Introduction to Sparsity

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Abstract

Starting from an innocently looking question of what it means for a graph to be sparse, we will develop the basics of the structural theory of sparse graph classes: a relatively new and rapidly growing area of graph theory. We will explore different combinatorial aspects of sparsity, expressed via seemingly different, but actually equivalent definitions: density of shallow minors, generalized coloring numbers, low treedepth colorings, neighborhood complexity, and maybe others. An example result that we will be able to derive from our toolbox is the following: there is a constant c such that every planar graph can be colored with c colors so that every two vertices that are at distance exactly three from each other have different colors.

During this mini-course we will explore the theory of structural sparsity for graphs. This theory was initiated by Jaroslav Nešetřil and Patrice Ossona de Mendez around 2008 and since then it has been developed by many graph theorists from all around the globe. It turns out that fundamental results about sparsity somehow bring together three facets of the subject:

- **combinatorics**: studying different graph-theoretical viewpoints on sparsity and finding links between them;
- **algorithm design**: applying combinatorial tools to design efficient algorithms on sparse graphs, and ask combinatorial questions based on expected algorithmic applications;
- **logic and model theory**: use abstract notions of sparsity to establish boundaries of expressibility and tractability of variants of logical formalisms.

Here we will focus on the combinatorial aspect, but it may happen that some algorithms will be produced on the way.

There are freely available lecture notes that cover everything that will be presented during this mini-course, and much more: `https://www.mimuw.edu.pl/~mp248287/sparsity/` .

1. Degeneracy

For a graph G, by $V(G)$ and $E(G)$ we denote the vertex and edge set of G, respectively. A graph H is a *subgraph* of G, denoted $H \subseteq G$, if H can be obtained from G by deleting vertices and edges.

Definition 1.1. The *density* of a graph G is the ratio between the numbers of edges and vertices:

$$\mathsf{density}(G) = \frac{|E(G)|}{|V(G)|}.$$

1.1. Prove that the density of a graph G is equal to half of the average degree in G.

Definition 1.2. The *hereditary density* of a graph G is the largest density among its subgraphs:

$$\mathsf{density}^{\downarrow}(G) = \max_{H \subseteq G} \mathsf{density}(H).$$

Definition 1.3. A graph G is *d-degenerate* if every its subgraph contains a vertex of degree at most d. The *degeneracy* of G, denoted $\mathsf{degeneracy}(G)$, is the smallest d for which G is d-degenerate.

1.2. Prove that for every graph G we have

$$\mathsf{density}^{\downarrow}(G) \leq \mathsf{degeneracy}(G) \leq 2 \cdot \mathsf{density}^{\downarrow}(G).$$

1.3. Prove that a graph is 1-degenerate if and only if it is a forest (that is, an acyclic graph).

1.4. Prove that a graph G is d-degenerate if and only if there is a linear ordering σ of vertices of G such that every vertex has at most d neighbors that are smaller in σ.

Definition 1.4. The *arboricity* of a graph G, denoted $\mathsf{arboricity}(G)$, is the minimum number a such that the edges of G can be partitioned into a forests.

1.5. Prove that for every graph G we have

$$\mathsf{arboricity}(G) \leq \mathsf{degeneracy}(G) \leq 2 \cdot \mathsf{arboricity}(G).$$

1.6. Prove that planar graphs are 5-degenerate. Is the number 5 optimum? Is this also true for planar multigraphs?

1.7. Prove that every d-degenerate graph admits a proper coloring with $d+1$ colors. Here, a *proper coloring* of a graph is a coloring of its vertices where the endpoints of every edge receive different colors.

1.8. Prove that a d-degenerate graph on n vertices has at most $1+2^d \cdot n$ different cliques, where a clique is a set of pairwise adjacent vertices.

Definition 1.5. A set of vertices I in a graph G is *independent* if there are no edges with both endpoints in I. A set of vertices D is *dominating* if every vertex of G either belongs to D or has a neighbor in D. The size of a largest independent set in G is denoted by $\alpha(G)$, while the size of a smallest dominating set in G is denoted by $\gamma(G)$.

1.9. Prove that every d-degenerate graph on n vertices contains an independent set of size at least $\frac{n}{d+1}$.

1.10. Suppose G is a graph and σ is a linear ordering of G witnessing that G is d-degenerate, as in Problem 1.4. For a vertex u, let $N^+[u]$ denote the set consisting of u and all its neighbors that are smaller in σ. Consider the following algorithm:

- Let H be a graph with the same vertex set as G, where we consider a pair of vertices u and v adjacent if and only if the set $N^+[u] \cap N^+[v]$ is not empty.
- Let I be an inclusion-wise maximal independent set in H.
- Let $D = \bigcup_{u \in I} N^+[u]$.

Prove that D is a dominating set in G that satisfies $|D| \leq (d+1)^2 \cdot \gamma(G)$.

1.11. Prove that in a d-degenerate graph at least half of the vertices have degree at most $2d$.

1.12. For a graph G, the 1-*subdivision* of G is obtained by replacing every edge of G by a path of length 2, where the middle vertex has degree exactly 2. Prove that the 1-subdivision of any graph is 2-degenerate.

2. Shallow minors and notions of sparsity

Definition 2.1. A graph H is a *minor* of G, denoted $H \preceq G$, if there exists a *minor model* of H in G. The minor model consists of a connected subgraph $I_u \subseteq G$ for each vertex u of H, called the *branch set* of u, so that the following conditions are satisfied: branch sets I_u are pairwise vertex-disjoint and whenever uv is an edge in H, there is an edge between I_u and I_v in G.

2.1. Prove that H is a minor of G if and only if H can be obtained from G using the following operations: vertex removal, edge removal, and edge contraction.

2.2. Prove that planar graphs are minor-closed (i.e., every minor of a planar graph is also planar).

Theorem 2.2. A graph is planar if and only if it does not contain K_5 nor $K_{3,3}$ as a minor.

2.3. Prove that every graph that does not contain K_t as a minor has density smaller than 2^t.

Theorem 2.3. Every K_t-minor-free graph has density bounded by $\mathcal{O}(t\sqrt{\log t})$ and this bound is asymptotically tight.

Definition 2.4. A graph H is a *depth-d minor* of G, denoted $H \preceq_d G$, if there exists a minor model of H in G where every branch set has radius at most d. Here, a connected graph has radius at most d if it contains a vertex that is at distance at most d from any other vertex of the graph.

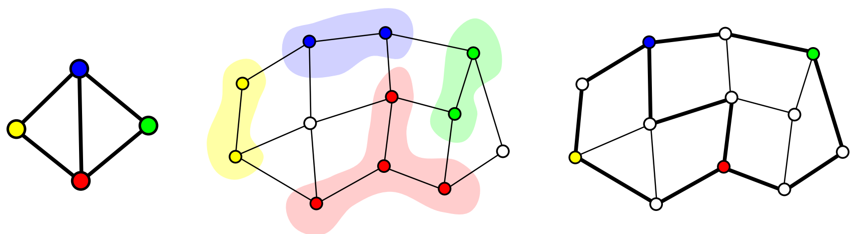

Figure 2.1: A diamond graph and its minor and topological minor models in a larger graph. Both models have depth 1.

2.4. Prove that if $J \preceq_a H$ and $H \preceq_b G$, then $J \preceq_{2ab+a+b} G$.

Definition 2.5. For a graph class $\mathcal{C}$, the *depth-d reduct* of $\mathcal{C}$ is the class

$$\mathcal{C} \triangledown d := \{H : H \preceq_d G \text{ for some } G \in \mathcal{C}\}.$$

The *greatest reduced average density* at depth d in $\mathcal{C}$ is the quantity

$$\nabla_d(\mathcal{C}) := \sup_{H \in \mathcal{C} \triangledown d} \mathsf{density}(H).$$

Definition 2.6. A class of graph $\mathcal{C}$ *has bounded expansion* if $\nabla_d(\mathcal{C}) < +\infty$ for all $d \in \mathbb{N}$. In other words, there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that $\mathsf{density}(H) \leq f(d)$ whenever H is a depth-d minor of a graph from $\mathcal{C}$.

Definition 2.7. A class of graph $\mathcal{C}$ is *nowhere dense* for every $d \in \mathbb{N}$, the class $\mathcal{C} \triangledown d$ does not contain all graphs. Equivalently, there exists a function $t : \mathbb{N} \to \mathbb{N}$ such that for every $d \in \mathbb{N}$, the clique $K_{t(d)}$ is not a depth-d minor of any graph from $\mathcal{C}$.

2.5. Prove that every proper (that is, not containing all graphs) minor-closed class of graphs has bounded expansion.

2.6. Prove that for any $\Delta \in \mathbb{N}$, the class of all graphs with maximum degree at most Δ has bounded expansion.

2.7. Prove that if a class of graphs $\mathcal{C}$ has bounded expansion, then for every $d \in \mathbb{N}$ the class $\mathcal{C} \nabla d$ also has bounded expansion. Prove the same implication for nowhere denseness as well.

2.8. Prove that if a class of graphs has bounded expansion then it is nowhere dense.

2.9. Prove that for every $t \in \mathbb{N}$ there exists a graph with density at least t and with no cycle of length at most t. Conclude that there exists a class of graphs that is nowhere dense but does not have bounded expansion.

Definition 2.8. A graph H is a *topological minor* of G, denoted $H \preceq^{\circ} G$, if there exists a *topological minor model* of H in G. The topological minor model η consists of a vertex $\eta(u)$ of G for each vertex u of H, and a path $\eta(e)$ in G for each edge e of H, so that the following conditions are satisfied: for each $uv \in E(H)$, the path $\eta(uv)$ has endpoints $\eta(u)$ and $\eta(v)$, and paths $\{\eta(e) \colon e \in E(H)\}$ are vertex-disjoint apart from sharing endpoints whenever necessary.

Further, H is a *depth-d topological minor* of G, denoted $H \preceq^{\circ}_d G$, if there exists a topological minor model of H in G where every path has length at most $2d+1$.

2.10. Prove that if $H \preceq^{\circ} G$ then $H \preceq G$, and if $H \preceq^{\circ}_d G$ then $H \preceq_d G$.

2.11. Give a class of graphs that excludes K_5 as a topological minor, but whose closure under taking minors contains all graphs.

2.12. We define the notion of *topologically nowhere denseness* in the same way as nowhere denseness, but we use shallow topological minors instead of minors. Prove that a class of graphs is nowhere dense if and only if it is topologically nowhere dense.

Theorem 2.9. A class of graphs has bounded expansion if and only if it has topologically bounded expansion.

3. Generalized coloring numbers: basics

Definition 3.1. Let G be a graph, let σ be a vertex ordering of G, and let $d \in \mathbb{N}$. For vertices $u, v \in V(G)$ with $u \leq_\sigma v$, we say that:

- u is *strongly d-reachable* from v if there is a path of length at most d from v to u whose every internal vertex w satisfies $v <_\sigma w$; and
- u is *weakly d-reachable* from v if there is a path of length at most d from v to u whose every internal vertex w satisfies $u <_\sigma w$.

For a vertex v, we denote

$$\begin{aligned} \mathrm{WReach}_d[G,\sigma,v] &:= \{\, u : u \leq_\sigma v \text{ and } u \text{ is weakly } d\text{-reachable from } v \,\}, \\ \mathrm{SReach}_d[G,\sigma,v] &:= \{\, u : u \leq_\sigma v \text{ and } u \text{ is strongly } d\text{-reachable from } v \,\}. \end{aligned}$$

Note that every vertex is both weakly and strongly d-reachable from itself.

Definition 3.2. Let G be a graph, σ be a vertex ordering of G, and $d \in \mathbb{N}$. The *d-admissibility* of a vertex v of G, denoted $\mathrm{adm}_d(G,\sigma,v)$, is equal to the maximum size of a family of paths $\mathcal{P}$ with the following properties:

- every path from $\mathcal{P}$ has length at most d and leads from v to a vertex that is smaller in σ;
- paths from $\mathcal{P}$ are pairwise vertex-disjoint apart from sharing the endpoint v.

Definition 3.3. Let G be a graph and let $d \in \mathbb{N}$. For a vertex ordering σ of G, we define the *weak d-coloring number*, the *strong d-coloring number*, and the *d-admissibility* of σ as follows:

$$\begin{aligned}\mathrm{wcol}_d(G,\sigma) &:= \max_{v\in V(G)} |\mathrm{WReach}_d[G,\sigma,v]|,\\ \mathrm{col}_d(G,\sigma) &:= \max_{v\in V(G)} |\mathrm{SReach}_d[G,\sigma,v]|,\\ \mathrm{adm}_d(G,\sigma) &:= \max_{v\in V(G)} \mathrm{adm}_d(G,\sigma,v).\end{aligned}$$

The *weak d-coloring number*, the *strong d-coloring number*, and the *d-admissibility* of G are defined as the minimum among vertex orderings σ of G of the respective parameter for σ. That is, if by $\Pi(G)$ we denote the set of vertex orderings of G, then

$$\begin{aligned}\mathrm{wcol}_d(G) &:= \min_{\sigma\in\Pi(G)} \mathrm{wcol}_d(G,\sigma),\\ \mathrm{col}_d(G) &:= \min_{\sigma\in\Pi(G)} \mathrm{col}_d(G,\sigma),\\ \mathrm{adm}_d(G) &:= \min_{\sigma\in\Pi(G)} \mathrm{adm}_d(G,\sigma).\end{aligned}$$

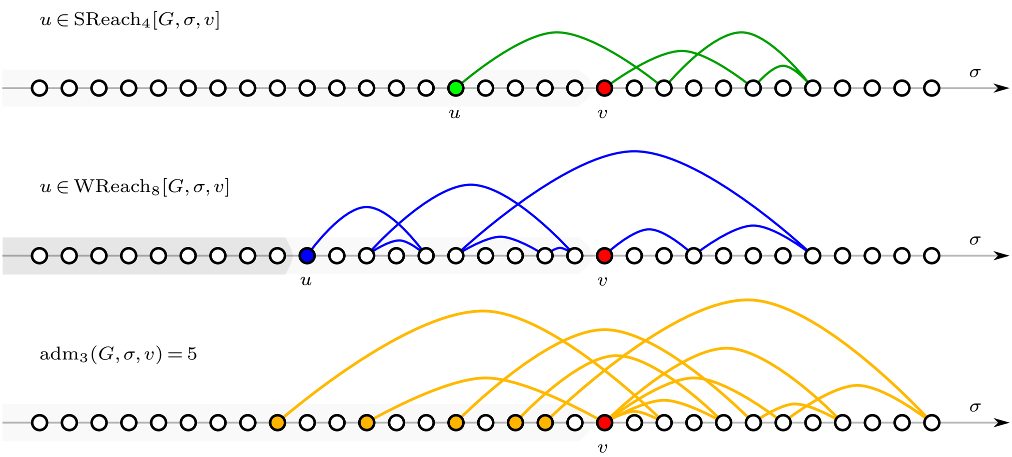

Figure 3.1: Different notions of reaching smaller vertices by short paths. In the first panel, u is strongly 4-reachable from v. In the second panel, u is weakly 8-reachable from v. In the third panel, the 3-admissibility of v is 5.

3.1. Prove that for every graph G we have

$$\mathrm{wcol}_1(G) - 1 = \mathrm{col}_1(G) - 1 = \mathrm{adm}_1(G) = \mathsf{degeneracy}(G).$$

3.2. Prove that for every graph G, its vertex ordering σ, and $d \in \mathbb{N}$ we have

$$\mathrm{adm}_d(G,\sigma) \le \mathrm{col}_d(G,\sigma) \le \mathrm{wcol}_d(G,\sigma).$$

3.3. For every $d \in \mathbb{N}$, compute

$$\mathrm{adm}_d(\text{Forests}), \qquad \mathrm{col}_d(\text{Forests}), \qquad \mathrm{wcol}_d(\text{Forests}).$$

3.4. Prove that for every graph G, its vertex ordering σ, and $d \in \mathbb{N}$ we have

$$\mathrm{col}_d(G,\sigma) \le 1 + (\mathrm{adm}_d(G,\sigma))^d.$$

3.5. Prove that for every graph G, its vertex ordering σ, and $d \in \mathbb{N}$ we have

$$\mathrm{wcol}_d(G,\sigma) \le \sum_{i=0}^{d} (\mathrm{col}_d(G,\sigma) - 1)^i.$$

3.6. Prove that for every graph G, its vertex ordering σ, $d \in \mathbb{N}$, and vertices $u, v \in V(G)$, the set

$$\mathrm{WReach}_d[G,\sigma,u] \cap \mathrm{WReach}_d[G,\sigma,v]$$

intersects all paths of length at most d connecting u and v.

3.7. Prove that for every graph G and $d \in \mathbb{N}$ we have

$$\nabla_d(G) \le \mathrm{wcol}_{4d+1}(G).$$

3.8. For a graph G, vertex subset S, vertex $u \in S$, and $d \in \mathbb{N}$, let $b_d(u,S)$ be the maximum number of paths of length at most d that all start in u, end in $S - \{u\}$, and are vertex-disjoint apart from u. Consider the following algorithm: starting with $S = V(G)$, iteratively remove from S any vertex u that has the smallest $b_d(u,S)$ until S becomes empty. Prove that if σ is the reversal of the order of removal of vertices by this algorithm, then σ achieves optimum d-admissibility, that is, $\mathrm{adm}_d(G,\sigma) = \mathrm{adm}_d(G)$.

Fact 3.4. For every graph G we have

$$\mathrm{adm}_d(G) \le 6d\,(\lceil \nabla_{d-1}(G) \rceil)^3.$$

3.9. Conclude the following Theorem 3.5.

Theorem 3.5. The following conditions are equivalent for a graph class $\mathcal{C}$:

(1) $\mathcal{C}$ has bounded expansion, that is, $\nabla_d(\mathcal{C})$ is finite for every $d \in \mathbb{N}$;

(2) $\mathrm{adm}_d(\mathcal{C})$ is finite for every $d \in \mathbb{N}$;

(3) $\mathrm{col}_d(\mathcal{C})$ is finite for every $d \in \mathbb{N}$;

(4) $\mathrm{wcol}_d(\mathcal{C})$ is finite for every $d \in \mathbb{N}$.

4. Generalized coloring numbers: applications

4.1. Let G be a graph, σ be a vertex ordering of G, and $d \in \mathbb{N}$. For every vertex $u \in V(G)$, define the *cluster* of u as follows:

$$C_u := \{v \in V(G) : u \in \mathrm{WReach}_{2d}[G,\sigma,v]\}.$$

Prove that the following conditions hold:

- each cluster has radius at most $2d$;
- each vertex of $V(G)$ appears in at most $\mathrm{wcol}_{2d}(G,\sigma)$ clusters; and
- for each vertex $u \in V(G)$, the ball $\{v : \mathrm{dist}(u,v) \le d\}$ is entirely contained in some cluster.

Such a family of clusters is called an *distance-d neighborhood cover* of G with radius $2d$ and overlap $\mathrm{wcol}_{2d}(G,\sigma)$.

Definition 4.1. A set of vertices I in a graph G is *distance-d independent* if vertices of I are pairwise at distance more than d. A set of vertices D is *distance-d dominating* if every vertex of G is at distance at most d from a vertex of D. The size of a largest distance-d independent set in G is denoted by $\alpha_d(G)$, while the size of a smallest distance-d dominating set in G is denoted by $\gamma_d(G)$.

4.2. Prove that for every graph G and $d \in \mathbb{N}$, we have

$$\alpha_d(G) \ge \gamma_d(G) \ge \alpha_{2d}(G).$$

4.3. Construct a sequence of graphs $G_1, G_2, G_3, \ldots$ satisfying the following: for every $i \in \mathbb{N}$,

$$\alpha_2(G_i) = 1 \qquad \text{and} \qquad \gamma_1(G_i) \ge i.$$

4.4. Consider the following algorithm applied on a graph G with a vertex ordering σ. For every vertex u, mark the vertex of $\mathrm{WReach}_d[G,\sigma,u]$ that is the smallest in σ. Letting D be the set of all marked vertices, prove that D is a distance-d dominating set of G that satisfies $|D| \le \mathrm{wcol}_{2d}(G,\sigma) \cdot \gamma_d(G)$.

4.5. Consider the following algorithm deployed on a graph G with a vertex ordering σ.

- Start with $A = \emptyset$, $D = \emptyset$, and $R = V(G)$.
- As long as R is not empty, do the following:
  - Let u be the vertex of R that is the smallest in σ.
  - Add u to A, add $\mathrm{WReach}_{2d}[G,\sigma,u]$ to D, and remove all vertices that are distance-d dominated by $\mathrm{WReach}_{2d}[G,\sigma,u]$ from R.

Prove that once the algorithm terminates, the following assertions hold:

(1) D is a distance-d dominating set in G that satisfies $|D| \le \mathrm{wcol}_{2d}(G,\sigma) \cdot |A|$.

(2) A contains a distance-$2d$ independent set of size at least $\frac{|A|}{\mathrm{wcol}_{2d}(G,\sigma)}$.

Conclude that for every graph G and $d \in \mathbb{N}$, we have

$$\alpha_{2d}(G) \le \gamma_d(G) \le \mathrm{wcol}_{2d}(G)^2 \cdot \alpha_{2d}(G).$$

5. Low treedepth colorings

Definition 5.1. A *rooted forest* is an acyclic graph in which for every connected component we pick one of its vertices to be the root. This imposes an ancestor/descendant relation on vertices: a vertex u is an ancestor of a vertex v if and only if u and v are in the same connected component and v lies on the unique path from u to the root of this component.

The *depth* of a rooted forest is the maximum number of vertices on a path from the root of some connected component to any of its vertices.

Definition 5.2. A *treedepth decomposition* of a graph G is a rooted forest F on the same vertex set as G that satisfies the following condition: for every edge uv of G, either u is an ancestor of v in F or vice versa. The *treedepth* of G, denoted $\mathrm{td}(G)$, is the minimum depth of a treedepth decomposition of G.

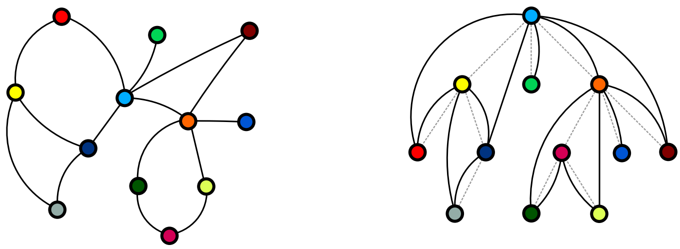

Figure 5.1: A graph and its tree-depth decomposition of depth 4.

5.1. Prove that the treedepth of a path on n vertices is at most $\lceil \log_2(n+1) \rceil$.

5.2. Consider the following game on a graph G played by two players: Connector and Splitter. In each round, Connector first picks a connected component C of G and then Splitter picks one vertex u of C. The game continues on the graph $C-u$, i.e., the component C with u removed. The Splitter wins when the graph becomes empty.

Prove that the treedepth of a graph is equal to the minimum number of rounds needed for the Splitter to win the game.

5.3. Prove that the treedepth of a path on n vertices is at least $\lceil \log_2(n+1) \rceil$.

Definition 5.3. The *depth-first search* is a search algorithm that given a connected graph G and any its vertex u_0, performs as follows. The search is always at some vertex v, initially set to be u_0. When the search enters v via some edge, it marks v as visited and inspects the neighbors of v in any order. For each neighbor w that upon inspection appears to be not yet marked as visited, the search enters w through vw, calls itself recursively on w, and having processed w withdraws back to v through vw. Once all neighbors are inspected, the search finishes processing v.

The *DFS forest* of a graph G is any forest obtained by running depth-first search in each connected component of G starting from any vertex, and including all the traversed edges. We may view such a DFS forest as rooted by taking the starting vertices of the searches to be the roots.

5.4. Prove that every DFS forest of a graph G is a treedepth decomposition of G.

5.5. For a graph G, let $\mathrm{lp}(G)$ be the length of the longest path in G. Prove that

$$\log_2 \mathrm{lp}(G) \leq \mathrm{td}(G) \leq \mathrm{lp}(G).$$

5.6. Prove that for every graph G we have

$$\mathrm{td}(G) = \mathrm{wcol}_\infty(G).$$

Definition 5.4. For $p \in \mathbb{N}$, a coloring λ of vertices of a graph G is *p-centered* if for every connected subgraph H of G, either the vertices of H receive more than p different colors under λ, or there exists a color c such that exactly one vertex of H is colored c.

5.7. Prove that 1-centered colorings of a graph are exactly its proper colorings.

5.8. Prove that any p-centered coloring of a graph G is a *treedepth-p coloring* in the following sense: For every $p' \le p$, every set of p' colors induces in G a subgraph of treedepth at most p'.

5.9. Prove that if G has a treedepth-p coloring using N colors, then G also has a p-centered coloring using $N \cdot p^{\binom{N}{p}}$ colors.

5.10. Let G be a graph, σ be its vertex ordering, and $p \in \mathbb{N}$. Let us color the vertices of G using $\mathrm{wcol}_{2^{p-1}}(G,\sigma)$ colors by inspecting vertices in the increasing order w.r.t. σ and assigning each vertex u a color that is not present among other vertices of $\mathrm{WReach}_{2^{p-1}}[G,\sigma,u]$. Prove that the coloring obtained in this way is p-centered.

5.11. Prove that a class of graphs $\mathcal{C}$ has bounded expansion if and only if the following condition holds: for every $p \in \mathbb{N}$ there exists a number $N(p)$ such that every graph from $\mathcal{C}$ admits a p-centered coloring using at most $N(p)$ colors.

Definition 5.5. For a graph G and $d \in \mathbb{N}$, we define a graph $G^{=d}$ as follows: the vertex set of $G^{=d}$ is the same as of G, while two vertices u, v are adjacent in $G^{=d}$ if and only if they are at distance *exactly* d in G.

5.12. Prove that every graph G of treedepth at most d admits a coloring using at most $2^d - 1$ colors with the following property: for any pair of vertices u and v, if the distance between u and v in G is finite and odd, then u and v receive different colors.

5.13. Prove that if $\mathcal{C}$ is a class of bounded expansion and $d \in \mathbb{N}$ is odd, then there is a number M such that for every graph $G \in \mathcal{C}$, the graph $G^{=d}$ admits a proper coloring with at most M colors.

5.14. Estimate the constant M given by the solution of Problem 5.13 for $\mathcal{C} = \text{Planar}$ and $d = 3$.

6. Neighborhood complexity

6.1. Prove that a bipartite planar graph on n vertices has less than $2n$ edges.

6.2. Prove that if G is a planar graph and A is a subset of its vertices, then the cardinality of the set family

$$\{N[v] \cap A \colon v \in V(G)\}$$

is at most $7|A|+1$. Here, $N[v]$ is the *closed neighborhood* of v which consists of v and its neighbors.

6.3. Let $\mathcal{C}$ be a graph class of bounded expansion. Prove that for every graph $G \in \mathcal{C}$ and subset of vertices $A \subseteq V(G)$, we have

$$|\{N[v] \cap A \colon v \in V(G)\}| \le (4^{\nabla_1(\mathcal{C})} + \nabla_1(\mathcal{C}) + 1) \cdot |A| + 1.$$

In the next few problems we will work out a proof of the following theorem

Theorem 6.1. Let $\mathcal{C}$ be a graph class of bounded expansion and let $d \in \mathbb{N}$ be fixed. Then there exists a constant C such that for every graph $G \in \mathcal{C}$ and nonempty subset of vertices $A \subseteq V(G)$, we have

$$|\{N^d[v] \cap A \colon v \in V(G)\}| \leq C \cdot |A|.$$

Here $N^d[v]$ is the *distance-d neighborhood* of v which consists of all vertices at distance at most d from v.

Definition 6.2. For a graph G, set of vertices A, and $v \in V(G)$, the *distance-d profile* of v on A is the function $\mathsf{profile}_d[v, A] \colon A \to \{0, 1, \ldots, d, +\infty\}$ defined as follows: for $a \in A$, we put

$$\mathsf{profile}_d[v, A](a) = \begin{cases} \mathrm{dist}(v, a) & \text{if } \mathrm{dist}(v, a) \leq d, \\ +\infty & \text{otherwise.} \end{cases}$$

6.4. Prove that in Theorem 6.1 it suffices to bound the number of different functions on A realized as distance-d profiles by $C \cdot |A|$.

Let σ be a vertex ordering of G with optimum weak $2d$-coloring number, that is,

$$\mathrm{wcol}_{2d}(G, \sigma) = \mathrm{wcol}_{2d}(G) \leq \mathrm{wcol}_{2d}(\mathcal{C}).$$

Let us define

$$B := \bigcup_{a \in A} \mathrm{WReach}_d[G, \sigma, a].$$

Further, for each $u \in V(G)$ let

$$X[u] := \mathrm{WReach}_d[G, \sigma, u] \cap B.$$

6.5. Prove that if for two vertices $u, v \in V(G)$ we have

$$X[u] = X[v] \qquad \text{and} \qquad \mathsf{profile}_d[u, X[u]] = \mathsf{profile}_d[v, X[v]],$$

then also

$$\mathsf{profile}_d[u, A] = \mathsf{profile}_d[v, A].$$

Conclude that it suffices to bound the cardinality of the family $\{X[u] \colon u \in V(G)\}$ by $C \cdot |B|$ for some constant C.

6.6. Prove that for every $u \in V(G)$ with non-empty $X[u]$, if b is the vertex of $X[u]$ that is the largest in the ordering σ, then we have

$$X[u] \subseteq \mathrm{WReach}_{2d}[G, \sigma, b].$$

Using this observation finish the proof of Theorem 6.1.