Sparsity — tutorial 8

FO model checking and shrubdepth

Problem 1. Prove that for every class structures $\mathcal{C}$ with bounded expansion and FO formula $\varphi(\bar{x})$, there exists an algorithm that, given a structure $\mathbb{A}$, computes the set $\varphi(\mathbb{A})$ in time $\mathcal{O}(\|\mathbb{A}\| + |\varphi(\mathbb{A})|)$.

Problem 2. Prove that for every class structures $\mathcal{C}$ with bounded expansion and FO formula $\varphi(\bar{x})$, there exists a linear-time algorithm that, given a structure $\mathbb{A}$, computes $|\varphi(\mathbb{A})|$.

Problem 3. Prove that for every class structures $\mathcal{C}$ with bounded expansion and FO sentence φ, there exists an algorithm that given a structure a structure $\mathbb{A}$, computes a data structure for $\mathbb{A}$ implementing the following operations in constant time:

- **Query**: Does $\mathbb{A} \models \varphi$?
- **Update**: Add/remove a given element u to/from a given unary relation R.

Definition 0.1. For a graph G, a *connection model* for G with a label set Λ is a pair consisting of:

- a labelling λ of vertices of G with labels from Λ,
- a rooted tree T whose leaves are the vertices of G, and
- for every non-leaf node x of T, a symmetric set $Z_x \subseteq \Lambda^2$,

such that the following condition holds. For every pair of vertices $u, v \in V(G)$, we have

$$uv \in E(G) \Leftrightarrow (\lambda(u), \lambda(v)) \in Z_x, \qquad \text{where } x \text{ is the lowest common ancestor of } u \text{ and } v \text{ in } T.$$

The *shrubdepth* of G is the lowest number d such that G has a connection model of depth at most d over a label set of size d.

Problem 4. Prove that the following classes have bounded shrubdepth:

- cliques;
- complements of matchings;

while the following classes do not have bounded shrubdepth:

- paths;
- half-graphs.

Here, a *half-graph* is a bipartite graph on vertex set $\{u_1, \ldots, u_n\} \cup \{v_1, \ldots, v_n\}$ where u_i and v_j are adjacent if and only if $i \leq j$.

Definition 0.2. For a graph G and $d \in \mathbb{N}$, by G^d we denote a graph on the vertex set $V(G)$ where u and v are adjacent if and only if $\text{dist}_G(u, v) \leq d$. For a class $\mathcal{C}$, we denote $\mathcal{C}^d = \{G^d : G \in \mathcal{C}\}$.

Problem 5. Prove that if a class $\mathcal{C}$ has bounded treedepth, then for every fixed $d \in \mathbb{N}$ the class $\mathcal{C}^d$ has bounded shrubdepth.

Problem 6. Prove that if a class $\mathcal{C}$ has bounded expansion and $d \in \mathbb{N}$ is fixed, then the class $\mathcal{C}^d$ admits *low shrubdepth colorings* in the following sense: for every $p \in \mathbb{N}$ there exists N such that every graph $G \in \mathcal{C}^d$ has a coloring with N colors in which every subset of p colors induced a subgraph of shrubdepth at most $f(p)$, for some function $f : \mathbb{N} \to \mathbb{N}$.