Sparsity — homework 3

Generalized coloring numbers

Deadline: November 28th, 2019, 23:00 CET

Problem 1. Let $\mathcal{C}$ be a class of bounded expansion and $p \in \mathbb{N}$ be fixed. Prove that there exists an integer $c \in \mathbb{N}$ such that for every $G \in \mathcal{C}$ there exists a directed graph D on the same vertex set as G that satisfies the following conditions:

- Every vertex u of D has at most c *outneighbors* in D, that is, vertices v such that the arc (u, v) is present in D.
- For every vertex subset $A \subseteq V(G)$ such that $|A| \leqslant p$ and $G[A]$ is connected, there exists $w \in A$ such that for every $u \in A \setminus \{w\}$ there is an edge (u, w) in D.

Problem 2. By $\{0,1\}^\star$ we denote the set of all binary strings of finite length. For a class of graphs $\mathcal{C}$ and $d \in \mathbb{N}$, a *distance-d labelling scheme* for $\mathcal{C}$ is a pair of algorithms:

- *Encoder*: given a graph $G \in \mathcal{C}$, the Encoder computes a *labelling* $\lambda\colon V(G) \to \{0,1\}^\star$ — a function that to each vertex v assigns its label $\lambda(v)$.
- *Decoder*: given two labels $\alpha, \beta \in \{0,1\}^\star$, the Decoder outputs a number in $\{0, 1, \ldots, d, \infty\}$.

The following property should be satisfied: if λ is the output of the Encoder on a graph $G \in \mathcal{C}$ and for two vertices $u, v \in V(G)$ we apply the Decoder to the labels $\lambda(u)$ and $\lambda(v)$, then the output of the Decoder should be $\mathrm{dist}_G(u, v)$ if this distance is at most d, and ∞ otherwise. Note that the Decoder is given only a pair of labels, it does not have access to the graph G. Both the Encoder and the Decoder should work in time polynomial in the size of their input.

Prove that if $\mathcal{C}$ has bounded expansion and $d \in \mathbb{N}$ is fixed, then there is a distance-d labelling scheme for $\mathcal{C}$ that on n-vertex graphs uses labels of length at most $\mathcal{O}(\log n)$. Here, the constant hidden in the $\mathcal{O}(\cdot)$ notation may depend on $\mathcal{C}$ and d.

Problem 3. Design a polynomial-time algorithm with the following specification. The input consists of integers $r, m \in \mathbb{N}$, a graph G, an ordering σ of $V(G)$, and a vertex subset $A \subseteq V(G)$ satisfying

$$|A| \geqslant 4 \cdot (2cm)^{c+1}, \qquad \text{where } c \coloneqq \mathrm{wcol}_r(G, \sigma).$$

The algorithm should return a pair of disjoint vertex subsets $S, B \subseteq V(G)$ such that

$$|S| \leqslant c, \quad B \subseteq A, \quad |B| \geqslant m, \quad \text{and } B \text{ is distance-}r\text{ independent in } G - S.$$

Note: Solutions achieving a worse upper bound on $|S|$ or working under the assumption of a higher lower bound on $|A|$ will receive partial score, inversely proportional to the severity of the relaxation.