Sparsity
Homework 3 – Problem 1

Jakub Boguta

November 27, 2019

Let $c := \mathrm{wcol}_p(\mathcal{C})$. Since $\mathcal{C}$ is of bounded expansion, we know that $c < \infty$.

Now let's take any $G \in \mathcal{C}$. There is a vertex ordering σ of G, such that $\mathrm{wcol}_p(G, \sigma) \leq c$. We define graph D so that arc (v, u) is present in D if and only if $u \in \mathrm{WReach}_p[G, \sigma, v] \setminus \{v\}$.

Clearly each $v \in V(D)$ has less than c outgoing edges, because it has

$$|\mathrm{WReach}_p[G, \sigma, v] \setminus \{v\}| < \mathrm{wcol}_p(G, \sigma) \leq p$$

outgoing edges.

Now let's take any vertex subset $A \subseteq V(G)$, such that $|A| \leq p$ and $G[A]$ is connected. Let w be the smallest vertex in A with respect to σ. For every $v \in A \setminus \{w\}$ there exists a path from v to w contained in A, so its length is at most p and w is the σ-smallest vertex on it, so $w \in \mathrm{WReach}_p[G, \sigma, v]$, which implies that arc (v, w) is in D.

Sparsity
Homework 3 – Problem 2

Jakub Boguta

November 27, 2019

The key ingredient of the solution is the following fact: Given a graph G with vertex ordering σ and any pair of vertices $u, v \in V(G)$, we know that:

$$\text{dist}(u,v) = \min_{k,l \geq 0}(\{k + l \mid \text{WReach}_k[G,\sigma,u] \cap \text{WReach}_l[G,\sigma,v] \neq \varnothing\})$$

.

Proof:

- ($\leq$) If $w \in \text{WReach}_k[G,\sigma,u] \cap \text{WReach}_l[G,\sigma,v]$, it follows that $\text{dist}(u,w) \leq k$ and $\text{dist}(v,w) \leq l$, so $\text{dist}(u,v) \leq k + l$.
- ($\geq$) Let's take the shortest path P (of length d) from u to v and let w be the σ-minimal vertex on this path. Vertex w splits P into two paths, one from u to w of length r, the other from w to v of length $d - r$. The former certifies that $w \in \text{WReach}_r[G,\sigma,u]$, the latter that $w \in \text{WReach}_{(d-r)}[G,\sigma,v]$, so RHS $\leq d$.

Let's allow labels to be from the set $\{0, 1, \$, \#\}^*$, we can trivially convert them into labels from $\{0,1\}^*$ while increasing their length by a factor of two.

Now the encoder starts by enumerating vertices from 0 to $n - 1$ and computing any vertex ordering σ such that $\text{wcol}_d(G,\sigma)$ is bounded by a constant dependent only on $\mathcal{C}$. It's easy to do it in polynomial time, for example using the algorithm from one of the tutorials that finds the ordering σ, s.t. $\text{adm}_d(G,\sigma) \leq d \cdot \text{adm}(G) \leq d \cdot \text{adm}(\mathcal{C})$. Since d-admissibility of such ordering is bounded by a constant, then d-weak coloring number is also bounded.

After the ordering is calculated, encoder finds $\text{WReach}_k[G,\sigma,v]$ for every $v \in V(G)$ and $0 \leq k \leq d$ and sets the label $\lambda(v)$ to be equal to:

$$\$\text{desc}(\text{WReach}_0[G,\sigma,v])\$\text{desc}(\text{WReach}_1[G,\sigma,v]) \ldots \$\text{desc}(\text{WReach}_d[G,\sigma,v])$$

where $\text{desc}(\text{WReach}_k[G,\sigma,v])$ is a string consisting of binary representations of numbers of vertices from $\text{WReach}_k[G,\sigma,v]$ separated by $'\#'$ characters.

Length of $\lambda(v)$ is trivially bounded by $d + d(\text{wcol}_d(G,\sigma) \cdot \mathcal{O}(\log n)) = \mathcal{O}(\log n)$

The decoder only needs to parse $\lambda(u)$ and $\lambda(v)$ and (assuming that $\min(\varnothing) = \infty$) compute:

$$\min_{k+l \leq d}(\{k + l \mid \text{WReach}_k[G,\sigma,u] \cap \text{WReach}_l[G,\sigma,v] \neq \varnothing\})$$

Note that it can easily determine whether $\mathrm{WReach}_k[G, \sigma, u] \cap \mathrm{WReach}_l[G, \sigma, v] = \varnothing$, because $k, l \leq d$, so it has bijective labels of vertices from $\mathrm{WReach}_k[G, \sigma, u]$ and $\mathrm{WReach}_l[G, \sigma, v]$. This algorithm obviously works in polynomial time.

Sparsity
Homework 3 – Problem 3

Jakub Boguta

November 27, 2019

The case when $m = 1$ is trivial, so let's assume that $m > 1$.

For any $v \in V(G)$ let's denote $\mathrm{WReach}_r[G, \sigma, v]$ by P_v. Note that we can easily determine sets P_v in polynomial time. Let's now prove that a following stronger condition implies that B is distance-r independet in $G \setminus S$:

$$\forall_{u,v \in B, u \neq v}(P_u \cap P_v \subseteq S)$$

assume that this condition is satisfied, but there exists a path of length at most r in $G \setminus S$ from u to v ($u \neq v$). Then there exists a σ-minimal vertex $w \notin S$ on this path, which splits it into two paths certifying that $w \in P_u$ and $w \in P_v$, so $P_u \cap P_v \nsubseteq S$ – a contradiction.

Let H be a graph on the same vertices as G with an edge between u and v if $u \prec_\sigma v$ and $u \in P_v$. Since $|P_v \setminus \{v\}| \leq c - 1$, we get that $\deg(H) \leq c - 1$ and we can color H with c colors (in polynomial time using a greedy algorithm). Let $I \subseteq A$ be the largest one-color subset in A, obviously $|I| \geq \frac{|A|}{c} > (cm)^c$.

Note now that family $\mathcal{F} = \{P_v \mid v \in I\}$ consists of pairwise different sets of size at most c (pairwise different because each P_v has different σ-maximal vertex). Since $|I| > (cm)^c > c!(m-1)^c$, we can apply the sunflower lemma and get a sunflower with m petals (and we can find it in time polynomial in $|I|, m, c \leq |V(G)|$, as described in the Platypus book). Let $B \subseteq I$ be the set of vertices such that family $\{P_v \mid v \in B\}$ is the said sunflower (so $|B| = m$). Finally let $S := \cap_{v \in B} P_v$.

Obviously $|S| \leq c$, because $|P_v| \leq c$. Also sets S and B are disjoint, because if there was some $u \in S \cap B$ then $u \in \cap_{v \in B} P_v$ and $u \in I$, but since $m > 1$ there is some $v \neq u, v \in B$, then $u \in P_v$ showing that u and v have to have different colors in H, so they can't be both in I. By the definition of a sunflower we know that for any pair of different $u, v \in B$ we have $P_u \cap P_v = S$, so B is a distance-r independent set in $G \setminus S$. $\square$