Sparsity — tutorial 11

VC dimension and friends

Problem 1. For a set system $\mathcal{S} = (U, \mathcal{F})$, we can consider the *dual set system* $\mathcal{S}^\star = (\mathcal{F}, U^\star)$, where the ground set consists of the sets from $\mathcal{F}$, while the family $U^\star \subseteq 2^{\mathcal{F}}$ consists of sets

$$\mathcal{F}_e\colon \{F \in \mathcal{F} \colon e \in F\} \qquad \text{for each element } e \in U.$$

Prove that if $\mathcal{S}$ has VC dimension d, then $\mathcal{S}^\star$ has VC dimension at most $2^{d+1} - 1$.

Problem 2. Determine the VC dimension of the following set systems:

- $U = \mathbb{R}^2$, $\mathcal{F}$ = lines in $\mathbb{R}^2$;
- $U = \mathbb{R}^2$, $\mathcal{F}$ = axis-aligned rectangles in $\mathbb{R}^2$;
- $U = \mathbb{R}^2$, $\mathcal{F}$ = convex polygons in $\mathbb{R}^2$;
- $U = \mathbb{R}^d$, $\mathcal{F}$ = half-spaces in $\mathbb{R}^d$.

Problem 3. A graph G is a *map graph* if with every vertex u of G we may associate a polygon R_u in the plane so that the interiors of regions R_u are pairwise disjoint and two vertices $u, v \in V(G)$ are adjacent in G if and only if the corresponding regions R_u and R_v share a point on their boundaries. Prove that the class of map graphs has bounded semi-ladder index, but is not nowhere dense.

In the solution of the next problem you may use the following fact.

Fact 1. Let G be an n-vertex graph such that $K_t \not\preccurlyeq_1 G$ for some constant $t \in \mathbb{N}$. Then G has at most $\binom{t+1}{2} \cdot n^{3/2}$ edges.

Problem 4. Let G be a graph such that $K_t \not\preccurlyeq_4 G$ for some $t \in \mathbb{N}$ and let $A \subseteq V(G)$. Suppose that every pair of elements of A has a common neighbor in $V(G) - A$. Prove that there exists a vertex v in $V(G) - A$ which has $\Omega_t(|A|^{1/4})$ neighbors in A.

In the next problems we will prove the following fact.

Fact 2. Let G be a bipartite graph with sides of the bipartition A and B. Let $m, t, d \in \mathbb{N}$ and suppose $|A| \geq (m + d)^{2t}$. Then at least one of the following assertions hold:

(a) A contains a distance-2 independent set of size m;

(b) $K_t \preccurlyeq_4 G$;

(c) B contains a vertex of degree at least $\Omega_t(d^{1/4})$.

For the proof of Fact 2, construct a binary tree labelled with all the elements of A, by inserting them one by one and using the following question when comparing the inserted vertex $v \in A$ with a vertex $u \in A$ already residing in the tree: do u and v have a common neighbor? If the answer is positive, proceed to the son of the current node, otherwise proceed to the daughter of the current node. Let τ be the obtained tree.

Problem 5. Prove that if τ has a node with m letters D, then A contains a distance-2 independent set of size m.

Problem 6. Prove that if τ has a node with d letters S, then either $K_t \preccurlyeq_4 G$ or B contains a vertex of degree at least $\Omega_t(d^{1/4})$.

Problem 7. Prove that if τ has a node with alternation $2t$ then $K_t \preccurlyeq_2 G$.

Problem 8. Prove Fact 2 and discuss how it can be used to show that the function $N_r(m)$ in uniform quasi-wideness is actually a polynomial in m (with degree depending on the class and r).

Problem 9. Let $\mathcal{C}$ be a nowhere dense class of graphs and let $G \in \mathcal{C}$. For $r \in \mathbb{N}$ and a vertex ordering σ of G, let

$$\mathcal{W}_{r,\sigma} = \{\mathrm{WReach}_r[G, \sigma, v] \ : \ v \in V(G)\}.$$

Prove that there exists a constant k, depending only on $\mathcal{C}$ and r (and not on G and σ), such that $\mathcal{W}_{r,\sigma}$ has VC dimension at most k.

Problem 10. Let $r \in \mathbb{N}$, $\varepsilon > 0$, and $\mathcal{C}$ be a nowhere dense class of graphs. Take any graph $G \in \mathcal{C}$, say with n vertices, and let σ be a vertex ordering of G with $\mathrm{wcol}_{2r}(G, \sigma) \leq n^{\varepsilon}$. For a set $A \subseteq V(G)$, let

$$B = \bigcup_{a \in A} \mathrm{WReach}_r[G, \sigma, a].$$

For a vertex $u \in V(G)$, let

$$X[u] = B \cap \mathrm{WReach}_r[G, \sigma, u].$$

Prove that

$$|\{X[u] \colon u \in V(G)\}| \leq |A| \cdot n^{k\varepsilon}$$

for some constant k depending only on $\mathcal{C}$ and k.

Problem 11. Discuss how the solution of the above problem can be used to prove that for a fixed nowhere dense class $\mathcal{C}$ and $r \in \mathbb{N}$, the distance-r neighborhood complexity in graphs from $\mathcal{C}$ is of the order $\mathcal{O}(|A|^{1+\varepsilon})$.