Sparsity — tutorial 14

Polynomial expansion and approximation schemes

Problem 1. A graph class has *bounded doubling dimension* if there exists a number $M \in \mathbb{N}$ such that the following holds: for every graph $G \in \mathcal{C}$ and every integer $r \geq 1$, every ball of radius r in G can be covered by at most M balls of radius $\lfloor r/2 \rfloor$. Prove that every class with bounded doubling dimension has polynomial expansion.

Note: A ball of radius r in a graph G is a set of vertices of the form $\{u\colon \mathrm{dist}_G(u,v) \leq r\}$ for some $v \in V(G)$. Note that balls of radius 0 are single vertices.

Problem 2. Consider the λ-local search algorithm for the r-Independent Set problem: start with an arbitrary solution and as long as there is an improvement step consisting of removing and adding at most λ vertices to the current solution, apply it.

Prove that for every $r \in \mathbb{N}$, $\epsilon > 0$, and class $\mathcal{C}$ of polynomial expansion there exists a constant $\lambda \in \mathbb{N}$, depending only on r, ϵ, and $\mathcal{C}$, such that λ-local search yields a $(1-\epsilon)$-approximation algorithm for r-Independent Set on graphs from $\mathcal{C}$. More precisely, λ-local search applied on any graph $G \in \mathcal{C}$ computes an r-independent set of size at least $1-\epsilon$ times the largest size of an r-independent set in G.

Problem 3. Fix a dimension $d \geq 1$. For a family $\mathcal{F}$ of (bounded, closed) subsets of $\mathbb{R}^d$, the intersection graph $G(\mathcal{F})$ has $\mathcal{F}$ as the vertex set and two elements $A, B \in \mathcal{F}$ are adjacent if they intersect. A family $\mathcal{F}$ is of *density* ρ if for every bounded closed $A \subseteq \mathbb{R}^d$ there are at most ρ elements $B \in \mathcal{F}$ with $A \cap B \neq \emptyset$ and $\mathrm{diam}(A) \leq \mathrm{diam}(B)$. Prove that, for fixed d and ρ, the class of intersection graphs of subsets of $\mathbb{R}^d$ of density ρ has polynomial expansion.

Problem 4. Consider the (c,r)-Dominating Set problem, which is a variant of standard r-Dominating Set where every vertex has to be r-dominated by c different elements of the solution. Let $\mathcal{C}$ be a class of polynomial expansion and $c, r \in \mathbb{N}$ be fixed. Prove that for each $\varepsilon > 0$ there exists $\lambda \in \mathbb{N}$, depending only on $\mathcal{C}, c, r, \epsilon$, such that λ-local search yields a PTAS on $\mathcal{C}$ for (c,r)-Dominating Set.

Problem 5. A graph class $\mathcal{C}$ is *weakly hyperfinite* if for every $\epsilon > 0$ there exists $C \in \mathbb{N}$ such that from every n-vertex graph $G \in \mathcal{C}$ one may delete at most ϵn vertices so that each connected component of the remaining graph has size at most C. Prove that every graph class of polynomial expansion is weakly hyperfinite.

Problem 6. Using a black-box result from previous lectures, prove that given an r-dominating set L, the existence of a λ-close r-dominating set of size smaller than L can be decided in time $f(\lambda) \cdot n$, for some computable function f. Conclude that on any class of polynomial expansion $\mathcal{C}$ and any $\varepsilon > 0$, there is a $(1+\varepsilon)$-approximation algorithm for r-Dominating Set on $\mathcal{C}$ with running time $f(\lambda) \cdot n^2$.