Sparsity — tutorial 5

Generalized coloring numbers

Definition 0.1. For a graph G, let $\Phi(G)$ be the set of all graphs H that can be obtained as follows:

- Compute any vertex ordering σ of G with optimum degeneracy.
- Let $\vec{G}$ be the orientation of G according to σ, i.e., every edge is oriented towards the endpoint that is smaller in σ.
- Let H be the supergraph of G obtained by adding all edges uv such that in $\vec{G}$ there either exist arcs (u, w) and (w, v) for some vertex w, or arcs (w, u) and (w, v) for some vertex w.

For a class $\mathcal{C}$, we denote $\Phi(\mathcal{C}) = \bigcup_{G \in \mathcal{C}} \Phi(G)$. By Φ^d we denote the d-fold composition of the operator Φ.

Problem 1. Prove that if $\mathcal{C}$ is a class of bounded expansion, then $\Phi(\mathcal{C})$ is also a class of bounded expansion.

Problem 2. Prove that if G is a graph and u, v are vertices of G such that $2 < \mathrm{dist}_G(u, v) < +\infty$, then for any $H \in \Phi(G)$ we have $\mathrm{dist}_H(u, v) < \mathrm{dist}_G(u, v)$.

Problem 3. Let $\mathcal{C}$ be a class of bounded expansion and let $d \in \mathbb{N}$. Prove that there exists a constant $c \in \mathbb{N}$ such that for every graph G and graph $H \in \Phi^d(G)$, if σ is an optimum degeneracy ordering of H, then $\mathrm{wcol}_d(G, \sigma) \leq c$.

Problem 4. For a fixed class $\mathcal{C}$ of bounded expansion and $d \in \mathbb{N}$, give a linear-time algorithm that given a graph $G \in \mathcal{C}$ computes a vertex ordering σ of G that satisfies $\mathrm{wcol}_d(G, \sigma) \leq c$, for some constant c depending only on $\mathcal{C}$ and d.

Problem 5. For all $n \in \mathbb{N}$, construct a graph G_n with $\mathrm{ind}_2(G) = 1$ and $\mathrm{dom}_1(G) \geq n$.

Problem 6. Prove that for every graph G and integer $d \in \mathbb{N}$, it holds that $\mathrm{dom}_{2d}(G) \leq \mathrm{ind}_d(G)$.

Problem 7. Let $\mathcal{C}$ be a class of bounded expansion and let $d \in \mathbb{N}$. Give a two-line proof that there exists a constant $c \in \mathbb{N}$ such that every distance-$2d$ independent set I in a graph $G \in \mathcal{C}$ contains a distance-$(2d + 1)$ independent set I' of size at least $|I|/c$. How would a similar argument work if one only assumed that $\mathcal{C}$ is nowhere dense?

Problem 8. Let $d \in \mathbb{N}$, let G be a graph, and let σ be a vertex ordering of G. Consider the following algorithm. Every vertex $u \in V(G)$ picks $v(u)$ to be the smallest vertex of $\mathrm{WReach}_d[G, \sigma, u]$ in the ordering σ. Then, define D as the set of those vertices that have been picked by any vertex:

$$D \coloneqq \{v(u) \colon u \in V(G)\}.$$

Prove that D is a distance-d dominating set of G that moreover satisfies $|D| \leq \mathrm{wcol}_{2d}(G, \sigma) \cdot \mathrm{dom}_d(G)$.

Problem 9. Suppose G is a graph and σ is a vertex ordering of G of degeneracy at most d. For a vertex u, let $N^+[u]$ denote the set consisting of u and all its neighbors that are smaller in σ. Consider the following algorithm:

- Let H be a graph with the same vertex set as G, where we consider a pair of vertices u and v adjacent if and only if the set $N^+[u] \cap N^+[v]$ is not empty.
- Let I be an inclusion-wise maximal independent set in H.
- Let $D = \bigcup_{u \in I} N^+[u]$.

Prove that D is a dominating set in G that satisfies $|D| \leq (d + 1)^2 \cdot \mathrm{dom}(G)$.