Sparsity — tutorial 9

Shrubdepth and uniform quasi-wideness

Definition 0.1. For a graph G, a *connection model* for G with a label set Λ is a pair consisting of:

- a labelling λ of vertices of G with labels from Λ,
- a rooted tree T whose leaves are the vertices of G, and
- for every non-leaf node x of T, a symmetric set $Z_x \subseteq \Lambda^2$,

such that the following condition holds. For every pair of vertices $u, v \in V(G)$, we have

$$uv \in E(G) \Leftrightarrow (\lambda(u), \lambda(v)) \in Z_x, \qquad \text{where } x \text{ is the lowest common ancestor of } u \text{ and } v \text{ in } T.$$

The *shrubdepth* of G is the lowest number d such that G has a connection model of depth at most d over a label set of size d.

Definition 0.2. For a graph G and $d \in \mathbb{N}$, by G^d we denote a graph on the vertex set $V(G)$ where u and v are adjacent if and only if $\mathrm{dist}_G(u, v) \leq d$. For a class $\mathcal{C}$, we denote $\mathcal{C}^d = \{G^d \colon G \in \mathcal{C}\}$.

Problem 1. Prove that if a class $\mathcal{C}$ has bounded treedepth, then for every fixed $d \in \mathbb{N}$ the class $\mathcal{C}^d$ has bounded shrubdepth.

Problem 2. Prove that if a class $\mathcal{C}$ has bounded expansion and $d \in \mathbb{N}$ is fixed, then the class $\mathcal{C}^d$ admits *low shrubdepth colorings* in the following sense: for every $p \in \mathbb{N}$ there exists N such that every graph $G \in \mathcal{C}^d$ has a coloring with N colors in which every subset of p colors induced a subgraph of shrubdepth at most $f(p)$, for some function $f \colon \mathbb{N} \to \mathbb{N}$.

Problem 3. Prove that a class of graphs is uniformly wide if and only if it has bounded maximum degree.

Problem 4. Prove that for every nowhere dense class $\mathcal{C}$ and integer $d \in \mathbb{N}$, there exists $\delta > 0$ such that for every n-vertex graph $G \in \mathcal{C}$, the graph G^d contains either a clique of size at least n^δ or an independent set of size at least n^δ.

Definition 0.3. A pair of sequences $a_1, \ldots, a_\ell$ and $b_1, \ldots, b_\ell$ of vertices of a graph G is called

- a *co-matching* if for all $i, j \in \{1, \ldots, \ell\}$: a_i is adjacent to b_j if and only if $i \neq j$;
- a *ladder* if for all $i, j \in \{1, \ldots, \ell\}$: a_i is adjacent to b_j if and only if $i > j$; and
- a *semi-ladder* if for all $i \in \{1, \ldots, \ell\}$: a_i is not adjacent to b_i, but is adjacent to all b_j satisfying $i > j$.

The *co-matching index* of a class $\mathcal{C}$ is the supremum of the lengths of co-matchings that can be found in graphs from $\mathcal{C}$. Define the *ladder index* and the *semi-ladder index* similarly.

Problem 5. Prove that a class of graphs has a finite semi-ladder index if and only if it has a finite co-matching index and a finite ladder index.

Problem 6. Prove that for every nowhere dense class $\mathcal{C}$ and $d \in \mathbb{N}$, the class $\mathcal{C}^d$ has a bounded semi-ladder index.

Problem 7. Suppose $r \in \mathbb{N}$, G is a graph, S is subset of vertices of G, and $(u_1, v_1), (u_2, v_2)$ are two pairs of vertices from G. We say that S *distance-r separates* (u_1, v_1) and (u_2, v_2) if every path of length at most r with one endpoint in $\{u_1, v_2\}$ and second in $\{u_2, v_2\}$ contains a vertex of S.

Prove that for every nowhere dense class $\mathcal{C}$ and integer $r \in \mathbb{N}$, there exist a constant $s_r \in \mathbb{N}$ and a function $N_r \colon \mathbb{N} \to \mathbb{N}$ such that the following holds. For every $m \in \mathbb{N}$, graph $G \in \mathcal{C}$, and set A of pairs of vertices of G with $|A| \geq N_r(m)$, there exist $S \subseteq V(G)$ and $B \subseteq A$ with $|S| \leq s_r$ and $|B| \geq m$ such that every pair of distinct pairs from B is distance-r separated by S.