Problem 1-3.

For a graph G, by $\mathrm{fvs}(G)$ we denote the minimum cardinality of a set $X \subseteq V(G)$ such that $G - X$ is acyclic and by $\mathrm{cp}(G)$ we denote the maximum cardinality of a family of vertex-disjoint cycles in G. Prove that there exists a universal constant c such that for every planar graph G it holds that $\mathrm{fvs}(G) \leq c \cdot \mathrm{cp}(G)$.

Solution. We will prove the statement with $c = 3$, i.e. that for every planar graph G there exists a pair $(X(G), \mathcal{C}(G))$, where $X(G) \subseteq V(G)$ has the property that $G - X(G)$ is a forest, $\mathcal{C}(G)$ is a family of vertex disjoint cycles in G, and $|X(G)| \leq 3|\mathcal{C}(G)|$.

Let us proceed by induction on number of vertices of graph G. For G being a forest it suffices to put $X(G) = \mathcal{C}(G) = \varnothing$, which settles the base case (although G empty would be enough).

If G is not connected, and $G = G_1 \sqcup G_2$, then it suffices to plug $X(G) = X(G_1) \sqcup X(G_2)$ and $\mathcal{C}(G) = \mathcal{C}(G_1) \sqcup \mathcal{C}(G_2)$. The same holds when G is connected and has a bridge. Therefore we may assume that G is connected and bridgeless.

If G has a vertex v of degree 1, then it suffices to put $X(G) = X(G - v)$ and $\mathcal{C}(G) = \mathcal{C}(G - v)$. If G has a vertex v of degree 2, whose neighbors u and w are not connected, then consider graph G/vw with v' corresponding to the contracted edge. If $v' \in X(G/vw)$, then we put $X(G) = X(G/vw) \setminus \{v'\} \cup \{w\}$; otherwise we put $X(G) = X(G/vw)$. If one of the cycles in $\mathcal{C}(G/vw)$ contains uv' then replace it with cycle containing uvw in $\mathcal{C}(G)$, keeping all the remaining cycles unchanged. Finally if G contains a triangle uvw, then put

$$X(G) = X(G - uvw) \sqcup \{u, v, w\} \quad \text{and} \quad \mathcal{C}(G) = \mathcal{C}(G - uvw) \sqcup \{uvw\}.$$

Now all deleted edges are hit by one of the vertices added to X and the new cycle is vertex-disjoint with all existing ones. By inductive assumption we have $|X(G)| = |X(G - uvw)| + 3 \leq 3|\mathcal{C}(G - uvw)| + 3 = 3|\mathcal{C}(G)|$. This allows to assume that $\delta(G) \geq 3$ and that G is triangle-free.

Suppose that G has a face $uvwx$ of length 4 with at least one vertex, say x, of degree 3. Then if we plug $X(G) = X(G - uvwx) \sqcup \{u, v, w\}$ and $\mathcal{C}(G) = \mathcal{C}(G - uvwx) \sqcup \{uvwx\}$, then $X(G)$ satisfies desired properties (every cycle hitting x hits also u or w), $\mathcal{C}(G)$ as well, and size of X increased by 3, size of $\mathcal{C}$ by 1, so the desired inequality holds. Similarly suppose that G has a face $uvwxy$ of length 5 with at least two non-adjacent vertices of degree 3, say w and y. Then plugging $X(G) = X(G - uvwxy) \sqcup \{u, v, x\}$ and $\mathcal{C}(G) = \mathcal{C}(G - uvwxy) \sqcup \{uvwxy\}$ does the job because each cycle potentially w hits also v or x, and each cycle hitting y hits also x or u.

Finally let us prove that a connected bridgeless triangle-free planar graph of minimum degree 3 in which neither of the structures described in the previous paragraph (4-face incident with some 3-vertex; 5-face incident with two non-adjacent 3-vertices) occur does not exist (and therefore, all possible cases are already considered). Suppose the contrary and use discharging.

Setup. Every vertex and every face receives a charge of $+12$, every edge receives a charge of -12. The Euler's formula asserts that the total charge is $+24$, hence positive.

First discharging phase. Every vertex and every face sends a charge of $+3$ to every incident edge. In this manner, every edge and every 4-face attains exactly zero charge, and every other face attains a charge of at most -3 (with equality for 5-faces). Vertices of degree 3 have $+3$ charge left, and all other vertices are of nonpositive charge.

Second discharging phase. Every vertex of degree 3 sends a charge of $+1$ to every incident face. After this phase all vertices have nonpositive charge. The total charge is positive, so there must exist faces with strictly positive charge. By assumption, no 4-face may have received extra charge in the second phase. For $k \geq 6$, the total charge on each k-face is at most $12 - 3k + k \leq 0$. Therefore there exists a 5-face with positive charge, so it must have received at least $+4$ during the second phase, meaning at least four incident 3-vertices. This contradicts the assumption, because among any three (so in particular among any four) vertices of a 5-cycle there is at least one pair of non-adjacent vertices.