Sparsity — tutorial 3

Bounded expansion and nowhere denseness continued

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Problem 1. Prove that if G is a graph with $\nabla_0(G) \leq k$ and A is a subset of its vertices, then there exists a vertex subset $B \supseteq A$ such that $|B| \leq 2|A|$ and every vertex of $V(G) - B$ has at most $2k$ neighbors in B.

Problem 2. Suppose $\mathcal{C}$ is a class of bounded expansion. Prove that for every $d \in \mathbb{N}$ there exists a constant c_d such that the following holds. For every graph $G \in \mathcal{C}$ and every subset A of its vertices, there exists a vertex subset $B \supseteq A$ such that $|B| \leq c_d|A|$ and for every vertex $u \in V(G) - B$, at most c_d vertices of B can be reached from u by a path of length at most d whose internal vertices do not belong to B.

Problem 3. Suppose $\mathcal{C}$ is a class of bounded expansion. Prove that for every $r \in \mathbb{N}$ there exists a constant c_r such that the following holds. For every graph $G \in \mathcal{C}$ and every its vertex subset $A \subseteq V(G)$, there exists a vertex subset $B \supseteq A$ with the following properties:

- $|B| \leq c_r|A|$, and
- for every pair of vertices $u, v \in A$, if $\mathrm{dist}_G(u,v) \leq r$ then $\mathrm{dist}_{G[B]}(u,v) = \mathrm{dist}_G(u,v)$.

Fact 1. If G is a graph and $r, c \in \mathbb{N}$, then

$$\nabla_r(G \bullet K_c) \leq 2c^2(r+1)^2\nabla_r(G) + c.$$

Problem 4. Prove that if a class $\mathcal{C}$ is nowhere dense, then for every constant $c \in \mathbb{N}$ the class $\mathcal{C} \bullet K_c$ is also nowhere dense.

Problem 5. Prove that every clique can be obtained as a congestion-2 minor in a planar graph.

Problem 6. Let G be a graph and let k be minimal such that G is a congestion-k minor of some tree. Show that $\mathsf{tw}(G)/2 \leq k \leq \mathsf{tw}(G)$.

Problem 7. Let G be a graph.

- Suppose that G has $2m$ edges. Show that there exists a bipartite subgraph $H \subseteq G$ of G with m edges.
- Suppose G has $n > 10^9$ vertices. Use the probabilistic method to prove that there exists a subgraph $H \subseteq G$ of G with $m/4$ edges and vertex partition $V(H) = A \cup B$ such that $||A| - n/2| \leq n/100$.
- Assume G has $2n$ vertices and $2m$ edges. Show that there exists a bipartite subgraph $H \subseteq G$ of G with m edges and vertex partition $V(H) = A \cup B$ such that $|A| = |B|$.