Selected solutions from tutorials

Problem 1 (Problem 2, Tutorial 3)**.** Suppose $\mathcal{C}$ is a class of bounded expansion. Prove that for every $r \in \mathbb{N}$ there exists a constant c_r such that the following holds. For every graph $G \in \mathcal{C}$ and every subset A of its vertices, there exists a vertex subset $B \supseteq A$ such that $|B| \leq c_r|A|$ and for every vertex $u \in V(G) - B$, at most c_r vertices of B can be reached from u by a path of length at most r whose internal vertices do not belong to B.

Solution. We first introduce some convenient notation. For any graph H, any set $X \subseteq V(H)$ and any $u \in V(H) - X$, the set of vertices of X which can be reached from u by a path of length at most r whose internal vertices do not belong to X will be called the *distance-r projection of u onto X in H*, and denoted by $\Pi^H_r(u, X)$. Thus, we need to prove that there is some superset $B \supseteq A$ whose size is bounded linearly in $|A|$, and such that r-projections onto B have bounded sizes.

Let us fix the constant $\xi = \lceil 2\nabla_{r-1}(\mathcal{C})\rceil$. We consider the following iterative procedure.

1. Start with $H = G$ and $Y = A$. We will maintain the invariant that $Y \subseteq V(H)$.
2. As long as there exists a vertex $u \in V(H) - Y$ with $|\Pi^H_r(u, Y)| \geq \xi$ do the following:
   - Select an arbitrary subset $Z_u \subseteq \Pi^H_r(u, Y)$ of size exactly ξ.
   - For each $w \in Z_u$, select a path P_w that starts at u, ends at w, has length at most r, and all its internal vertices are in $V(H) - Y$.
   - Modify H by contracting $\bigcup_{w\in Z_u}(V(P_w) - \{w\})$ onto u, and add the obtained vertex to Y.

Observe that in a round of the procedure above we always make a contraction of a connected subgraph of $H - Y$ of radius at most $r - 1$. Also, the resulting vertex falls into Y and hence does not participate in future contractions. Thus, at each point H is an $(r - 1)$-shallow minor of G. For any moment of the procedure and any $u \in V(H)$, by $\tau(u)$ we denote the subset of original vertices of G that were contracted onto u during earlier rounds. Note that either $\tau(u) = \{u\}$ when u is an original vertex of G, or $\tau(u)$ is a set of cardinality at most $1 + (r - 1)\xi$.

We claim that the presented procedure stops after at most $|A|$ rounds. Suppose otherwise, that we successfully constructed the graph H and subset Y after $|A| + 1$ rounds. Examine graph $H[Y]$. This graph has $2|A| + 1$ vertices: $|A|$ original vertices of A and $|A| + 1$ vertices that were added during the procedure. Whenever a vertex u is added to Y after contraction, then it introduces at least ξ new edges to $H[Y]$: these are edges that connect the contracted vertex with the vertices of Z_u. Hence, $H[Y]$ has at least $\xi(|A| + 1)$ edges, which means that

$$\frac{|E(H[Y])|}{|V(H[Y])|} \geq \frac{\xi(|A| + 1)}{2|A| + 1} > \nabla_{r-1}(\mathcal{C}).$$

This is a contradiction with the fact that H is an $(r - 1)$-shallow minor of G.

Therefore, the procedure stops after at most $|A|$ rounds producing (H, Y), where $|\Pi^H_r(u, Y)| < \xi$ for each $u \in V(H) - Y$. Define $B = \tau(Y) = \bigcup_{u\in Y}\tau(u)$. Obviously, we have $A \subseteq B$. Since $|\tau(u)| = 1$ for each original vertex $u \in A$ and $|\tau(u)| \leq 1 + (r - 1)\xi$ for each u that was added during the procedure, we have $|B| \leq ((r - 1)\xi + 2) \cdot |A|$. We are left with proving that distance-r projections are small.

By construction, we have $V(H) - Y = V(G) - B$. Take any $u \in V(H) - Y$ and observe that $\Pi^G_r(u, B) \subseteq \tau(\Pi^H_r(u, Y))$. Since $|\Pi^H_r(u, Y)| < \xi$ for each $u \in V(H) - Y$ and $|\tau(u)| \leq 1 + (r - 1)\xi$ for each $u \in V(H)$, we have $|\Pi^G_r(u, B)| \leq \xi(1 + (r - 1)\xi)$. Hence, we may conclude by defining $c_r = \xi((r - 1)\xi + 2)$. □

Problem 2 (Problem 3, Tutorial 3)**.** Suppose $\mathcal{C}$ is a class of bounded expansion. Prove that for every $r \in \mathbb{N}$ there exists a constant d_r such that the following holds. For every graph $G \in \mathcal{C}$ and every its vertex subset $A \subseteq V(G)$, there exists a vertex subset $B \supseteq A$ with the following properties:

- $|B| \leq d_r|A|$, and
- for every pair of vertices $u, v \in A$, if $\mathrm{dist}_G(u, v) \leq r$ then $\mathrm{dist}_{G[B]}(u, v) = \mathrm{dist}_G(u, v)$.

Solution. First, we apply the result of Problem 1 to the set A, yielding a set A' with the asserted properties: $|A'| \leq c_r|A|$ and for every vertex $u \in V(G) - A'$, at most c_r vertices of A' can be reached from u by a path of length at most r whose internal vertices do not belong to A'. Next, for each pair of distinct vertices $u, v \in A'$, select an arbitrary path $P_{u,v}$ that connects u and v, and whose internal vertices do not belong to A', and which is the shortest among the paths satisfying these properties; in case there is no such path, put $P_{u,v} = \emptyset$. Then define B to be A' plus the vertex sets of all paths $P_{u,v}$ that have length at most r.

We first claim that B has indeed the required property of preserving distances up to r. More precisely, take any distinct $u, v \in A$ with $\mathrm{dist}_G(u, v) \leq r$. Let R be a shortest path between u and v in G, and let $a_1, a_2, \ldots, a_q$ be consecutive vertices of A' visited on R, where $u = a_1$ and $v = a_q$. For each $i = 1, 2, \ldots, q-1$, let R_i be the segment of R between a_i and a_{i+1}. Then the existence of R_i certifies that some path of length at most $|R_i|$ between a_i and a_{i+1} was added when constructing B from A', and hence $\mathrm{dist}_{G[B]}(a_i, a_{i+1}) \leq |R_i|$. Consequently, by the triangle inequality we infer that

$$\mathrm{dist}_{G[B]}(u, v) \leq \sum_{i=1}^{q-1} \mathrm{dist}_{G[B]}(a_i, a_{i+1}) \leq \sum_{i=1}^{q-1} |R_i| = |R| = \mathrm{dist}_G(u, v).$$

However, the opposite inequality $\mathrm{dist}_{G[B]}(u, v) \geq \mathrm{dist}_G(u, v)$ follows directly from the fact that $G[B]$ is an induced subgraph of G. Hence indeed $\mathrm{dist}_G(u, v) = \mathrm{dist}_{G[B]}(u, v)$.

We are left with showing that $|B| \leq d_r|A|$ for some constant d_r. First, we have $|A'| \leq c_r|A|$, so we only need to upper bound the ratio $\frac{|B|}{|A'|}$. Let H be a graph on vertex set A', where $uv \in E(A')$ if and only if $P_{u,v}$ exists and has length at most r, and hence its vertex set was added in the costruction of B. Clearly $|B| \leq |A'| + (r-1) \cdot |E(H)|$, so it suffices to prove an upper bound on $|E(H)|$.

Take any $w \in B - A'$, and consider for how many pairs $\{u, v\}$ it can hold that $w \in P_{u,v}$. If $\{u, v\}$ is such a pair, then in particular both u and v can be reached from w by a path of length at most r that internally avoids A'. However, we know that the number of such vertices is at most c_r, so the number of such pairs $\{u, v\}$ is at most $\tau = \binom{c_r}{2}$. Consequently, we observe that graph H is a depth-$(r-1)$ congestion-τ minor of G: we can realize all the paths $P_{u,v}$ in G so that every vertex of $B - A'$ is used at most τ times. Now we know by Lemma 2.27 of Chapter 1 of Lecture Notes that the edge density in depth-$(r-1)$ congestion-τ minors of G is bounded by a function of $\nabla_{r-1}(G)$ and τ. Both $\nabla_{r-1}(G)$ and τ are bounded by constants, namely by $\nabla_{r-1}(\mathcal{C})$ and $\binom{c_r}{2}$ respectively. Hence $|E(H)|$ is bounded by a constant times $|A'|$. Since $|B| \leq |A'| + (r-1)|E(H)|$ and $|A'| \leq c_r|A|$, we are done. ☐

For the next problem, see Tutorial 5 for missing definitions and previous problems.

Problem 3 (Problem 3, Tutorial 5)**.** Let $\mathcal{C}$ be a class of bounded expansion and let $d \in \mathbb{N}$. Prove that there exists a constant $c \in \mathbb{N}$ such that for every graph G and graph $H \in \Phi^d(G)$, if σ is an optimum degeneracy ordering of H, then $\mathrm{wcol}_d(G, \sigma) \leq c$.

Solution. By constructions, we have a sequence of graphs

$$G = G_0 \subseteq G_1 \subseteq \ldots \subseteq G_{d-1} \subseteq G_d = H$$

such that $G_{i+1} \in \Phi(G_i)$, for $i \in \{0, 1, \ldots, d-1\}$. By Problem 1 of Tutorial 5, each of the classes $\Phi^i(\mathcal{C})$ for $i \leq d$ has bounded expansion, so in particular it is k_i-degenerate for some constant k_i depending only on $\mathcal{C}$ and i. In particular, G_{d-1} is k_{d-1}-degenerate and G_d is k_d-degenerate.

Take any $u \in V(G)$. We are going to prove that

$$|\mathrm{WReach}_d[G, \sigma, u]| \leq 1 + k_d + k_d^2 + k_{d-1} \cdot k_d; \tag{1}$$

this will give a constant upper bound on $\mathrm{wcol}_d(G, \sigma)$. To this end, we consider any $v \in \mathrm{WReach}_d[G, \sigma, u]$ and give an upper bound on the number of possible vertices v, where different cases of the alignment of u and v give rise to different summands in the bound (1). The summand $+1$ in (1) corresponds to the case $u = v$, hence from now on we assume that $u \neq v$.

Since $v \in \mathrm{WReach}_d[G, \sigma, u]$, in G there is a path P from u to v of length at most d, whose all vertices are not smaller in σ than v. In particular we have $\mathrm{dist}_G(u, v) \leq d$. By Problem 2 of Tutorial 5 we infer that $\mathrm{dist}_{G_{d-1}}(u, v) \leq 2$. We now consider various cases.

First assume that $\mathrm{dist}_{G_d}(u, v) = 1$, that is, u and v became adjacent in G_d. Hence v is a neighbor of u in G_d that is smaller in the order σ. But there can be only k_d such neighbors — they correspond to the summand k_d in (1).

Hence, from now assume that $\mathrm{dist}_{G_d}(u, v) = 2$, so also $\mathrm{dist}_{G_{d-1}}(u, v) = 2$, which means that u and v have a common neighbor w in G_d. Since in Problem 2 of Tutorial 5 we argued that every path of length larger than 2 gets shortcutted in each iteration of Φ, we may assume that w lies on P. In particular $v <_\sigma w$.

The next case is when $v <_\sigma w <_\sigma u$. Then in G_d we have that w is a neighbor of u smaller in σ, and v is a neighbor of w smaller in σ. Hence, there are at most k_d options for choosing w based on u, and at most k_d options for choosing v based on w, resulting in k_d^2 options in total. So this case corresponds to the summand k_d^2 in the bound (1).

Finally, we are left with the case when $u <_\sigma w$. Now comes the crucial observation. Let $\vec{G}_{d-1}$ be the orientation of G_{d-1} used to construct G_d; in particular, all outdegrees in $\vec{G}_{d-1}$ are bounded by k_{d-1}. Observe that the only case when the edge uv is not added in the construction of G_d (as we assume $\mathrm{dist}_{G_d}(u, v) = 2$) is when in $\vec{G}_{d-1}$, the edges uw and vw are both oriented towards w. But then w is an outneighbor of u in $\vec{G}_{d-1}$, and there can be only k_{d-1} outneighbors of u in $\vec{G}_{d-1}$. Similarly as before, v is a neighbor of w in G_d that is smaller in σ, so for fixed w there are at most k_d options for v. We conclude that this case gives rise to at most $k_{d-1} \cdot k_d$ possible vertices v, which gives the last summand in (1) and finishes the proof. $\square$