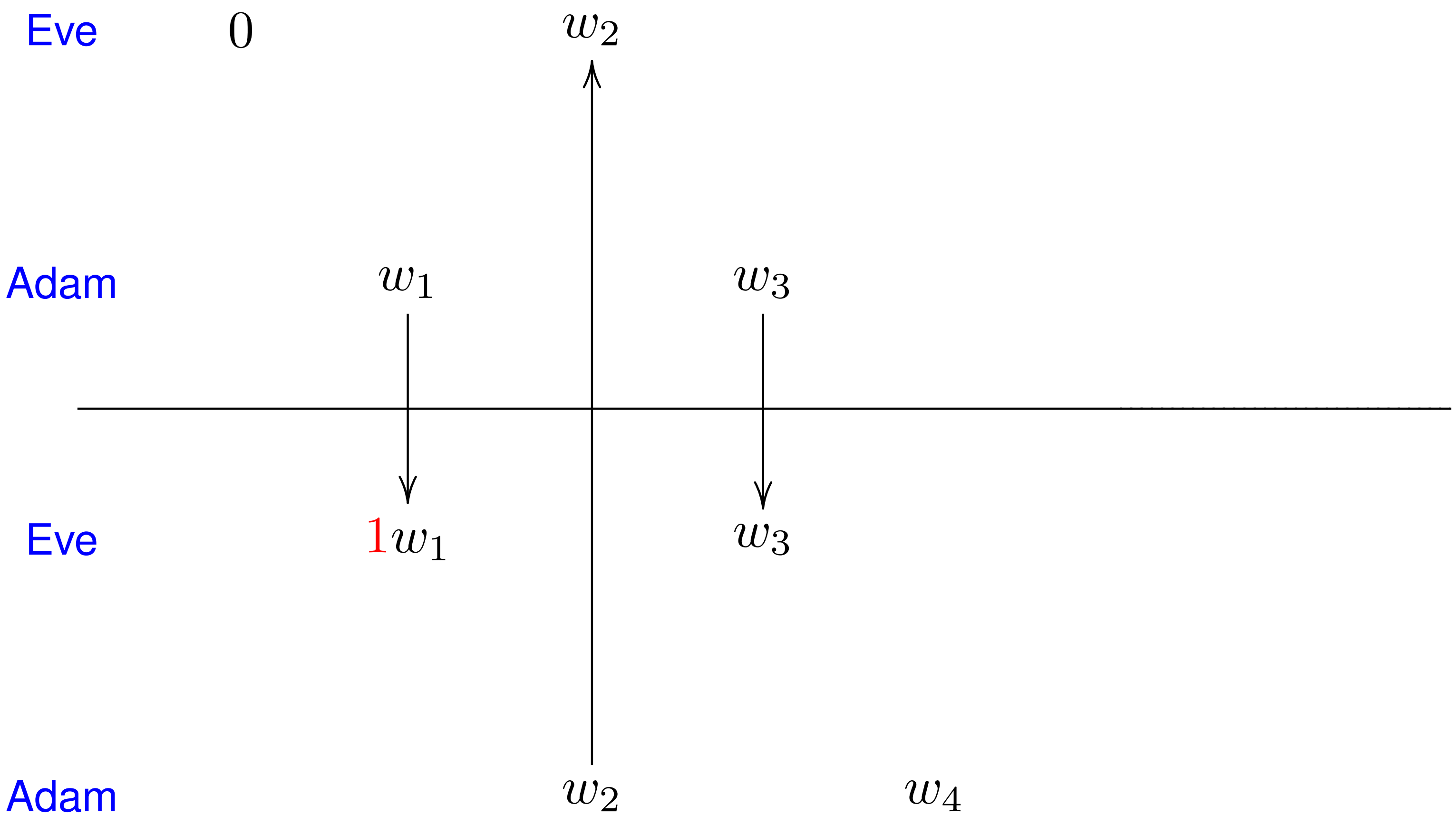

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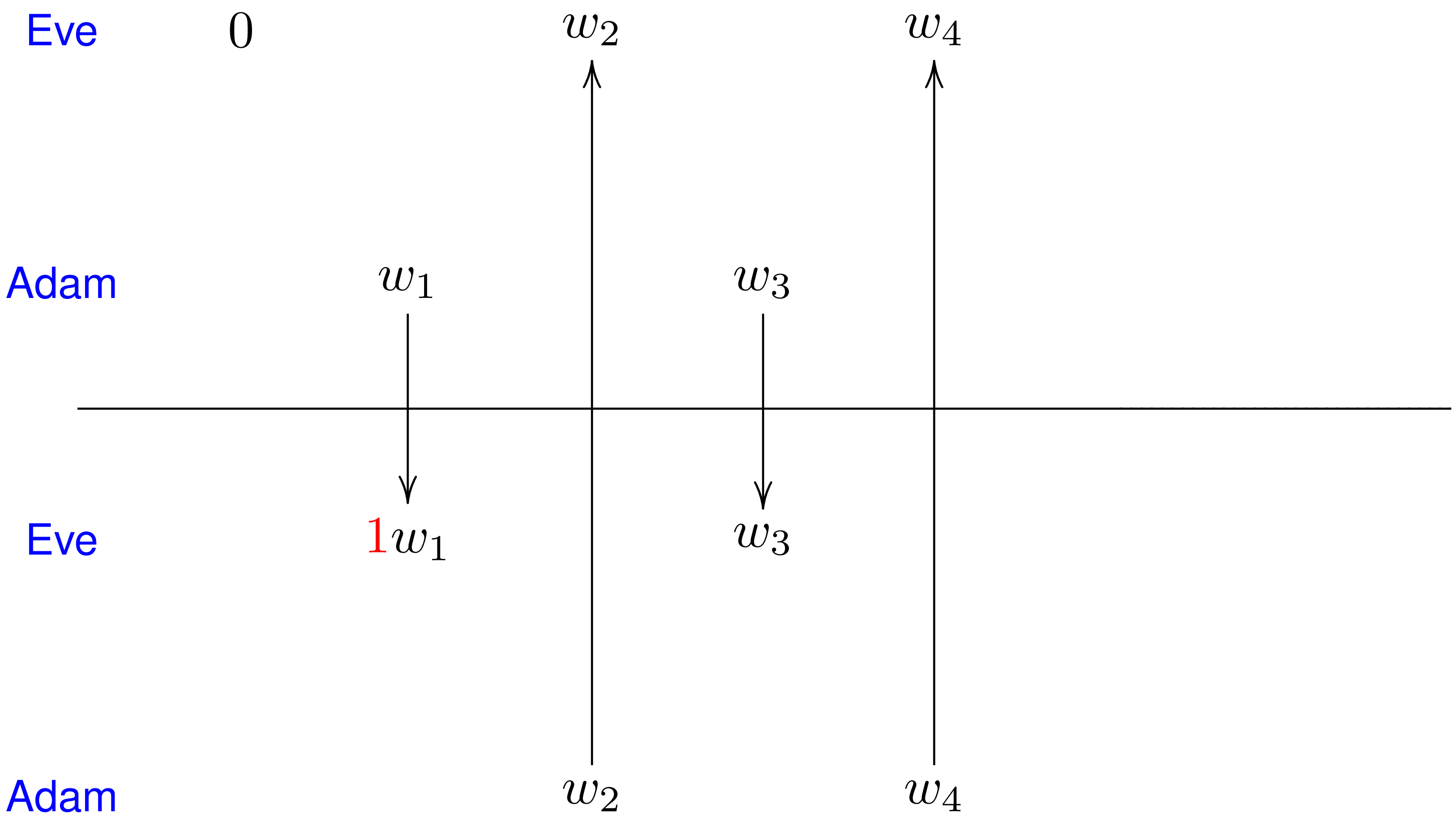

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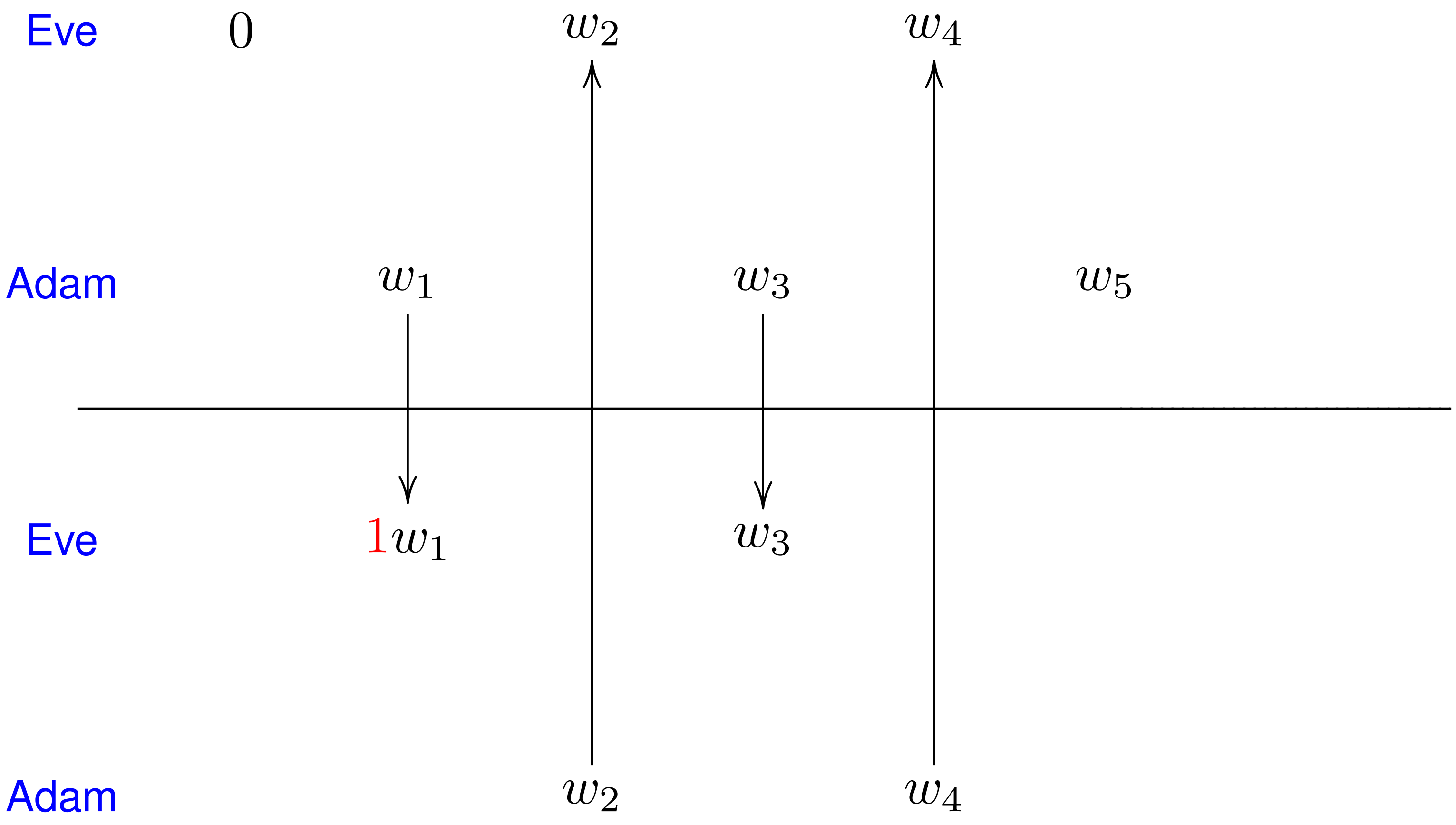

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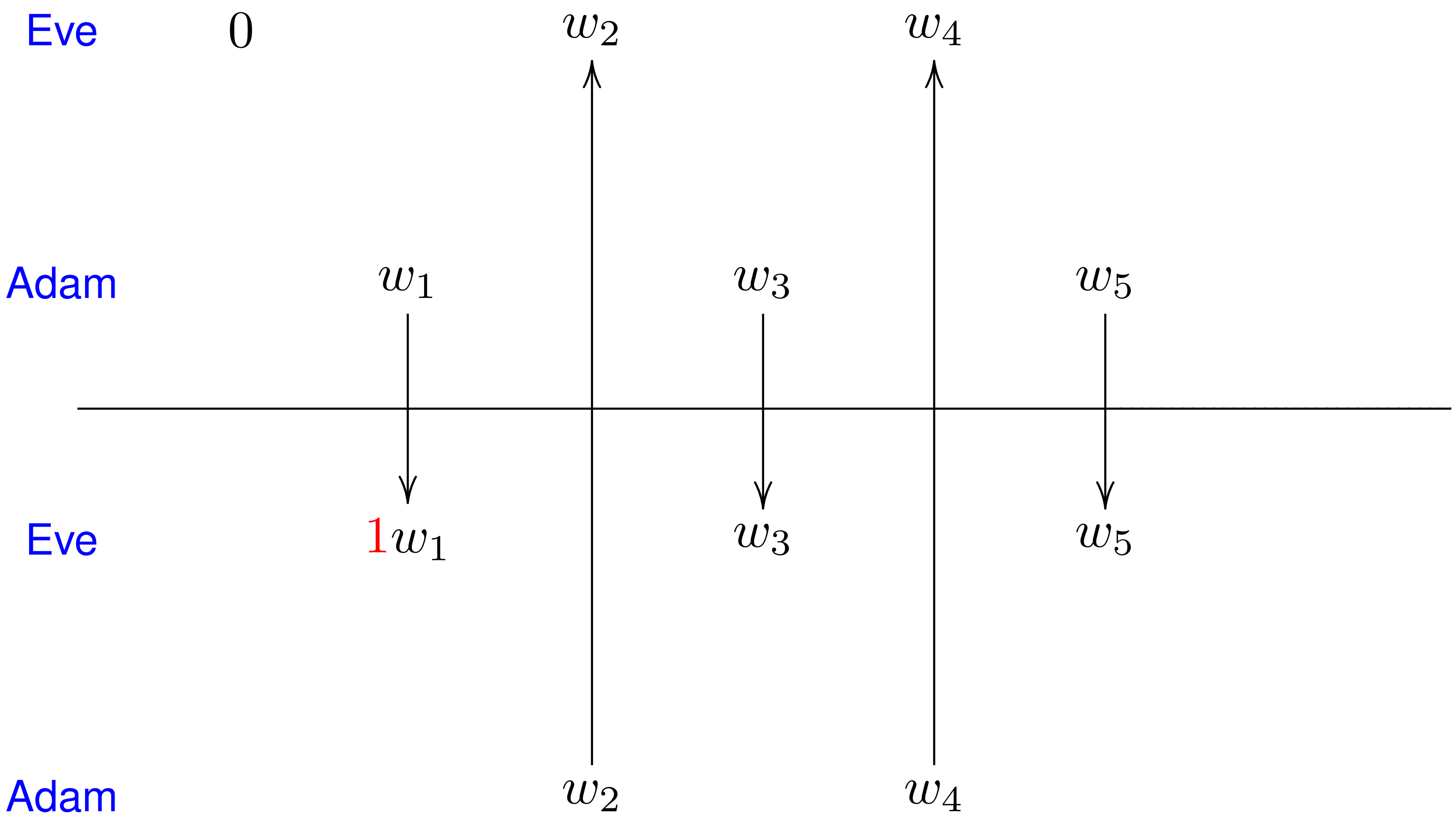
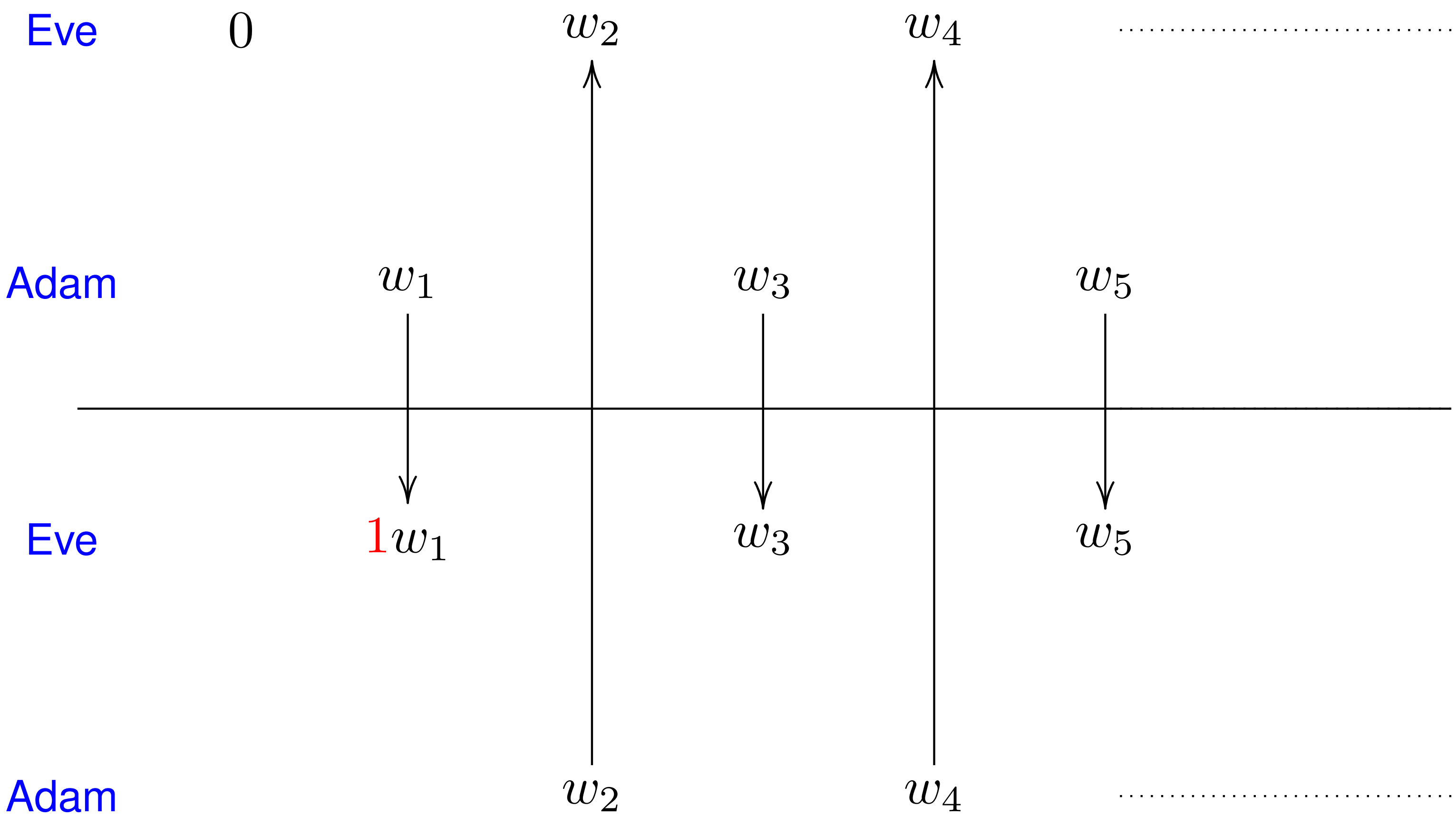


Suppose Adam wins

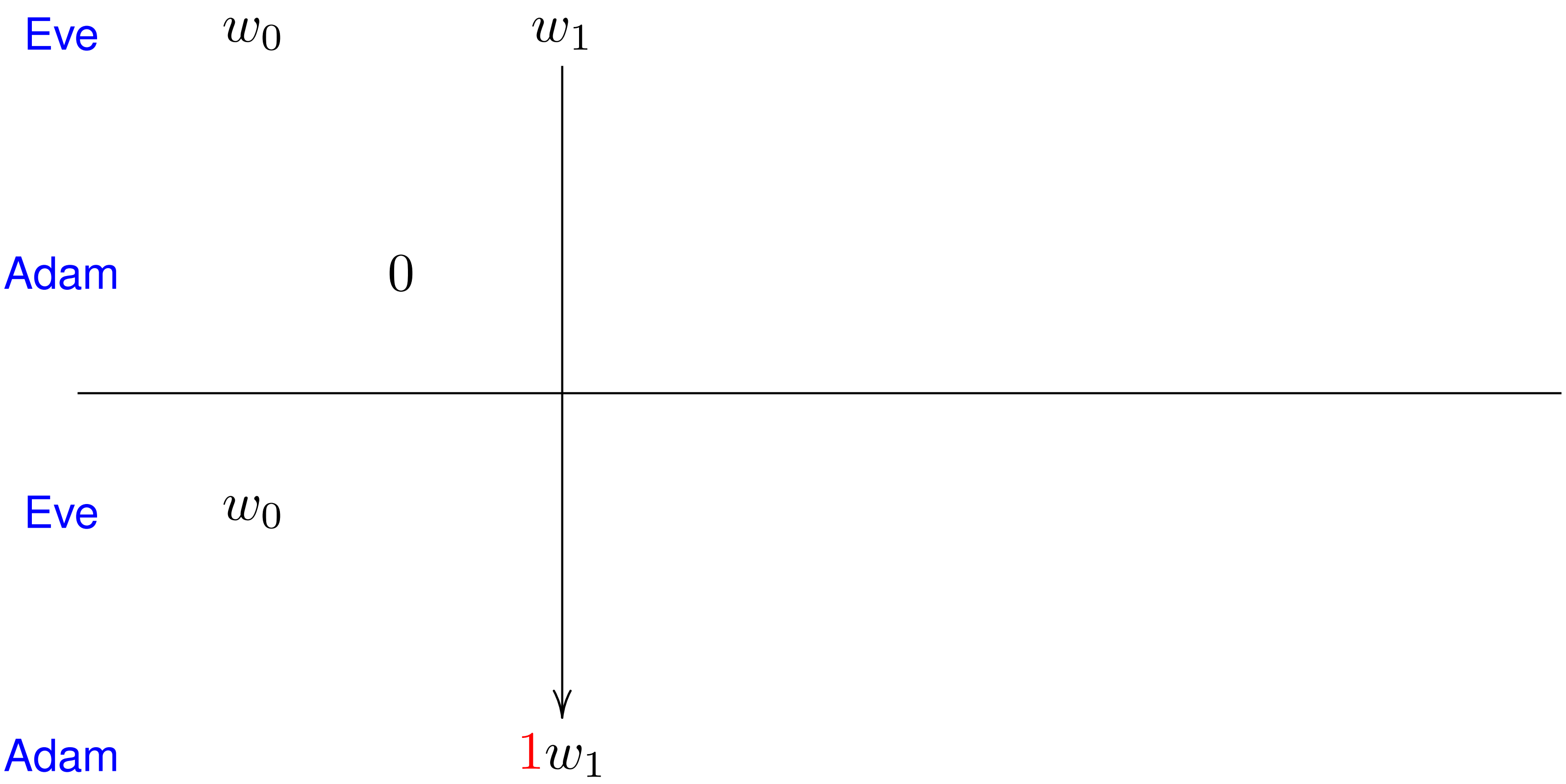


Suspose Eve wins

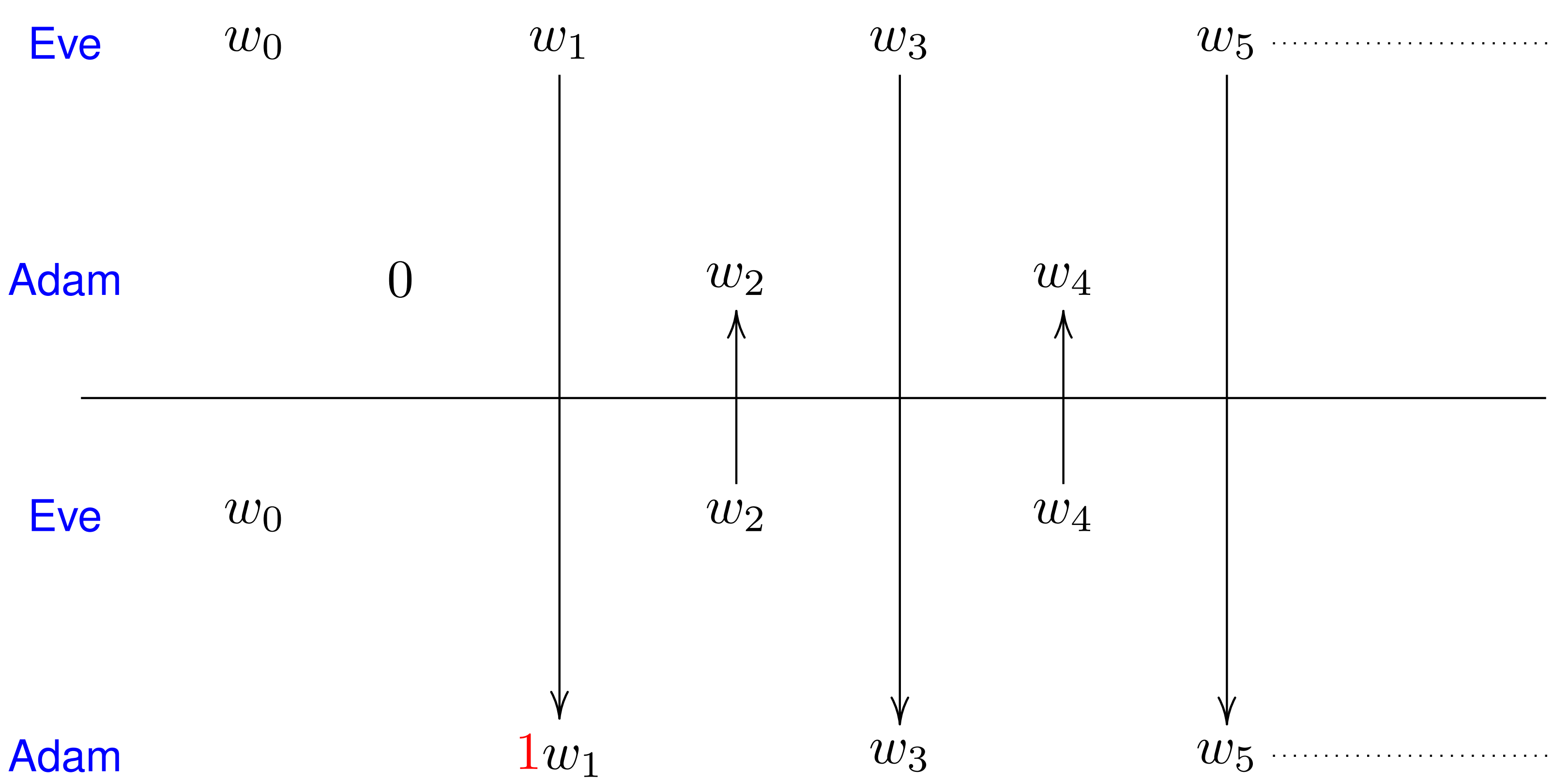
Eve	w_0		w_1
Adam		0	

Eve	w_0
Adam	

Suppose Eve wins



Suppose Eve wins



However, most of “natural” games *are* determined.

By **Martin’s Theorem** (1975), games with **Borel** criteria are always determined.

This includes the **parity games**. Colors: $0, 1, 2, \ldots, n$.

Eve wins if the *highest* color that occurs infinitely often is **even**.
Adam wins if the highest color that occurs infinitely often is **odd**.

The winning sets satisfy the game equations.

$$
\begin{aligned}
W_E &= (V_E \cap \Diamond W_E) \cup (V_A \cap \Box W_E) &= \quad Eve(W_E) \\
W_A &= (V_A \cap \Diamond W_A) \cup (V_E \cap \Box W_A) &= \quad Adam(W_A)
\end{aligned}
$$

But they are neither least nor greatest solutions.

For example, W_E in the game with ranks $0, 1, 2, 3$, equals

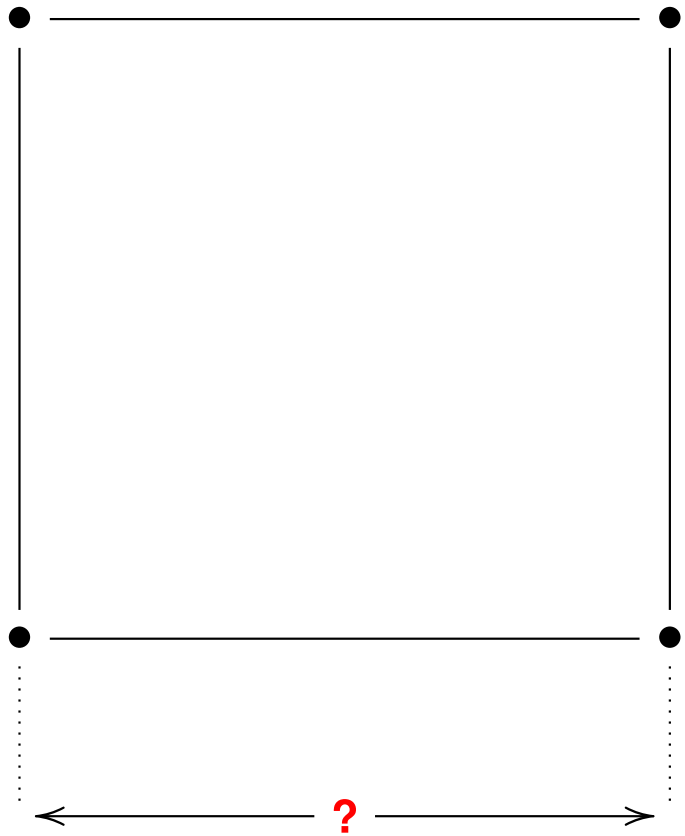
$$
\begin{aligned}
\mu X_3.\nu X_2.\mu X_1.\nu X_0.\quad & (V_E \cap rank_0 \cap \Diamond X_0) \cup \\
& (V_E \cap rank_1 \cap \Diamond X_1) \cup \\
& (V_E \cap rank_2 \cap \Diamond X_2) \cup \\
& (V_E \cap rank_3 \cap \Diamond X_3) \cup \\
& (V_A \cap rank_0 \cap \Box X_0) \cup \\
& (V_A \cap rank_1 \cap \Box X_1) \cup \\
& (V_A \cap rank_2 \cap \Box X_2) \cup \\
& (V_A \cap rank_3 \cap \Box X_3)
\end{aligned}
$$

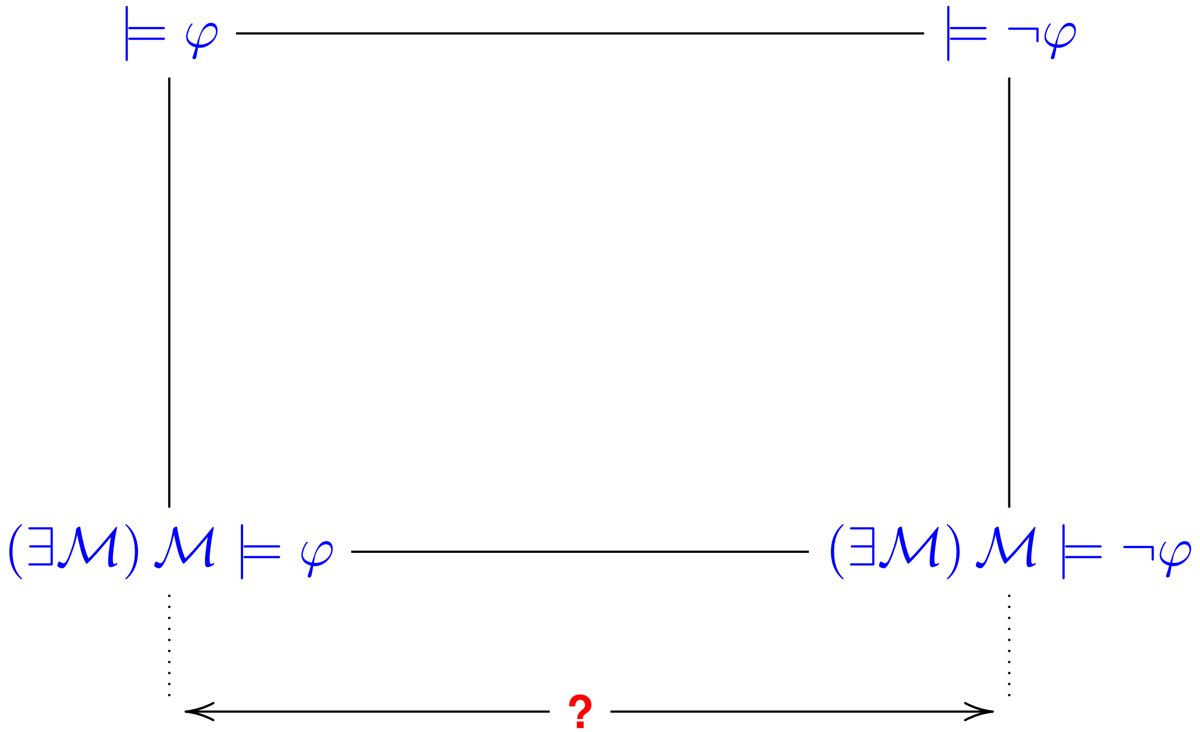
Modal μ-calculus

It is an extension of the propositional modal logic by the **least** (μ) and **greatest** (ν) fixed point operators, introduced by Kozen in 1982.

$$p \mid \neg p \mid X \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi \mid \Box\varphi \mid \mu X.\varphi \mid \nu X.\varphi$$

Parity games are for the modal μ-calculus like Hintikka games for first order logic.





To distinguish between the corners is **undecidable** for FO logic and **NP/co-NP hard** for propositional logic.

For infinite games, we may use arguments from descriptive set theory.

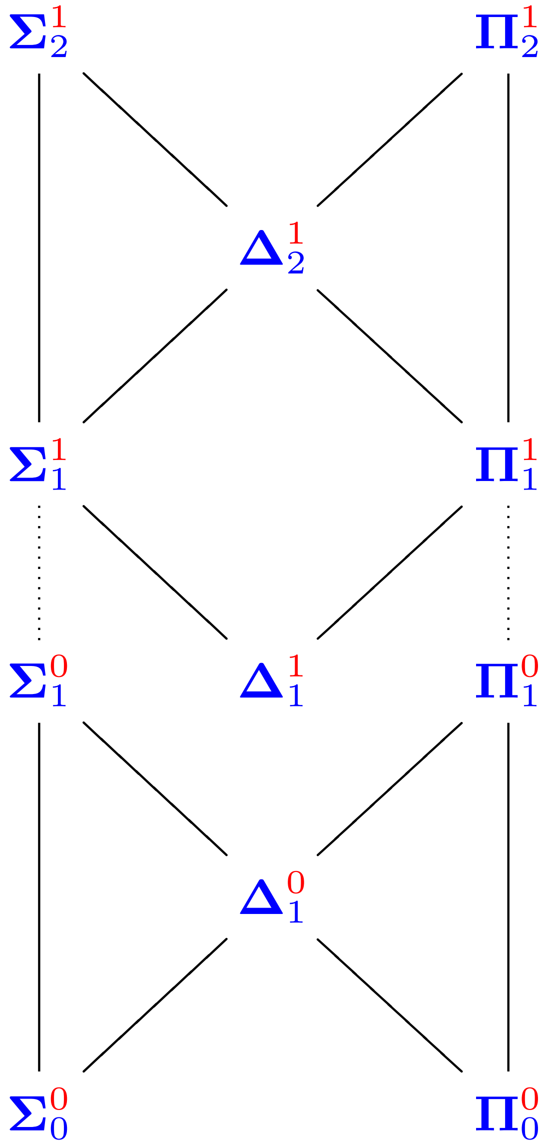
Classical definability theory

1900 Borel, Baire, Lebesgues

1917 Lusin, Suslin

1929 Tarski, Kuratowski

1940 Mostowski, Kleene



For $R \subseteq \omega^k \times (\{0,1\}^\omega)^\ell$, let

$$\exists^0 R = \{\langle \mathbf{m}, \alpha \rangle : (\exists n)\, R(\mathbf{m}, n, \alpha)\}$$

$$\exists^1 R = \{\langle \mathbf{m}, \alpha \rangle : (\exists \beta)\, R(\mathbf{m}, \alpha, \beta)\}$$

Arithmetical hierarchy			Analytical hierarchy		
Σ^0_0	$=$	recursive relations	Σ^1_0	$=$	arithmetical relations
Π^0_n	$=$	$\{\overline{R} : R \in \Sigma^0_n\}$	Π^1_n	$=$	$\{\overline{R} : R \in \Sigma^0_n\}$
Σ^0_{n+1}	$=$	$\{\exists^{\mathbf{0}} R : R \in \Pi^0_n\}$	Σ^1_{n+1}	$=$	$\{\exists^{\mathbf{1}} R : R \in \Pi^1_n\}$
Δ^0_n	$=$	$\Sigma^0_n \cap \Pi^0_n$	Δ^1_n	$=$	$\Sigma^1_n \cap \Pi^1_n$

Arithmetical hierarchy

$$\begin{aligned}\Sigma^0_0 &= \text{recursive relations}\\ \Pi^0_n &= \{\overline{R} : R \in \Sigma^0_n\}\\ \Sigma^0_{n+1} &= \{\exists^0 R : R \in \Pi^0_n\}\\ \Delta^0_n &= \Sigma^0_n \cap \Pi^0_n\end{aligned}$$

Analytical hierarchy

$$\begin{aligned}\Sigma^1_0 &= \text{arithmetical relations}\\ \Pi^1_n &= \{\overline{R} : R \in \Sigma^0_n\}\\ \Sigma^1_{n+1} &= \{\exists^1 R : R \in \Pi^1_n\}\\ \Delta^1_n &= \Sigma^1_n \cap \Pi^1_n\end{aligned}$$

Relativized (boldface) hierarchies

For $\beta \in \{0,1\}^\omega$, let $R[\beta] = \{\langle \mathbf{m}, \alpha\rangle : R(\mathbf{m}, \alpha, \beta)\}$.

$$\boldsymbol{\Sigma}^i_n = \{R[\beta] : R \in \Sigma^i_n, \beta \in \{0,1\}^\omega\} \qquad \boldsymbol{\Delta}^i_n = \boldsymbol{\Sigma}^i_n \cap \boldsymbol{\Pi}^i_n$$

$$\boldsymbol{\Pi}^i_n = \{R[\beta] : R \in \Pi^i_n,\ \beta \in \{0,1\}^\omega\} \qquad i \in \{0,1\}$$

$$\begin{aligned}\boldsymbol{\Sigma}^0_1 &= \textit{open}\\ \boldsymbol{\Pi}^0_1 &= \textit{closed}\\ \boldsymbol{\Delta}^1_1 &= \textit{Borel}\end{aligned}$$

Topological complexity of the game

We consider tree-like arenas which can be identified with elements of the Cantor discontinuum $\{0,1\}^\omega$.

Sets of arenas can be therefore classified in the arithmetical and analytical hierarchy.

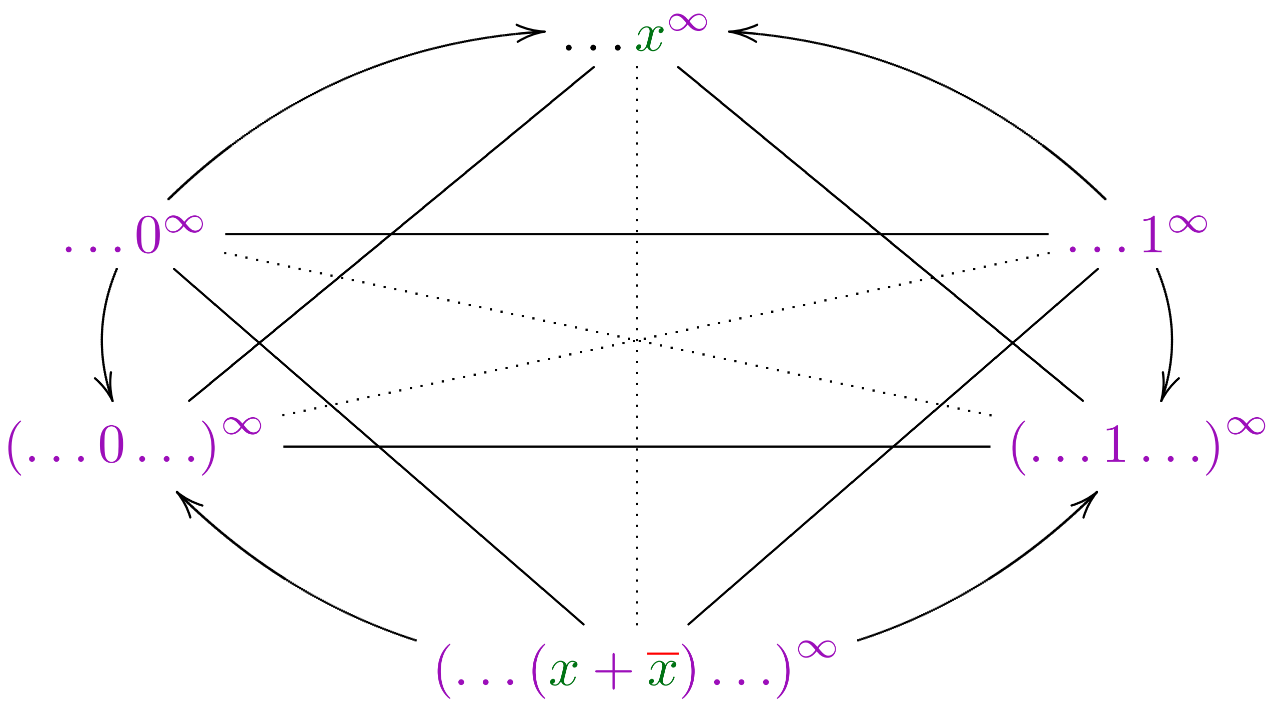
For a given winning criterion, we can ask what is the complexity of the set of those arenas, where Eve has a winning strategy.

Consider the game with colors $0, 1$. Possible outcomes:

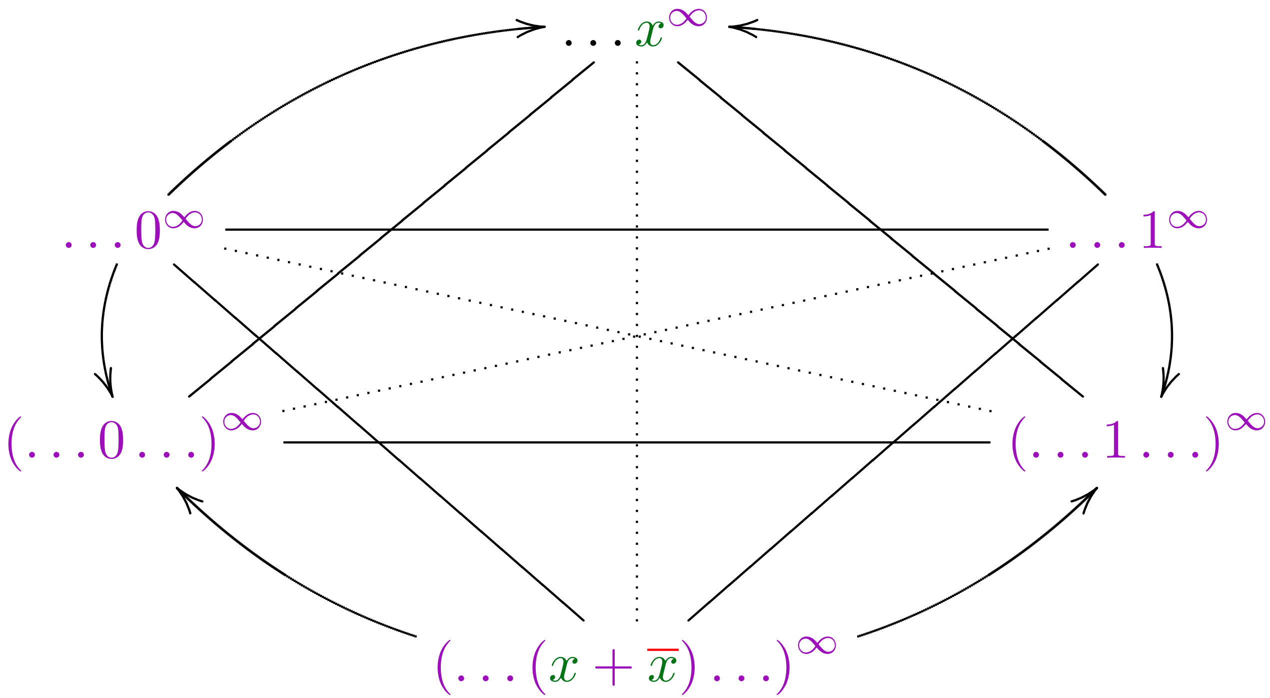
1 occurs only finitely often, *eventually obligatory* 0, in symbols: $\ldots 0^\infty$.

0 occurs only finitely often, *eventually prohibited* 0, in symbols: $\ldots 1^\infty$.

x occurs infinitely often, in symbols: $(\ldots x \ldots)^\infty$, $x = 0, 1$.

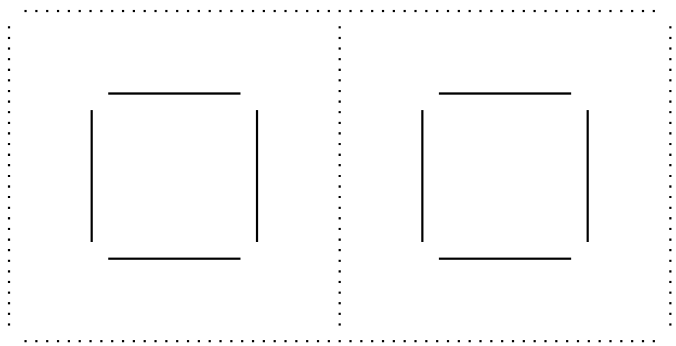


How far is *obligatory* from *prohibited* ?

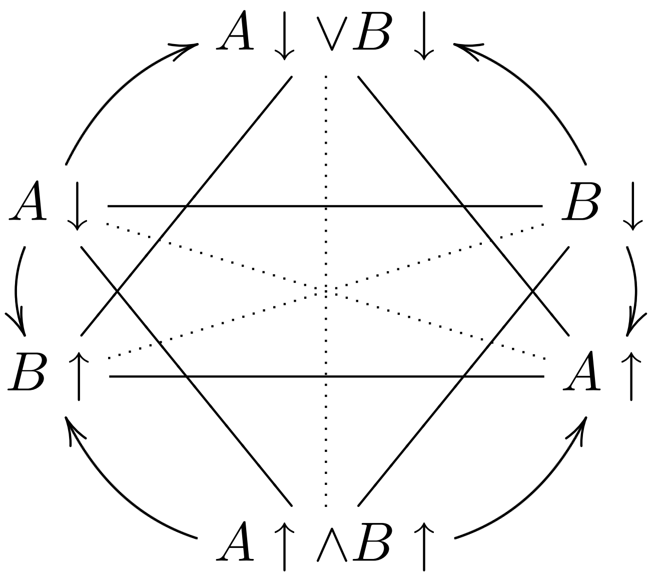


The set of arenas, where Eve has a strategy to ensure $\ldots 0^{\infty}$ is complete in the class $\mathbf{\Pi}^1_1$.

Moreover, this set and the set of trees where Adam has a strategy to ensure $\ldots 1^{\infty}$, cannot be separated by any Borel measurable set (Hummel, Michalewski, N., 2009).

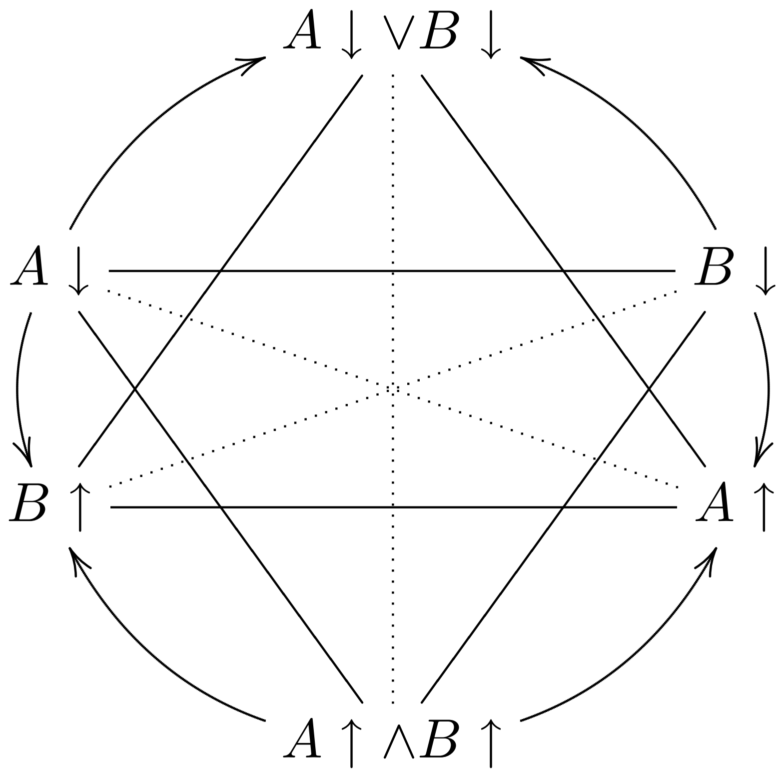


Moral: although *obligatory* and *prohibited* are seemingly opposite, the boundary is sometimes hard to delineate.



Conclusion

- The winning scenarios in classical games exhibit the pattern of the *square*, and determinacy theorem takes the form of the *triangle*.
- These patterns can be explained by the dualities in the μ-calculus.
- The new realizations give rise to the problem of the complexity of the *square*.


$A\downarrow \vee B\downarrow$
$A\downarrow$
$B\downarrow$
$B\uparrow$
$A\uparrow$
$A\uparrow \wedge B\uparrow$