Sparsity - homework 1

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1 Problem 1

From problem 4. from tutorial 2 we see that there exist some d for which d-subdivision of every K_n is a subgraph G' of some $G \in \mathcal{C}$. For every k let's take $n = k + \frac{k(k-1)}{2}$. First k nodes will be used as a proper nodes of a $2d+1$-subdivision of a k-clique, so to form each new path vu between them we go through some unused remaining node w (there are paths vw and wu in G', each with d internal nodes). And we remove all unused edges. We obtained this new graph G'' as a subgraph of G', so it is also a subgraph of G. And since there is odd number of internal nodes on each clique edge, it is bipartite so $G'' \in \mathcal{C}'$. And I showed that for some number $2d+1$ arbitrary K_k has $2d+1$ subdivision as an (improper) subgraph of some $G'' \in \mathcal{C}'$ so this class is somewhere dense.

2 Problem 2

For the drawing from the statement of any k-planar graph G let's consider putting new node on each edges crossing and switching the two original edges to four new edges incident to the new node. This new graph G' is planar (we have the drawing with no more edge crossings). Now consider it's depth-k ply-2 minor G'': we put all new internal nodes into two branch sets: branch set of smaller end of first original edge it lies on and branch set of smaller end of second original edge it lies on. Then each original node is a center of branch set of depth-k, and each new node in in 2 branch sets. Family of this minors has bounded expansion (since class $\mathcal{P}$ of planar graphs has bounded expansion) from Corollary 2.28 (chapter 1, p. 24): $\mathcal{P}\nabla^2 k$ also has bounded expansion.

Now for each depth-d minor of G let's see that we can obtain the same graph as a depth-d minor of G''. For every branch set in G for every it's edge (v, u) $(v \leq u)$, in G'' all internal edges are already contracted to v, so there is an edge between last of them and u, which we contract now. So we have some bijection on branch sets and we see that if there was an edge in G between two nodes from different branch sets these nodes are also in G' in different branch sets and internal nodes on the edge are already contracted to one of them so there still exists an edge between last of the contracted nodes and the second end of original edge, so these branch sets are still connected. So depth-d minors of G have no more edges than corresponding depth-d minors over same branch sets from G'', which are bounded.

3 Problem 3

For every d there exists a graph G in $\mathcal{C}$, such that $\nabla_d(G) \geq 2^d$. To prove this let's consider $K_{2^{d+1}+1}$, but in place of every vertex there is a rooted binary tree of depth d. Such a tree has 2^d leaves: let's connect all 2^{d+1} edges incident to the original vertex to the leaves, 2 edges per leaf. Resulting graph G is in $\mathcal{C}$, because every internal vertex of binary tree has degree 3 (root has 2), and every leaf has 1 to the parent and 2 to the outside. If we contract each binary tree into one branch set, each has radius d from the root and the resulting graph is the original $K_{2^{d+1}+1}$ so it has $\nabla_d(G) \geq \frac{(2^{d+1}+1)\cdot(2^{d+1})/2}{2^{d+1}+1} = 2^d$.

From the lemma 3.1. (chapter 2, p. 6) we see that since edge density grows exponentially so does the weak coloring number. More precisely: $wcol_d(\mathcal{C}) \geq \nabla_{\lfloor \frac{d-1}{4} \rfloor} \geq 2^{\lfloor \frac{d-1}{4} \rfloor} \geq 2^{\frac{d}{8}}$ (at least for $d \geq 4$, but also for $d = 0$: $wcol_0(\mathcal{C}) \geq 1 \geq 2^{\frac{0}{8}}$ and for $1 \leq d \leq 3$: $wcol_d(\mathcal{C}) \geq 2 \geq 2^{\frac{d}{8}}$), so the thesis holds for $\epsilon = \frac{1}{8}$.