GAME SOLUTION, EPISTEMIC DYNAMICS, AND FIXED-POINT LOGICS

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Abstract Logic, game theory and computer science share many themes. E.g., methods for solving games are a form of 'procedural rationality' that invites logical analysis. Our case study is Backward Induction for extensive games, replacing the static epistemic foundations of game theory by dynamic ones. We will analyze some recent views of game solution, viz. iterated announcements of players' rationality, and belief revision by plausibility upgrade with 'rationality-in-beliefs'. These views turn out equivalent, pointing at underlying invariant structure. We explore how standard fixed-point logics on finite trees fit such game-theoretic equilibria. We end with questions interfacing computational logic and game theory.

References J. van Benthem, 2002, *Logic in Games*, lecture notes, Institute for Logic, Language & Computation ILLC, Amsterdam. JvB 2010, *Logical Dynamics of Information and Interaction*, Cambridge University Press. A. Gheerbrant, 2010, *Logics of Finite Trees,* Ph. D. thesis, ILLC, UvA.

1 Logic and rational agency

To be rational is to reason intelligently. All available sources of information. Logic as the study of explicit informational processes (inference, observation, communication)

Zhi: Wen, Shuo, Qin 知 问 说 亲 (Fenrong Liu & J. Zhang, *A Note on Mohist Logic*)

To be rational is to act intelligently. Add goals, preferences, decisions, actions.

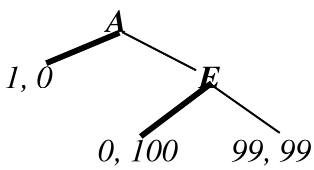
To be rational is to interact intelligently. Argumentation, communication, games.





2 Where it all comes together: logic and games

Games are a microcosm for logics of rational agency. Here is a minimal social scenario – but many things involved logically in explaining or predicting such behaviour:



Many contacts today: *logics for analyzing games*, *games for performing logical tasks*. Key result (Aumann 1995): characterization of *BI* by common knowledge of rationality. This lecture shifts the focus to analyzing *game solution methods as rational procedures*.

3 Running example: Backward Induction

The *BI* procedure for 'distinguished' extensive games:

> "At end nodes, players already have their values marked. At further nodes, once all daughters are marked, the player to move gets her maximal value that occurs on a daughter, while the other, non-active player gets his value on that maximal node."

A *strategy* for a player is a map selecting one move at each turn for that player. The *BI* procedure can create delicate cases, and its conceptual justification is still under debate.

4 Statics: defining the *BI* outcome in modal preference logic

Modal preference language $<pref_i>\phi$: *i* prefers some node with ϕ to the current one.

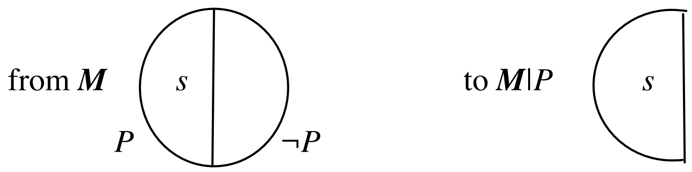
Theorem The *BI* strategy is as the unique relation σ satisfying the following axiom for all propositions *P* – seen as sets of nodes –, for all players *i*:

$(turn_i \ \& <\sigma^*>(\mathbf{end} \ \& \ P)) \rightarrow [move\text{-}i]<\sigma^*>(\mathbf{end} \ \& <pref_i>P)$.

5 Dynamics I: the *BI* procedure in logic of public announcement

Dynamic logics of public announcement: *Information update as model change*.

Learning that *P* eliminates the worlds where *P* is false (hard information). Picture:



Static epistemic logic Formulas $p \mid \neg\phi \mid \phi\vee\psi \mid K_i\phi \mid C_G\phi$, models $\boldsymbol{M} = (W, \{\sim_i \mid i\in G\}, V)$, worlds *W*, accessibility relations $\sim_i$, valuation *V*. Truth: $\boldsymbol{M}, s \models K_i\phi$ iff for all *t* with $s \sim_i t$: $\boldsymbol{M}, t \models \phi$, $\boldsymbol{M}, s \models C_G\phi$ iff for all *t* reachable from *s* by finite sequence of ~ steps: $\boldsymbol{M}, t \models \phi$. ***Dynamic logic of public announcement*** *PAL*: action expressions: *!P* for all formulas *P*, and modal operators describing their effects (one simultaneous recursion):

$$\boldsymbol{M}, s \models [!P]\,\phi \quad \text{iff} \quad \text{if } \boldsymbol{M}, s \models P, \text{ then } \boldsymbol{M}|P, s \models \phi$$

Theorem PAL axiomatized completely by epistemic logic plus *recursion axioms*:

$$[!P]q \leftrightarrow P \rightarrow q \qquad \text{for atomic facts } q$$
$$[!P]\neg\phi \leftrightarrow P \rightarrow \neg[!P]\phi$$
$$[!P]\phi\wedge\psi \leftrightarrow [!P]\phi \wedge [!P]\psi$$
$$[!P]K_i\phi \leftrightarrow P \rightarrow K_i(P \rightarrow [!P]\phi)$$

Backward Induction as repeated public announcement (cf. Muddy Children!):

Theorem The Backward Induction solution for extensive games is fixed-point of repeated announcement of "No player plays a strictly dominated move".

6 Dynamics II: the *BI* procedure as iterated belief revision

Conditional logic of relative plausibility: base logic for 'soft information':

$\boldsymbol{M}, s \models B_i\phi$ iff $\boldsymbol{M}, t \models \phi$ for all worlds t *minimal* in the ordering $\lambda xy. \leq_{i, s} xy$.

Soft information just *changes the plausibility ordering* of the existing worlds. E.g.,

Lexicographic upgrade $\Uparrow P$ changes the current model $\boldsymbol{M}$ to $\boldsymbol{M}\Uparrow P$:

P-worlds now better than all $\neg P$-worlds; within zones, old order remains.

Complete dynamic logics of belief revision exist – but we will not need these here.

BI as a limit of belief change *BI creates expectations* among branches. Start: empty relation. At a turn for i, x 'strictly dominates' sibling y 'in beliefs' if all currently most plausible end nodes after x are worse for i than all most plausible end nodes after y.

Theorem The *BI* procedure is the limit of iterated soft updates with the assertion that "No player plays a move strictly dominated in beliefs at her turns".

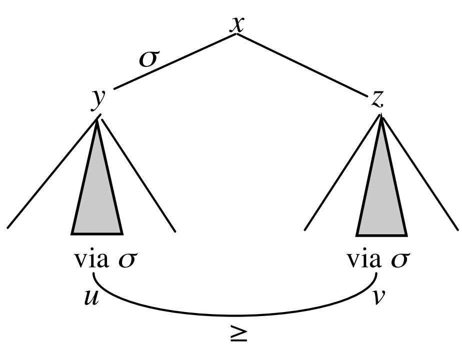
7 Logical analysis: from functional to relational strategies

Generalized strategy: any subrelation of the *move* relation. Generalizations of *BI*: say, take minimum for passive player on maximal nodes for the active player. *Such solution algorithms all make assumptions about players' preference*. One minimal version:

> First, mark all moves as 'active'. Call a move *a dominated* if it has a sibling move all of whose reachable endpoints via active nodes are preferred by the current player to all reachable endpoints via *a* itself. Now, at each stage, mark dominated moves in this sense of set preference as 'passive', leaving all others active. In the preference comparison, the 'reachable endpoints' by an active move are all those that can be reached via a sequence of further moves that are still active at this stage.

8 Defining BI as a unique static relation: the background

Fact A game model makes $(turn_i \wedge \langle\sigma^*\rangle(\boldsymbol{end} \wedge p)) \rightarrow [move\text{-}i]\langle\sigma^*\rangle(\boldsymbol{end} \wedge \langle pref_i\rangle p)$ true for all i at all nodes iff it has this property for all i:



The modal axiom is equivalent to a *confluence property* for action and preference:

$$CF1 \quad \&_i\ \forall x\,(Turn_i(x) \to \forall y\,(\,x\,\sigma y \to (x\ move\ y \wedge \forall u\,((end(u) \wedge y\,\sigma^*\,u)) \to \forall z\,(x\ move\ z \to \exists v\,(end(v) \wedge z\,\sigma^*\,v \wedge v \leq_i u))))$$

Capturing BI But now look at our relational version of *BI*. Consider the $\forall\forall\exists\exists$ form

$$CF2 \quad \&_i\ \forall x\ \forall y\,((Turn_i(x) \wedge x\,\sigma y) \to (x\ move\ y \wedge \forall z\,(x\ move\ z \to \exists u\,\exists v\,(end(u) \wedge end(v) \wedge y\,\sigma^*\,v \wedge z\,\sigma^*\,u \wedge u \leq_i v)))$$

Theorem *BI* is the largest subrelation of the *move* relation in a finite game tree satisfying (a) the relation has a successor at each intermediate node, (b) *CF2*.

Inspecting the syntax of *CF2*, we can use the *first-order fixed-point logic LFP(FO)*:

Theorem The *BI* relation is definable in *LFP(FO)*.

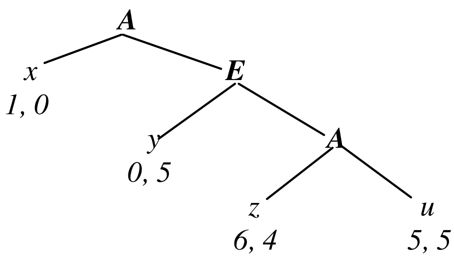
$$BI\,(x, y) = \nu S, xy\bullet\, x\ move\ y \wedge \&_i\,(Turn_i(x) \to \forall z\,(x\ move\ z \to \exists u\,\exists v\,(end(u) \wedge end(v) \wedge y\,S^*\,v \wedge z\,S^*\,u \wedge u \leq_i v)))$$

The decreasing *approximation stages* S^k are exactly the 'active move stages' of the above algorithm. Fixed-point logics analyze both statics and dynamics of game solution.

9 A dynamic-epistemic scenario: iterated announcement of rationality

Sharpening up the earlier story. At a turn for player *i*, move *a* is *dominated* by sibling *b* (a move available at the same node) if every history through *a* ends worse, in terms of *i*'s preference, than every history through *b*. Now ***rat*** says that "at the current node, no player has chosen a strictly dominated move in the past coming here". As with Muddy Children, iterated announcement of ***rat*** must reach a limit, where no node is dominated:

Example Solving games through iterated assertions of Rationality.



and the next stages…

Definition (Announcement limit). For each epistemic model ***M*** and each proposition φ, the *announcement limit* $(\varphi, \boldsymbol{M})^\#$ is the first model reached by successive announcements $!\varphi$ that no longer changes after the last announcement is made. Either this model is non-empty, in which case ***rat*** holds in all nodes, so it has become common knowledge (*self-fulfilling*), or it is empty, and the negation $\neg$***rat*** is common knowledge (*self-refuting*).

Both occur in concrete puzzles, though rationality assertions like ***rat*** tend to be self-fulfilling, while the ignorance statement driving the Muddy Children is self-refuting.

Theorem In any game tree $\boldsymbol{M}$, $(!\boldsymbol{rat}, \boldsymbol{M})^{\#}$ is the actual subtree computed by *BI*.

Explanation: sets of nodes as relations Each subrelation R of the *move* relation induces a set of nodes *reach(R)*: the range of R plus the root. And each set X of nodes has a matching relation consisting of all moves in the tree that end in X.

Fact For each k, in each game $\boldsymbol{M}$, $BI^k = rel((!\boldsymbol{rat})^k, \boldsymbol{M})$, $reach(BI^k) = ((!\boldsymbol{rat})^k, \boldsymbol{M})$.

Open problem Logics for defining the fixed-point limits of announcement procedures. In general, inflationary epistemic fixed-point logic fits, but sometimes μ-calculus works.

10 Once again, beliefs and iterated plausibility upgrade

Definition (Rationality in beliefs). Move x *dominates* sibling move y *in beliefs* if the *most plausible* end nodes reachable after x along any path in the game are all better for the active player than all *most plausible* end nodes reachable after y. *Rationality** (***rat****) says that no player plays a move that is dominated in beliefs.

Theorem On finite trees, the Backward Induction strategy is encoded in the plausibility order for end nodes created by iterated radical upgrade with rationality-in-belief.

And at the end of this procedure, players have acquired *common belief in rationality*.

Strategies as special plausibility relations We first observe that each subrelation R of the total move relation induces a total plausibility order *ord(R)* on leaves x, y of the tree:

> We put x *ord(R)* y iff, looking upward at the first node z where the histories of x, y diverged, if x was reached via an R move from z, then so is y.

Fact The relation *ord(R)* is a total re-order on leaves.

Ord(R) is *tree-compatible* in an obvious sense. Any tree-compatible total order $\leq$ on leaves induces a subrelation $rel(\leq)$ of the *move* relation, selecting just those moves at a node z whose histories lead only to $\leq$-maximal leaves in the total set of reachable leaves.

Fact For any game tree $\boldsymbol{M}$ and any k, $rel((\Uparrow rat^*)^k, \boldsymbol{M})) = BI^k$.

11 Midway conclusion: the stability of Backward Induction

All analyses are the same: *extensional equivalence, intensional difference*. In particular, we find an underpinning for ***dynamic instead of static foundations for game theory.*** Common knowledge or belief of rationality is not assumed, but *produced* by the logic.

12 Test case: variants of Backward Induction

This analysis also works for other variants of *BI* (Gheerbrant 2010) Natural definitions use a *first-order inflationary fixed-point logic IFP(FO)* with simultaneous fixed-points.

13 Alternative fixed-point analyses

Fixed-point recursion on well-founded tree order This exploits the tree structure:

Example Consider the definition $p \leftrightarrow \neg\Box p$. On a 3-node linear order

$$\begin{tikzcd} 1 \arrow[r] \arrow[rr, bend left] & 2 \arrow[r] & 3 \end{tikzcd}$$

starting from any set as a value for p, this will stabilize to the fixed-point $p = \{2\}$. There is no definition for this in the modal μ–calculus, and not even in its inflationary variant.

Fixed-point logic on games in strategic form Take *BI* output as set of strategy profiles.

14 Hiding the machinery: modal fragments of fixed-point logics for games

Which *fragments* are needed for game solution, and which fixed-point operators for describing its mechanics? One drastic way to go is modal, hiding the procedure again:

Open problem Can we axiomatize the modal logic of finite game trees with a *move* relation and its transitive closure, turns and *preference* relations for players, and a new relation *best* as computed by Backward Induction?

Fact The following modal axiom corresponds to *CF2* by standard techniques:

$$(turn_i \wedge \langle best\rangle[best^*](\mathbf{end} \rightarrow p)) \rightarrow [move\text{-}i]\langle best^*\rangle(\mathbf{end} \wedge \langle pref_i\rangle p)$$

Potential problem: the complexity of rationality In logics of *action* and *knowledge*, apparently harmless assumptions such as Perfect Recall make the logic undecidable, and sometimes Π^1_1-complete. *PR* generates commuting diagrams for *move* and uncertainty $\sim$

$$\forall x\ \forall y((x\ move\ y \wedge y \sim z) \rightarrow \exists u\ (x \sim u \wedge u\ move\ z))$$

that drive encodings of Tiling Problems. But Rationality assumptions also create grids:

$$\forall x\ \forall y\ ((Turn_i(x) \wedge x\ \sigma\, y) \rightarrow \forall z\ (x\ move\ z \rightarrow\ \forall u\ ((end(u) \wedge y\ \sigma^*\ u)$$
$$\rightarrow \exists v\ (end(v) \wedge z\ \sigma^*\ v \wedge v \leq_i u))))$$

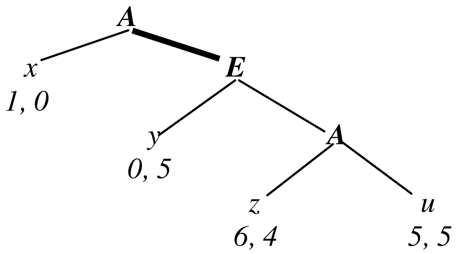
Does Rationality, meant to make behaviour predictable, really make its logic complex?

15 Further issues and extended game logics

Language design and game equivalence, Infinite games, Imperfect information.

16 Coda: Paradox of Backward Induction and more belief dynamics

Our mathematics may be elegant. But does it also make sense? Consider the so-called Paradox of Backward Induction, as discussed by philosophers:



Backward Induction tells us that ***A*** will go left at the start, on the basis of logical reasoning that is available to both players. But then, if ***A*** plays *right* (as marked by the black line) what should ***E*** conclude? Does not this mean that ***A*** is not following the *BI* reasoning, and hence that all bets are off as to what he will do later on in the game?

Agent types Responses to this difficulty vary. The characterization result of Aumann 1995 assumes that players know that rationality prevails throughout, a stubborn belief that players will act rationally later on, even if they have not done so up until now.

Belief revision A richer analysis should add an account of the types of agent that play a game. We must represent the *belief revision* policies by the players, that determine what they will do when making an observation contradicting their beliefs in the course of a game. There are many different options for such policies in the above example, such as

> 'It was just an error, and ***A*** will go back to being rational',
> '***A*** is telling me that he wants me to go right, and I will be rewarded for that',
> '***A*** is an automaton with a general rightward tendency', and so on.

Open problem How to merge our fixed-point analysis with belief revision structure?

17 Conclusion

Games a rich mixture of computational logic and philosophical logic. The future;

> *Logic + Game Theory = Theory of Play*