Sparsity — homework 5

Uniform quasi-wideness and friends

Deadline: January 16th, 2020, 23:00 CET

Problem 1. The *Flipper Game*[1] on a graph G is played between two players: Flipper and Connector. Every round starts with a move of Flipper, who chooses a subset A of vertices of G and flips the adjacency relation within A: for every pair of vertices $u, v \in A$, we remove the edge uv if there was one, and we add the edge uv if there was none. Then Connector picks one connected component C of the obtained graph and the game proceeds to the next round with the arena (graph on which the game is played) restricted to C. Flipper wins the game once the arena gets restricted to a single vertex.

Prove that for a class of graphs $\mathcal{C}$, the following conditions are equivalent:

- There exists $\ell \in \mathbb{N}$ such that for every $G \in \mathcal{C}$, Flipper can win the Flipper Game on G within at most ℓ rounds.
- The class $\mathcal{C}$ has bounded shrubdepth.

Note: The definition of shrubdepth can be found in Tutorials 8 *and* 9*.*

Problem 2. For $p \in \mathbb{N}$, a family of sets $\mathcal{F}$ is said to have the *p-Helly property* if the following condition holds. For every subfamily $\mathcal{G} \subseteq \mathcal{F}$ such that every p sets in $\mathcal{G}$ have a nonempty intersection, actually the whole subfamily $\mathcal{G}$ has a nonempty intersection as well.

Prove that for every nowhere dense class of graphs $\mathcal{C}$ and $r \in \mathbb{N}$, there exists $p \in \mathbb{N}$ such that for every graph $G \in \mathcal{C}$, the family of distance-r balls $\mathcal{F}_r^G = \{N_r^G[u] \colon u \in V(G)\}$ has the p-Helly property.

Problem 3. For a graph G and $d \in \mathbb{N}$, by G^d we denote the graph on the same vertex set as G where two distinct vertices u and v are considered adjacent if and only if the distance between u and v in G is at most d. For a class $\mathcal{C}$, we denote $\mathcal{C}^d = \{G^d \colon G \in \mathcal{C}\}$. A clique K in a graph G is *maximal* if there is no clique in G that would strictly contain K.

Prove that for every nowhere dense class $\mathcal{C}$ and $d \in \mathbb{N}$ there exists a constant $c \in \mathbb{N}$ such that the following holds: for every $H \in \mathcal{C}^d$, there are at most $|V(H)|^c$ different maximal cliques in H.

[1] The name is due to Szymon Toruńczyk.