Sparsity — tutorial 6

Dominating sets, independent sets, and neighborhood complexity

Problem 1. Prove that for every graph G and integer $d \in \mathbb{N}$, it holds that $\mathrm{dom}_{2d}(G) \leq \mathrm{ind}_d(G)$.

Problem 2. Let $\mathcal{C}$ be a class of bounded expansion and let $d \in \mathbb{N}$. Give a two-line proof that there exists a constant $c \in \mathbb{N}$ such that every distance-$2d$ independent set I in a graph $G \in \mathcal{C}$ contains a distance-$(2d+1)$ independent set I' of size at least $|I|/c$. How would a similar argument work if one only assumed that $\mathcal{C}$ is nowhere dense?

Problem 3. Suppose G is a graph and σ is a vertex ordering of G of degeneracy at most d. For a vertex u, let $N^+[u]$ denote the set consisting of u and all its neighbors that are smaller in σ. Consider the following algorithm:

- Let H be a graph with the same vertex set as G, where we consider a pair of vertices u and v adjacent if and only if the set $N^+[u] \cap N^+[v]$ is not empty.
- Let I be an inclusion-wise maximal independent set in H.
- Let $D = \bigcup_{u \in I} N^+[u]$.

Prove that D is a dominating set in G that satisfies $|D| \leq (d+1)^2 \cdot \mathrm{dom}(G)$.

Problem 4. Let $\mathcal{C}$ be a somewhere dense graph class that is closed under taking subgraphs. Prove that there exists $r \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ there exists a graph $G \in \mathcal{C}$ and a vertex subset $A \subseteq V(G)$ of size n with the following property: for each subset $B \subseteq A$ there exists some vertex $u \in V(G)$ such that $B = N^r_G[u] \cap A$.

Problem 5. Prove that if $\mathcal{C}$ is the class of d-degenerate graphs, where $d \in \mathbb{N}$ is fixed, then for every graph $G \in \mathcal{C}$ and nonempty vertex $A \subseteq V(G)$ we have

$$|\{N(u) \cap A \colon u \in V(G)| \leq \mathcal{O}(|A|^d).$$

Moreover, prove that the degree of the polynomial on the right hand side cannot be lower than d.

Problem 6. Suppose H is a bipartite graph with bipartition (X, Y) which satisfies the following conditions: vertices of Y have pairwise different neighborhoods in X, and every vertex of Y has at least one neighbor in X. Prove that there exists a mapping $\phi \colon Y \to X$ with the following properties:

- for each $y \in Y$, we have $\phi(y)y \in E(H)$; and
- for each $x \in X$, we have $|\phi^{-1}(x)| \leq \mathrm{wcol}_2(G) + 2^{\mathrm{wcol}_2(G)}$.

Definition 0.1. Fix $r \in \mathbb{N}$, a graph G, and a vertex subset $A \subseteq V(G)$. Let $u \in V(G) - A$ and $a \in A$. A path P connecting u and a is called A-*avoiding* if all its vertices apart from a do not belong to A. The *distance-r projection profile* of a vertex $u \in V(G) - A$ on A is the function $\mathrm{projprofile}_r[u, A] \colon A \to \{1, \ldots, r, \infty\}$ defined as follows: for $a \in A$, the value $\mathrm{projprofile}_r[u, A](a)$ is the length of a shortest A-avoiding path connecting u and a, or ∞ if this length is larger than r. A function $f \colon A \to \{1, \ldots, r, \infty\}$ is *realized* as a distance-r projection profile on A if there exists $u \in V(G) - A$ such that $f = \mathrm{projprofile}_r[u, A]$.

Problem 7. Prove that for every $r \in \mathbb{N}$, graph G, subset of its vertices $A \subseteq V(G)$, and $u, v \in V(G) - A$, if u and v have the same distance-r projection profile on A then they also have the same distance-r profile on A.

Problem 8. Prove that for every $r \in \mathbb{N}$ and class $\mathcal{C}$ of bounded expansion there exists a constant c, depending only on $\mathcal{C}$ and r, such that for every $G \in \mathcal{C}$ and nonempty $A \subseteq V(G)$, the number of different functions from A to $\{1, \ldots, r, \infty\}$ realized as distance-r projection profiles on A is at most $c \cdot |A|$.