Sparsity
Solutions to homework 1

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Problem 1. Statement 1. is true.

We use the fact that bounded degeneracy is equivalent to bounded maximum average degree (mad). Let us assume that classes $\mathcal{G}_1$ and $\mathcal{G}_2$ have mad bounded by A_1 and A_2, respectively.

Let us consider a graph $G \in \mathcal{G}_1 \oplus \mathcal{G}_2$ and one of its subgraph $H \subseteq G$. We know that G is a *sum* of graphs $G_1 \in \mathcal{G}_1$ and $G_2 \in \mathcal{G}_2$ and let π_1, π_2 be corresponding bijections. We construct subgraphs $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$. First, we set $V(H_1) = \pi_1(V(H))$ and $V(H_2) = \pi_2(V(H))$. To construct edges we consider every edge $e \in E(H)$ and if e comes from G_1 we put $\pi_1(e)$ in $E(H_1)$, otherwise we put $\pi_2(e)$ in $E(H_2)$.[1] Then, we obtain:

$$\begin{aligned}\operatorname{avgdeg}(H) = \frac{2|E(H)|}{|V(H)|} &= \frac{2|E(H_1)| + 2|E(H_2)|}{|V(H)|} = \frac{2|E(H_1)|}{|V(H_1)|} + \frac{2|E(H_2)|}{|V(H_2)|} \\ &= \operatorname{avgdeg}(H_1) + \operatorname{avgdeg}(H_2) \leqslant \operatorname{mad}(G_1) + \operatorname{mad}(G_2) \leqslant A_1 + A_2\end{aligned}$$

Hence class $\mathcal{G}_1 \oplus \mathcal{G}_2$ has mad bounded by $A_1 + A_2$.

On the other hand, statements 2. and 3. are false and we will provide one counterexample to both of them.

We define undirected graphs G_n and H_n as follows:

$$\begin{aligned} V(G_n) = V(H_n) &= \{v_{ij} \mid i,\ j \in \{1, \dots, n\}\} \\ E(G_n) &= \{v_{ii}v_{ij} \mid i \neq j\} \\ E(H_n) &= \{v_{ij}v_{ji} \mid i \neq j\} \end{aligned}$$

Clearly, classes $\mathcal{G}$ and $\mathcal{H}$ are of bounded expansion (hence they are nowhere dense as well) because every graph belonging to them is a forest[2] – graphs G_n are sets of n stars and graphs H_n are partial matchings.

It remains to prove that class $\mathcal{G} \oplus \mathcal{H}$ is somewhere dense (and thus it cannot be of bounded expansion). Let us take graph $J = G_n \cup H_n$ (both

[1] If both $\pi_1(e)$ and $\pi_2(e)$ are in corresponding graphs we choose to put $\pi_1(e)$ in $E(H_1)$

[2] and every minor of a forest is a forest as well

graphs have the same set of vertices and we consider union of their sets of edges). First $J \in \mathcal{G} \oplus \mathcal{H}$ and it is easy to observe that graph J is a 2-subdivision of a clique K_n – vertices v_{ii} correspond to clique vertices and each edge $v_{ii}v_{jj}$ is divided by vertices v_{ij} and v_{ji}. Hence every clique K_n is a 1-shallow minor of some graph in $\mathcal{G} \oplus \mathcal{H}$ which ends the proof.