Sparsity — homework 4

Structural measures

Deadline: December 12th, 2019, 23:00 CET

Problem 1. Let G be a graph with edges labelled with integers. An oriented path P in G is *increasing* if the sequence of labels of the edges on P is strictly increasing. For a vertex u, let $M(u)$ be the set of all vertices v such that there is an increasing path from u to v.

Prove that for every class of graphs $\mathcal{C}$ of bounded expansion and integer $d \in \mathbb{N}$, there exists a constant c such that the following holds: for every graph $G \in \mathcal{C}$, any labelling of its edges with integers in $\{1, \ldots, d\}$, and any non-empty vertex subset $A \subseteq V(G)$, we have

$$|\{M(u) \cap A \colon u \in V(G)\}| \leqslant c \cdot |A|.$$

Note: By a path we mean a simple path, that is, a walk that visits every vertex at most once.

Problem 2. Prove that for every $d \in \mathbb{N}$ there exists a first-order formula φ_d in the signature of graphs such that the following holds: for every graph G, φ_d is true in G if and only if the treedepth of G is at most d.

Problem 3. In the Maximum Weight Distance-d Independent Set problem we are given a graph G together with a weight function on vertices $\mathbf{w} \colon V(G) \to \mathbb{R}_{\geqslant 0}$, and the goal is to find a distance-d independent set in G with the largest possible total weight.

Prove that for every class of bounded expansion $\mathcal{C}$ and fixed integer $d \in \mathbb{N}$, this problem admits a constant-factor approximation algorithm on graphs from $\mathcal{C}$: there exists a polynomial-time algorithm that given a graph $G \in \mathcal{C}$ outputs a distance-d independent set in G whose weight is at least $1/c$ times the maximum possible weight of a distance-d independent set in G, for some constant $c \in \mathbb{N}$ depending only on $\mathcal{C}$ and d.