Sparsity — tutorial 12

VC dimension, neighborhood complexity, and approximation

neighborhood complexity in nowhere dense graph classes

Problem 1. Let $\mathcal{C}$ be a nowhere dense class of graphs and let $G \in \mathcal{C}$. For $r \in \mathbb{N}$ and a vertex ordering σ of G, let

$$\mathcal{W}_{r,\sigma} = \{\mathrm{WReach}_r[G, \sigma, v] \;:\; v \in V(G)\}.$$

Prove that there exists a constant k, depending only on $\mathcal{C}$ and r (and not on G and σ), such that $\mathcal{W}_{r,\sigma}$ has VC dimension at most k.
Hint: use uniform quasi-wideness. How many elements of a $2r$-scattered set may a WReach_r set contain?

Problem 2. Let $r \in \mathbb{N}$, $\varepsilon > 0$, and $\mathcal{C}$ be a nowhere dense class of graphs. Take any graph $G \in \mathcal{C}$, say with n vertices, and let σ be a vertex ordering of G with $\mathrm{wcol}_{2r}(G, \sigma) \leq n^{\varepsilon}$. For a set $A \subseteq V(G)$, let

$$B = \bigcup_{a \in A} \mathrm{WReach}_r[G, \sigma, a].$$

For a vertex $u \in V(G)$, let

$$X[u] = B \cap \mathrm{WReach}_r[G, \sigma, u].$$

Prove that

$$|\{X[u] \colon u \in V(G)\}| \leq |A| \cdot n^{k\varepsilon}$$

for some constant k depending only on $\mathcal{C}$ and r.

Problem 3. Discuss how the solution of the above problem can be used to prove that for a fixed nowhere dense class $\mathcal{C}$ and $r \in \mathbb{N}$, the distance-r neighborhood complexity in graphs from $\mathcal{C}$ is of the order $\mathcal{O}(|A|^{1+\varepsilon})$.

approximation

Problem 4. Let G be a graph. Consider the following linear program for fractional dominating sets.

<u>Dominating Set LP</u>

- **Variables:** x_v for all $v \in V(G)$
- **Objective:** minimize $\sum_{v \in V(G)} x_v$
- **Constraints:**
  - $\sum_{w \in N[v]} x_w \geq 1$ for all $v \in V(G)$; and
  - $x_v \geq 0$ for all $v \in V(G)$.

Assume G is d-degenerate. Let $D_1 = \{v \in V(G) \;:\; x_v \geq 1/(3d)\}$ and $D_2 = V(G) \setminus N[D_1]$, that is, D_1 consists of all vertices which get weight at least $1/(3d)$ and D_2 are the vertices that are not dominated by D_1. Clearly, $D_1 \cup D_2$ is a dominating set of G.

Prove that $|D_1 \cup D_2| \leq 3dk$, where k is the size of a minimum dominating set of G.

Problem 5. Let $\mathcal{F}$ be a set family on ground set A. A *set cover* for A with sets from $\mathcal{F}$ is a subset $\mathcal{G} \subseteq \mathcal{F}$ such that $\bigcup_{F \in \mathcal{G}} F = A$. Prove that there exists a randomized polynomial-time algorithm which given a set system $\mathcal{F}$ of VC dimension $d \geq 2$, computes a set cover for $\mathcal{F}$ that has size at most $\mathcal{O}(2^d \cdot \tau(\mathcal{F}) \ln \tau(\mathcal{F}))$ with probability at least $\frac{1}{2}$.