Problem 2

Define the family of trees $\mathcal{G}_k$ inductively as follows: $\mathcal{G}_1$ consists of a single tree on a single vertex; $\mathcal{G}_{k+1}$ contains all trees that can be obtained by taking a disjoint union of two trees $T_1, T_2 \in \mathcal{G}_k$ and adding a single edge between the vertex sets $V(T_1)$ and $V(T_2)$. Show that for any tree $T \in \mathcal{G}_k$, its treedepth is exactly k, but any proper minor of T has got treedept strictly less than k.

Proof. We proceed by induction on k. The base case of $k = 1$ is obvious, so assume it holds for some k, and take any $T \in \mathcal{G}_{k+1}$, with $T_1, T_2 \in \mathcal{G}_k$ and the edge $e = uv$ ($u \in V(T_1)$, $v \in V(T_2)$) certifing this fact. Clearly treedepth of T is at most $k + 1$, as removing u splits T into conneceted components $\{C_1, \ldots, C_t\}$ such that each C_i is a subgraph of a tree in $\mathcal{G}_k$, and thus of treedepth at most k. It cannot be strictly less than $k + 1$ for if that were the case, there would exist a vertex w (WLOG $w \in V(T_1)$), such that removal of w splits T into connected components $\{C_1, \ldots, C_t\}$ each of treedepth strictly less than k. However, then it would follow that the treedepth of T_2 is strictly less than k as there would exists i such that $T_2 \subseteq C_i$, contradicting the inductive assumption.

Now, it will suffice to get the desired result, if we prove that any edge contraction, edge deletion or vertex deletion drops the treewidth by at least one. However, as we recall from the tutorial, treedepth of an n vertex tree is at most $1 + \lfloor \log_2 n \rfloor$. And any of the aforementioned operations applied to a tree either drops the number of vertices by 1 (edge contraction and vertex deletion) or splits the tree into conneceted components each having at most $n - 1$ vertices (edge deletion). Obviously the size of each tree in $\mathcal{G}_{k+1}$ is exactly 2^k (by trivial induction), thus any of the minor operation will yield a graph with each connected component having strictly less than 2^k vertices and so, by the tutorial inequality, its treedepth is at most k, concluding the proof. □