SPARSITY 2019/20 COURSE HOMEWORK 2

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Problem 1

Let $\mathcal{C}$ be a somwhere dense graph class closed under taking subgraphs. Prove that the class $\mathcal{C}'$ consisting of bipartite graphs belonging to $\mathcal{C}$ is also somwhere dense.

Proof. During the tutorials, we established that whenever $\mathcal{C}$ is a somwhere dense graph class, then for some $d \in \mathbb{N}$, we can find as a subgraph in some graph in $\mathcal{C}$ an exact d subdivision of arbitrarily large clique (meaning each edge is subdivided exactly d times). If d is odd, we are done, as then this implies in $\mathcal{C}'$ contains exact d subdivisions of arbitrarily large cliques (any cycle in this subdivision is even, thus it is a bipartite graph). Otherwise d is even, and find an exact d subdivision of a clique of size $t + \binom{t}{2}$. Now first t vertices of original clique will form branch vertices of our t clique, let the other branch vertices form the set A. We start constructing H as follows: for each pair of new branch vertices, say u and v, pick any $a \in A$ and add the path from u through a to v to the graph H (observe that this path is of length $2(d+1)$ and is thus even), and remove a from A. At the end of this process we will obtain a subgraph which learly is a $2d+1$ exact subdivision of K_t, thus belongs to $\mathcal{C}'$ and this concludes the proof. □

Problem 2

A graph G is k-planar, if there exists a drawing of G, such that

- *every pair of edges intersect at no more than 1 point which moreover cannot be an endpoint of either of them (presumably applies to edges not sharing an endpoint),*
- *no three edges intersect at a single point,*
- *every edge intersects at most k other edges.*

Prove that for any fixed k the class of all k-planar graphs has bounded expansion.

Proof. Fix a drawing of our graph G satisfing the conditions of k-planarity. Now consider a pair of corssing edges of our graph (say uv and xy). We may replace the crossing with a vertex — ie. add a vertex t to the graph incident to $\{u, v, x, y\}$ and remove the edges uv and xy from G while preserving the drawing (and the number of crossings drops by 1). Repeatedly apply this procedure as long as we can find a pair of crossing edges and call the resulting graph G'. Clearly G' is planar (as we removed all crossings). Now, I claim that G is a depth k congestion 2 minor of G'. To this end, fix any orientation of our graph G. Now for any vertex v define the branch set of v (call it $\phi(v)$) as follows: for each out-neighbour u of v, let $t_1, \ldots, t_l$ be the 'crossing vertices' added on the image of the edge $\overrightarrow{vu}$ in the drawing (clearly $l \leqslant k$) and simply add them all to branch set of v.

Now each branch set $\phi(v)$ is a star (centred at v), and each vertex is at distance at most k from v. Moreover each vertex belongs to at most 2 different branch sets. Thus, let H be the graph modeled by ϕ. Clearly $G \subseteq H$, and H is a depth h congestion 2 minor of a planar graph, equivalently $G \in \mathcal{C}\nabla^2 k$ where $\mathcal{C}$ denotes the class of all planar graphs, thus proving that k-planar graphs have bounded expansion. □

Problem 3

Let $\mathcal{C}$ be the class of all graphs with degree bounded by 3. Prove that, there is a constant $\varepsilon > 0$, such that for any $d \in \mathbb{N}$, $\mathrm{wcol}_d(\mathcal{C}) \geqslant 2^{\varepsilon d}$.

Proof. During the lecture we established the following inequality

$$\nabla_d(G) \leqslant \mathrm{wcol}_{4d+1}(G),$$

which when combined with the monotonicity of wcol (ie $r \leqslant s \implies \mathrm{wcol}_r(G) \leqslant \mathrm{wcol}_s(G)$) provides us with a constructive means of proving lower bounds for wcol — finding a dense shallow minor.

So, fix $d \in \mathbb{N}$ and let G be the graph consisting of $2^{d+1} + 1$ disjoint full binary trees, each of depth d exactly (call them T_i with the root r_i and the set of leaves $\{a_{i,j} \colon 1 \leqslant j \leqslant 2^d\}$ for $i \in \{1, \dots, 2^{d+1} + 1\}$). We will be adding some edges to our graph, but we will maintain the invariant $\Delta(G) = 3$. For each pair (i, j) where $i, j \in \{1, \dots, 2^{d+1} + 1\}$ and $i < j$ we will add an edge between some leaf of T_i and T_j. To each leaf we can add 2 edges without increasing the maximum degree, and each tree has got exactly 2^d leafs giving us 2^{d+1} possible 'edge slots'.

So, we may find a map from the set $\{1, \dots, 2^{d+1} + 1\} \setminus \{i\}$ to the set of leaves of T_i in which each leaf is assigned to exactly two indices. Fix one such map for each i and call it f_i. Now, for each pair of indices $i < j$, add to G the edge between $a_{i,f_i(j)}$ and $a_{j,f_j(i)}$. Clearly the degree of each $a_{i,j}$ is now precisely 3, and there is exactly 1 edge between each pair of trees T_i. As the depth of each of these trees is exactly d, so is its radius. Thus letting H to be the minor of G resulting from contracting each tree T_i, it is a depth d shallow minor of G. By construction, H is a clique on $2^{d+1} + 1$ vertices, thus of edge density exactly 2^d. By the inequality from the lecutre, we get $\mathrm{wcol}_{4d+1}(\mathcal{C}) \geqslant 2^d$. Now, we observe that for $d > 3$ we get $\mathrm{wcol}_d(\mathcal{C}) \geqslant \mathrm{wcol}_{d-1}(\mathcal{C}) \geqslant \mathrm{wcol}_{d-2}(\mathcal{C}) \geqslant \mathrm{wcol}_{d-3}(\mathcal{C})$ and out of these, one is of the form $4k + 1$, in the worst case it is the last one, thus $\mathrm{wcol}_d(\mathcal{C}) \geqslant 2^{(d-4)/4}$. Thus, we get the desired result (we can get rid of multiplicative constant by lowering our ε enough, if we so desire; for smaller d, the inequality is trivial). $\square$