Sparsity — homework 6

VC dimension and polynomial expansion

Deadline: January 30th, 2020, 23:00 CET

Problem 1. Prove that if a class of graphs $\mathcal{C}$ has polynomial expansion, then the class $\mathcal{C}^2$ either has polynomial expansion as well, or is not nowhere dense.

Note: The square of a class $\mathcal{C}$, i.e. $\mathcal{C}^2$, was defined at the beginning of Chapter 5 of the lecture notes.

Problem 2. For two set systems $\mathcal{F}_1$ and $\mathcal{F}_2$ over the same grounds set U, we define the set system $\mathcal{F}_1 \oplus \mathcal{F}_2$ over U as follows:

$$\mathcal{F}_1 \oplus \mathcal{F}_2 \coloneqq \{ A_1 \cup A_2, A_1 \cap A_2 : A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2 \}.$$

That is, $\mathcal{F}_1 \oplus \mathcal{F}_2$ consists of all the unions and all the intersections of pairs of sets, where the first set in the pair comes from $\mathcal{F}_1$ and the second one comes from $\mathcal{F}_2$.

Prove that for every pair of integers $d_1, d_2 \in \mathbb{N}$ there exists $d \in \mathbb{N}$ such that the following holds: if the VC dimension of $\mathcal{F}_1$ is at most d_1 and the VC dimension of $\mathcal{F}_2$ is at most d_2, then the VC dimension of $\mathcal{F}_1 \oplus \mathcal{F}_2$ is at most d.

Problem 3. A *triangle* in a graph is a triple of pairwise adjacent vertices, while a *triangle packing* is a set of pairwise vertex-disjoint triangles. A *triangle packing* $\mathcal{P}$ in a graph G is *maximal* if every triangle in G intersects a triangle in $\mathcal{P}$.

Prove that there exists a constant $c \in \mathbb{N}$ such that the following holds. Suppose G is an n-vertex planar graph and $\mathcal{P}, \mathcal{R}$ are two maximal triangle packings in G, where $|\mathcal{P}| \leqslant |\mathcal{R}|$. Then for every integer k satisfying $|\mathcal{P}| \leqslant k \leqslant |\mathcal{R}|$ there exists a maximal triangle packing $\mathcal{Q}$ in G such that

$$||\mathcal{Q}| - k| \leqslant c \cdot n^{2/3}.$$