Sparsity — homework 1

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Solution 2

In several places of the solution we implicitly use the following properties of treedepth:

- if H is a minor of a graph G, then $\mathsf{td}(H) \leqslant \mathsf{td}(G)$;
- if v is a vertex of a graph G, then $\mathsf{td}(G) \leqslant \mathsf{td}(G - v) + 1$;
- if H_1 and H_2 are disjoint subgraphs of some graph, then $\mathsf{td}(H_1 \cup H_2) = \max\{\mathsf{td}(H_1), \mathsf{td}(H_2)\}$.

(The first property was proved on the lecture, and the remaining two follow immediately from the recursive definition of treedepth)

We prove the claim from the problem statement by induction on k. For $k = 1$, the only graph in $\mathcal{G}_k$ is a one-vertex tree T, and by definition of treedepth we have

$$\mathsf{td}(T) = 1 + \min_{v \in V(T)} \mathsf{td}(T - v) = 1 + \mathsf{td}(K_0) = 1,$$

where K_0 denotes the empty graph. The only proper minor of T is K_0, and $\mathsf{td}(K_0) = 0$, so the claim indeed holds for $k = 1$.

We show for every $k \geq 2$ that if the claim holds for $k - 1$, then it also holds for k. Let $T \in \mathcal{G}_k$. By definition of the class $\mathcal{G}_k$, there exists disjoint subtrees T_1 and T_2 of T and vertices $v_1 \in V(T_1)$, $v_2 \in V(T_2)$ such that T_1 and T_2 are isomorphic to some trees from $\mathcal{G}_{k-1}$ and $T = (T_1 \cup T_2) + v_1 v_2$. We first show that every proper minor U of T has treedepth smaller than k. It is enough to show it for the maximal proper minors, which are obtained be removing one vertex or edge, or contracting one edge.

Suppose first that $U = T - v$ for some $v \in V(T)$, and let $i \in \{1, 2\}$ be such that $v \in V(T_i)$. By induction hypothesis we have $\mathsf{td}(T_i - v) < k - 1$ and $\mathsf{td}(T_{3-i} - v_{3-i}) < k - 1$, so

$$\begin{aligned}
\mathsf{td}(U) &\leqslant 1 + \mathsf{td}(U - v_{3-i}) \\
&= 1 + \mathsf{td}((T_i - v) \cup (T_{3-i} - v_{3-i})) \\
&= 1 + \max\{\mathsf{td}(T_i - v), \mathsf{td}(T_{3-i} - v_{3-i})\} \\
&< 1 + (k - 1) = k.
\end{aligned}$$

Now suppose that $U = T - e$ for some $e \in V(T)$. If $e = v_1v_2$, then

$$\mathsf{td}(U) = \mathsf{td}(T_1 \cup T_2) = \max\{\mathsf{td}(T_1), \mathsf{td}(T_2)\} = k - 1 < k.$$

Hence we assume that $e \neq v_1v_2$, and we let $i \in \{1, 2\}$ be such that $e \in E(T_i)$. By induction hypothesis we have $\mathsf{td}(T_i - e) < k - 1$ and $\mathsf{td}(T_{3-i} - v_{3-i}) < k - 1$, so

$$\begin{aligned}\mathsf{td}(U) &\leqslant 1 + \mathsf{td}(U - v_{3-i})\\ &= 1 + \mathsf{td}(T_i - e) \cup (T_{3-i} - v_{3-i}))\\ &= 1 + \max\{\mathsf{td}(T_i - e), \mathsf{td}(T_{3-i} - v_{3-i})\}\\ &< 1 + (k - 1) = k.\end{aligned}$$

Finally suppose that $U = T/e$. Let us first consider the case $e = v_1v_2$. Let v denote the vertex of U representing the contracted edge e. By induction hypothesis we have $\mathsf{td}(T_i - v_i) < k - 1$ for $i \in \{1, 2\}$, so

$$\begin{aligned}\mathsf{td}(U) &\leq 1 + \mathsf{td}(U - v)\\ &= 1 + \mathsf{td}((T_1 - v_1) \cup (T_2 - v_2))\\ &= 1 + \max\{\mathsf{td}(T_1 - v_1), \mathsf{td}(T_2 - v_2)\}\\ &< 1 + (k - 1) = k\end{aligned}$$

Now consider the case $e \neq v_1v_2$, and let $i \in \{1, 2\}$ be such that $e \in E(T_i)$. By induction hypothesis we have $\mathsf{td}(T_i/e) < k - 1$ and $\mathsf{td}(T_{3-i} - v_{3-i}) < k - 1$, so

$$\begin{aligned}\mathsf{td}(U) &\leq 1 + \mathsf{td}(U - v_{3-i})\\ &= 1 + \mathsf{td}(T_i/e \cup (T_{3-i} - v_{3-i}))\\ &= 1 + \max\{\mathsf{td}(T_i/e), \mathsf{td}(T_{3-i} - v_{3-i})\}\\ &< 1 + (k - 1) = k\end{aligned}$$

It remains to show that $\mathsf{td}(T) = k$. For any $v \in V(T)$, either $T_1 \subseteq T - v$, or $T_2 \subseteq T - v$, so by induction hypothesis we have $\mathsf{td}(T - v) \geqslant \min\{\mathsf{td}(T_1), \mathsf{td}(T_2)\} = k - 1$, so

$$\mathsf{td}(T) = 1 + \min_{v \in V(T)} \mathsf{td}(T - v) \geqslant 1 + (k - 1) = k$$

Since $T - v_1$ is a proper minor of T, we have $\mathsf{td}(T - v_1) \leq k - 1$, so

$$\mathsf{td}(T) \leqslant 1 + \mathsf{td}(T - v_1) \leq 1 + (k - 1) = k.$$

Hence indeed $\mathsf{td}(T) = k$. This completes the inductive proof.