Sparsity — tutorial 1

Planar graphs and beyond

October 4th, 2019

Problem 1. Given a graph G embedded in a torus, show that there is a vertex $v \in V(G)$ of degree ≤ 6.

Problem 2. Show that if we can draw a 6-regular graph on a torus, it is triangularized after drawing.

Problem 3. Find maximum integer t such that K_t is embeddable on a torus.

Problem 4. Show that a planar bipartite graph has a vertex of degree at most 3.

Problem 5. The *arboricity* of a graph G, denoted $\mathrm{arb}(G)$, is the minimum number k such that the edge set of G can be partitioned into k sets, each inducing a forest. Prove that

$$\mathrm{arb}(G) \leq \mathsf{deg}(G) \leq 2 \cdot \mathrm{arb}(G) - 1.$$

Problem 6. Let G be a graph such that every subgraph of G has average degree strictly less than 3.

1. Show that G has either a vertex of degree at most 1 or a vertex of degree 2 with a neighbor of degree at most 5.
2. Show that there exists a coloring f of $V(G)$ with six colors such that:
   - for every color i, $f^{-1}(i)$ is an independent set in G (i.e., f is a proper coloring of G), and
   - for every two colors i, j, $f^{-1}(\{i, j\})$ is a forest.

Problem 7. Prove that for every n-vertex graph G and an integer c, G contains at most n/c vertices of degree at least $2c \cdot \mathsf{deg}(G)$.

Problem 8. Prove that for every graph G and a set $A \subseteq V(G)$, there exists a set $A \subseteq B \subseteq V(G)$ of size at most $2|A|$ such that every $v \in V(G) \setminus B$ has at most $2 \cdot \mathsf{deg}(G)$ neighbors in B.

Problem 9. Prove that every planar graph has either a vertex of degree at most 2 or two adjacent vertices whose sum of degrees is at most 13.

Problem 10. Show that for every planar graph G there exists an integer k such that G is a minor of a $k \times k$ grid.

Problem 11. Prove that if H is a minor of G, then $\mathsf{tw}(H) \leq \mathsf{tw}(G)$ and $\mathsf{td}(H) \leq \mathsf{td}(G)$.

Problem 12. Show that G is a forest if and only if its treewidth is at most 1.

Problem 13. Prove that the treedepth of an n-vertex tree is at most $1 + \log_2(n)$.

Problem 14. Compute the treedepth of an n-vertex path.

Problem 15. Show a tree decomposition of width 2 of an n-vertex cycle.

Problem 16. What is the treewidth of the clique K_n?

Problem 17. A graph is *outerplanar* if it can be drawn in the sphere such that there is a face incident with every vertex of the graph. Prove that an outerplanar graph has treewidth at most 2.

Problem 18. Show that the treewidth of a $k \times k$ grid is at most k.

Problem 19. Show that if a graph G does not contain a k-vertex path as a subgraph, then the treedepth of G is less than k.

Problem 20. Show that for every graph G it holds that

$$\mathsf{td}(G) \leq (\mathsf{tw}(G) + 1) \cdot \log_2(1 + |V(G)|).$$

Problem 21. Show that if T is a tree, $\beta : V(T) \to 2^{V(G)}$ is a function such that $\bigcup_{t \in V(T)} \beta(t) = V(G)$ and for every $t_1 t_2 \in E(T)$, if T_i is the connected component of $T - \{t_1 t_2\}$ containing t_i, then $\beta(t_1) \cap \beta(t_2)$ is a separator between $\bigcup_{t \in T_1} \beta(t)$ and $\bigcup_{t \in T_2} \beta(t)$, then (T, β) is a tree decomposition of G.

Problem 22. Let $W \subseteq V(G)$ and let (T, β) be a tree decomposition of G. Show that either there exists $t \in V(T)$ with $W \subseteq \beta(t)$ or two vertices $w_1, w_2 \in W$ and an edge $t_1 t_2 \in E(T)$ such that $\beta(t_1) \cap \beta(t_2)$ separates w_1 from w_2.