Sparsity — homework 1

Introduction and basic notions

Deadline: October 25th, 2019, 12:15 CET

Problem 1. Consider two classes of graphs $\mathcal{G}_1$, $\mathcal{G}_2$ and an operator $\mathcal{G}_1 \oplus \mathcal{G}_2$ defined as follows. A graph G belongs to $\mathcal{G}_1 \oplus \mathcal{G}_2$ if there exist graphs $G_1 \in \mathcal{G}_1$ and $G_2 \in \mathcal{G}_2$ with $|V(G)| = |V(G_1)| = |V(G_2)|$ and bijections $\pi_1 : V(G) \to V(G_1)$ and $\pi_2 : V(G) \to V(G_2)$ such that $uv \in E(G)$ if and only if $\pi_1(u)\pi_1(v) \in E(G_1)$ or $\pi_2(u)\pi_2(v) \in E(G_2)$. Prove or disprove the following statements:

1. If $\mathcal{G}_1$ and $\mathcal{G}_2$ are of bounded degerenacy, then $\mathcal{G}_1 \oplus \mathcal{G}_2$ is of bounded degeneracy.
2. If $\mathcal{G}_1$ and $\mathcal{G}_2$ are of bounded expansion, then $\mathcal{G}_1 \oplus \mathcal{G}_2$ is of bounded expansion.
3. If $\mathcal{G}_1$ and $\mathcal{G}_2$ are nowhere dense, then $\mathcal{G}_1 \oplus \mathcal{G}_2$ is nowhere dense.

Problem 2. A family $\mathcal{G}_k$ of trees is defined inductively as follows. $\mathcal{G}_1$ consists only of the one-vertex tree. For $k \geqslant 1$, every tree in $\mathcal{G}_{k+1}$ is formed by taking a disjoint union of two trees $T_1, T_2 \in \mathcal{G}_k$ and adding an edge between an arbitrary vertex of T_1 and an arbitrary vertex of T_2.

Prove that for every $k \geqslant 1$, every tree $T \in \mathcal{G}_k$ has treedepth k, but every proper minor of T has treedepth strictly smaller than k.

Problem 3. For a graph G, by $\mathrm{fvs}(G)$ we denote the minimum cardinality of a set $X \subseteq V(G)$ such that $G - X$ is acyclic and by $\mathrm{cp}(G)$ we denote the maximum cardinality of a family of vertex-disjoint cycles in G. Prove that there exists a universal constant c such that for every planar graph G it holds that $\mathrm{fvs}(G) \leqslant c \cdot \mathrm{cp}(G)$.

Your score in this problem will depend on the obtained constant c. Obtaining any $c \leqslant 3$ gives maximum score, any universal constant c gives at least 2 points. Obtaining $c = 2$ gives a very good publication.