Sparsity — tutorial 2

Measuring sparsity

Definition 1. For graphs G, H, their *lexicographic product* $G \bullet H$ is defined as the graph on the vertex set $V(G) \times V(H)$ where (u,v) and (u',v') are adjacent if either $u \neq u'$ and $uu' \in E(G)$, or $u = u'$ and $vv' \in E(H)$. The *c-blowup* of a graph G is the graph $G \bullet K_c$.

Problem 1. Prove that if G is a graph and $r, c \in \mathbb{N}$, then

$$\widetilde{\nabla}_r(G \bullet K_c) \leq 2rc^3 \cdot \widetilde{\nabla}_r(G) + c^2.$$

Infer that there for each r there is a polynomial $P_r(\cdot,\cdot)$ such that

$$\nabla_r(G \bullet K_c) \leq P_r(c, \nabla_r(G)).$$

Conclude that if a class $\mathcal{C}$ has bounded expansion, then for every constant $c \in \mathbb{N}$ the class $\mathcal{C} \bullet K_c = \{G \bullet K_c \colon G \in \mathcal{C}\}$ also has bounded expansion.

Solution. We prove only the first statement, the remaining ones follow from the fact that grads and topological grads are polynomially bounded by one another.

Let $G' \coloneqq G \bullet K_c$. Vertices of G' are of the form (u,i), where $u \in V(G)$ and $i \in \{1,\ldots,c\}$; two such vertices (u,i) and (v,j) are adjacent in G' iff $u = v$ or $uv \in E(G)$. For a fixed $u \in V(G)$, the set of vertices $\{(u,i) \colon i \in \{1,\ldots,c\}\}$ will be called the *fiber* of u.

Since we need to bound $\widetilde{\nabla}_r(G')$ from above, let ϕ be a depth-r topological minor of some graph H in G'. Say that H has n vertices and m edges; we need to bound m by a linear function of n. The strategy is to turn H into some depth-r topological minor of G while losing only a constant multiplicative factor on the density; then we may used the assumed upper bound on $\widetilde{\nabla}_r(G)$.

For every vertex $u \in V(H)$, the vertex $\phi(u) \in V(G')$ will be called the *nail* of u; thus, edges of H are mapped by ϕ to internally vertex-disjoint paths of length at most $2r+1$ connecting corresponding nails. There are n nails in G, hence they are located in at most n fibers.

Call an edge $uv \in E(H)$ *self-problematic* if both nails $\phi(u)$ and $\phi(v)$ are in the same fiber. Observe that each fiber with a nail contains nails of the endpoints of at most $\binom{c}{2} \leq c^2$ self-problematic edges of H, hence the number of self-problematic edges in H is at most $c^2 n$. Construct a subgraph H_1 of H by removing all self-problematic edges; then we may construct a depth-r topological minor model ϕ_1 of H_1 in G by just dropping the images of self-problematic edges in ϕ. It follows that H_1 has at least $m - \binom{c}{2}n$ edges.

Call an edge $uv \in E(H_1)$ *other-problematic* if some internal vertex of $\phi(uv)$ belongs to a fiber in which there is a nail. Observe that each fiber with a nail gives rise to at most $c-1$ other-problematic edges, hence the number of other-problematic edges in H_1 is at most $(c-1)n$. Construct a subgraph H_2 of H_1 by removing all other-problematic edges; again, we may construct a depth-r topological minor model ϕ_2 of H_2 in G by just dropping the images of other-problematic edges in ϕ_1. It follows that H_2 has at least $m - cn - \binom{c}{2}n \geq m - c^2 n$ edges.

Next, we shall say that two edges $e, e' \in E(H_2)$ are *in conflict* if $\phi(e)$ and $\phi(e')$ use vertices from the same fiber. Observe that each edge of $e \in E(H_1)$ is in conflict with at most $(2r-1)(c-1) < 2rc$, since, by the previous steps, only the at most $2r-1$ internal vertices of $\phi(e)$ can give rise to conflicts with other edges, each with at most $c-1$ other. Draw a conflict graph on the $E(H_2)$, where two edges are adjacent iff they are in conflict. This graph has maximum degree smaller than $2rc$, so it admits a proper coloring with $2rc$ colors. By taking the largest color class in this coloring, we find a subset of edges $F \subseteq E(H_2)$ of size at least $|E(H_2)|/2rc \geq \frac{m-c^2n}{2rc}$ such that the edges of F are pairwise not in conflict. Construct a subgraph H_3 of H_2 by removing all the edges apart from those from F; again, dropping the removed edges in ϕ_2 yields a depth-r topological minor model ϕ_3 of H_3 in G. It follows that H_3 has at least $\frac{m-c^2n}{2rc}$ edges.

We now construct a graph H_4 that is a depth-r minor of G. First, for every fiber, say of a vertex $w \in V(G)$, that contains a nail of ϕ_3, we add w to the vertex set of H_4; thus H_4 has at most n vertices.

Next, for every edge uv of H_3, say with $\phi(u)$ belonging to the fiber of $w_u \in V(G)$ and $\phi(v)$ belonging to the fiber of $w_v \in V(G)$, we add the edge $w_u w_v$ to the edge set of H_4. Observe that thus, one edge $w_u w_v$ can be added to H_4 at most c^2 times, once per each pair of vertices from cartesian product of the fiber of w_u and the fiber of w_v. Of course, in H_4 we keep only one copy of this edges, thus H_4 has at least $|E(H_3)|/c^2 \geq \frac{m-c^2n}{2rc^3}$ edges.

Finally, observe that if we map every edge $w_u w_v$ added to H_4 to the natural projection of the path $\phi(uv)$ onto G (for every edge $w_u w_v \in E(H_4)$ we arbitrarily fix one edge $uv \in E(H_3)$ which gave rise to $w_u w_v$), then we obtain a depth-r topological minor model of H_4 in G. This is because these projections are pairwise internally vertex-disjoint and do not pass through vertices whose fibers contained nails of ϕ_3, by all the previous steps. Since H_4 is a depth-r topological minor of G, we have $\frac{|E(H_4)|}{|V(H_4)|} \leq \widetilde{\nabla}_r(G)$. Since $|E(H_4)| \geq \frac{m-c^2n}{2rc^3}$ and $|V(H_4)| \leq n$, we have

$$\widetilde{\nabla}_r(G) \geq \frac{m - c^2 n}{2rc^3 \cdot n} = \frac{1}{2rc^3}\frac{m}{n} - \frac{c^2}{2rc^3}.$$

This implies that $\frac{m}{n} \leq 2rc^3\widetilde{\nabla}_r(G) + c^2$ and concludes the proof. □

Fact 1. If G is a graph and $r, c \in \mathbb{N}$, then

$$\nabla_r(G \bullet K_c) \leq 2c^2(r+1)^2\nabla_r(G) + c.$$

Proof. Let $G' \coloneqq G \bullet K_c$. Vertices of G' are of the form (u, i), where $u \in V(G)$ and $i \in \{1, \ldots, c\}$; two such vertices (u, i) and (v, j) are adjacent in G' iff $u = v$ or $uv \in E(G)$. For a fixed $u \in V(G)$, the set of vertices $\{(u, i) \colon i \in \{1, \ldots, c\}\}$ will be called the *fiber* of u.

Take any depth-r minor H of G, and let ϕ be a depth-r minor model of H in G'. Say that H has m edges and n vertices; we need to give an upper bound on $\frac{m}{n}$. For every vertex $u \in V(H)$, we select any vertex $\gamma(u) \in V(\phi(u))$ that is at distance at most r from every vertex of $\phi(u)$ within this branch set. Moreover, for every edge $uv \in E(H)$, we fix any path P_{uv} in G' that has length at most $2r + 1$ that connects $\gamma(u)$ with $\gamma(v)$.

Call an edge $uv \in E(H)$ *self-problematic* if the fiber to which $\gamma(v)$ belongs also contains some vertex of $\phi(u)$, or vice versa: the fiber to which $\gamma(u)$ belongs also contains some vertex of $\phi(v)$. Observe that every vertex u of H gives rise to at most $c - 1$ self-problematic edges: these are those edges $uv \in E(H)$, for which $\phi(v)$ contains a vertex of the fiber of $\gamma(u)$, and there can be at most $c - 1$ such neighbors v. Therefore, the number of self-problematic edges in H is at most cn.

Next, we construct a depth-r minor H' of G by a random process as follows. First, let σ be a permutation of $V(G)$ chosen uniformly at random. We define the vertex set of H' to comprise all vertices $u \in V(H)$ that satisfy the following condition: if F_u is the fiber of $\gamma(u)$ in G', then among vertices $\{v \colon \phi(v) \cap F_u \neq \emptyset\}$ the vertex u is the earliest with respect to σ. Next, we define the edge set of H' to comprise all non-self-problematic edges $uv \in E(H)$ that satisfy the following condition: there is no vertex $w \in V(H) - \{u, v\}$ that is earlier than both u and v in σ and $\phi(w)$ contains a vertex of some fiber used also by a vertex of P_{uv}. Observe that we may construct a depth-r minor model ϕ' of H' in G as follows: for every $u \in V(H')$, we construct $\phi'(u)$ by taking $\gamma(u)$ and adding for every edge of the form $uv \in E(H')$, the prefix of the path P_{uv} contained in P_u. It is easy to see that this is indeed a minor model: no vertex of G is used in more than one branch set, because only the vertex that is earlier in σ is allowed to use it.

We are left with estimating the density of H'. Observe that for every edge $uv \in E(H)$, there are at most $(2r + 2)(c - 1) \leq 2c(r + 1)$ other vertices $w \in V(H)$ whose branch sets $\phi(w)$ contain a vertex that is in the same fiber as a vertex of P_{uv}. The probability that u and v are before all these vertices u is at least $\frac{2}{(2c(r+1))^2} = \frac{1}{2c^2(r+1)^2}$. This means that a non-problematic edge remains in H' with probability at least $\frac{1}{2c^2(r+1)^2}$, so by linearity of expectation, the expected number of edges in H' is at least $\frac{1}{2c^2(r+1)^2}$ times the number of non-problematic edges in H. Therefore, for some run of the experiment we have that H' contains at least

$$\frac{m - cn}{2c^2(r+1)^2}$$

edges. Since H' is a depth-r minor of G, we have that this number divided n is not larger than $\nabla_r(G)$, which implies

$$\frac{m}{n} \leq 2c^2(r+1)^2\nabla_r(G) + c.$$

This concludes the proof. □